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Nonlocal elasticity in nanobeams: the stress-driven integral model

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ABSTRACT

Nonlocal elastic models have attracted an increasing amount of attention in the past years, due to the promising feature of providing a viable simulation for scale effects in nano-structures and especially in nano-beams designed for use as actuators or sensors. In adapting ERINGEN's nonlocal model of elasticity to flexure of nano-beams, the bending field is expressed as convolution between the elastic curvature field and an averaging kernel. Basic difficulties are involved in this approach due to conflicting requirements imposed on the bending field by equilibrium on one hand and by constitutive conditions on the other one. In the newly proposed constitutive theory the bending field is placed in the proper position of input variable, giving to the elastic curvature field the role of output of the constitutive law, evaluated by convolution between the bending field and an averaging kernel. Conflicting restrictions on the bending field are thus eliminated and existence and uniqueness of the solution are assured under any data. Equivalence between integral and differential constitutive relations is proven to hold for nonlocal laws with a special kernel, under constitutive boundary conditions stemming naturally from the integral relation. When compared with the local limit, a stiffer elastic response is evaluated by the stress-driven nonlocal model, due to normalisation of the kernel. The theory provides an effective methodology for investigating small scale effects in nanobeams, by well-posed problems.

1. Introduction

Early ideas about nonlocal models were contributed in Kröner (1967), Krumhansl, (1968) and Kunin (1968). In literature, the predominant and still basic reference for nonlocal elastic models is Eringen (1983). The strain-driven nonlocal elastic law there proposed is described by the integral convolution[1]

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_{\boldsymbol{\Omega}} \phi_\lambda\big(\mathbf{x}-\boldsymbol{\xi}\big) \cdot (\mathbf{E}\cdot\boldsymbol{\varepsilon}_{el})\big(\boldsymbol{\xi}\big)\, d\boldsymbol{\Omega}_{\boldsymbol{\xi}}\,, \tag{1}$$

and was originally conceived in investigating screw dislocations and RAYLEIGH surface waves. In these nonlocal elasticity problems with unbounded domains, the integral convolution was replaced with an equivalent differential equation under boundary conditions of vanishing at infinity.

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[1] Here $\mathbf{E}$ is the stiffness of local elasticity theory and ϕ_λ is an *averaging kernel*, with the physical dimension of the inverse of a volume, depending on a *nonlocal parameter* λ. The elastic strain $\boldsymbol{\varepsilon}_{el}$ is the input and the stress $\boldsymbol{\sigma}$ is the output.

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This differential formulation was later adopted to analyse simple beam models with standard boundary constraints of structural mechanics (Aydogdu, 2009; Peddieson, Buchanan, & McNitt, 2003; Reddy, 2007; 2010; Reddy & El-Borgi, 2014; Reddy, El-Borgi, & Romanoff, 2014; Reddy & Pang, 2008).

The analysis of statics, dynamics and buckling of elastic nanobeams performed by this approach, faced however serious difficulties and unexpected outcomes even in simplest cases. The situation was actively debated, with recent attempts of overcoming paradoxical results (Barretta, Feo, Luciano, & Marotti de Sciarra, 2016; Fernández-Sáez, Zaera, Loya, & Reddy, 2016; Tuna & Kirca, 2016; Wang & Liew, 2007; Xu, Deng, Zhang, & Xu, 2016).

In order to get well-posed problems, adjustments were proposed in Pisano and Fuschi (2003), Challamel and Wang (2008), Benvenuti and Simone (2013) and Khodabakhshi and Reddy (2015) by following the idea of a local-nonlocal mixture early contributed in Eringen (1972, 1987)

Further models, based on gradient and couple stress formulations, were also adopted in order to analyse size-dependent behaviour of beam-like components in NEMS technologies (Akgöz and Civalek, 2015; Şimşek, 2016; Şimşek and Reddy, 2013; Lam, Yang, Chong, Wanga, and Tonga, 2003; Sedighi, 2014; Sedighi, Keivani, and Abadyan, 2015; Tsiatas, 2009; Yang, Chong, Lam, & Tong, 2002).

Collections of results on this topic can be found in recent review articles and books (Eltaher, Khater, & Emam, 2016; Gopalakrishnan & Narendar, 2013; Rafiee & Moghadam, 2014; Wang, Wang, & Kitamura, 2016; Wang & Arash, 2014) and references cited therein.

Strain and stress gradient models and their relationships with Eringen nonlocal law were discussed in Aifantis (2003, 2009, 2011) and further investigated in Polizzotto (2014, 2015, 2016).

In applying the nonlocal strain-driven integral elastic constitutive law to linearized plane and straight beam models according to Bernoulli–Euler theory, the bending interaction field[2] was assumed to be the output of an integral convolution over the beam length, between an averaging kernel ϕ_λ and the local response to the elastic curvature field χ_{el}:

$$M(x) = \int_a^b \phi_\lambda(x-y) \cdot (K \cdot \chi_{el})(y)\, dy\,, \tag{2}$$

where $K = C^{-1} = I_E$ is the standard local elastic bending stiffness with I_E second moment of the field E of Euler elastic moduli on the beam cross section, in the bending direction.

The many warnings indicating that something was not going in the right way did however not focus on the basic contradiction existing between the strain-driven constitutive model and equilibrium requirements.

It was in Romano, Barretta, Diaco, and Marotti de Sciarra (2017) that the difficulties met in solving even simple beam problems with a nonlocal elasticity governed by the strain-driven integral law (Eq. (2)), were clearly detected as due to impossibility of reconciling the constitutive expression of the bending interaction field with the conditions imposed by equilibrium.

The recent revision of elasticity theory contributed in Romano and Barretta (2013) and Romano, Barretta, and Diaco (2014a, 2014b) has played a decisive role in suggesting a natural way for escaping this concurrence of conflicting rules.

The starting point, in the new model there proposed, is the notion of *geometric stretching* as Lie derivative $\mathcal{L}_{\mathbf{v}_\varphi}(\mathbf{g})$ of the covariant metric tensor field $\mathbf{g}$ along a space-time motion $\boldsymbol{\varphi}_\alpha$, with velocity $\mathbf{v}_\varphi = \partial_{\alpha=0}\boldsymbol{\varphi}_\alpha$ (Romano et al., 2014a).

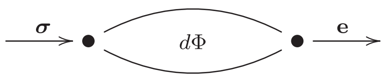
Constitutive laws generate *incremental* responses of the material which depend on the value of a set of state variables (e.g. stress and temperature) and of their increments (Lie derivatives) along the motion.

Kinematic compatibility requires that all incremental responses sum up to the geometric stretching.

In geometrically linearized treatments Lie derivatives are replaced with time-derivatives. Essential to the definition of an elastic model is the existence, of a convex Green's stress-potential Φ such that[3]

$$\mathbf{e} := d\Phi(\boldsymbol{\sigma})\,, \tag{3}$$

a property graphically expressed by:



(4)

The input field $\boldsymbol{\sigma}$ is the *natural stress state*, a contravariant tensor field performing on the dual covariant geometric stretching field, corresponding to any virtual motion, a virtual power per unit mass. According to the original idea attributed to Piola (1833) by Truesdell and Toupin (1960), the natural stress state $\boldsymbol{\sigma}$ is defined, in modern geometrical terms, as Lagrange multiplier of the rigidity constraint expressed by vanishing of the *geometric stretching*.

The *elastic state* $\mathbf{e}$ of the material is the output of the elastic law, characterised by the nonlinear elastic response operator $d\Phi$.

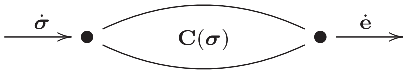
The stress-state increment $\dot{\boldsymbol{\sigma}}$ is the corresponding Lie-derivative of the natural stress field:

$$\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}_\varphi}(\boldsymbol{\sigma})\,. \tag{5}$$

[2] We adopt here the term *bending interaction field* or simply *bending field* instead of the usual term *field of bending moments*.

[3] The symbol d denotes the fiber derivative, according to the time-fibration of the trajectory manifold, that is the derivative performed at a material point while keeping the time fixed (Spivak, 1970).

Taking the LIE time-derivative of Eq. (3), the property of time-invariance of the elastic response $d\Phi$ yields the following constitutive scheme (Romano, Barretta, & Diaco, 2014b):



(6)

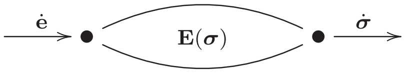
where the natural stress-state increment $\dot{\boldsymbol{\sigma}}$ is the input and the elastic-state increment $\dot{\mathbf{e}} := \mathcal{L}_{\mathbf{v}_\varphi}(\mathbf{e})$ is the output. By EULER-SCHWARZ theorem, the incremental response is expressed by the stress-dependent linear, symmetric and positive definite tangent compliance operator

$$\mathbf{C}(\boldsymbol{\sigma}) = d^2\Phi(\boldsymbol{\sigma}). \tag{7}$$

The formulation in Eq. (6) is the converse of the classical HOOKE's law:

vt tensio sic vis

where the elastic stiffness operator $\mathbf{E}(\boldsymbol{\sigma}) = \mathbf{C}^{-1}(\boldsymbol{\sigma})$, acting on the input elastic-state increment $\dot{\mathbf{e}}$, yields the stress-state increment $\dot{\boldsymbol{\sigma}}$ as output:



(8)

The formulation in Eq. (6) is consistent with the requirement that in an explicit law the input variable should be an otherwise defined notion while the output can well be defined by the constitutive law.

By appealing to the principle of conservation of mass it can then be shown (Romano et al., 2014b) that the elastic model fulfils the basic property that the work of elastic deformation done along any stress-closed path vanishes, as expressed by the implication

$$\boldsymbol{\sigma} = \boldsymbol{\varphi}_{\Delta t}\!\downarrow\!\boldsymbol{\sigma} \;\Rightarrow\; \int_0^{\Delta t} dt \int_{\boldsymbol{\varphi}_t(\boldsymbol{\Omega})} \langle \boldsymbol{\sigma}, \dot{\mathbf{e}} \rangle \, \mathbf{m} = 0. \tag{9}$$

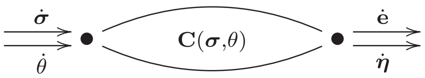
Here $\downarrow$ denotes the pull-back operator and $\langle , \rangle$ is the duality pairing between contravariant and covariant tensor fields. In geometrically linearized treatments the pull-back reduces to a time-translation and hence the l.h.s of Eq. (9) can be simply expressed by the equality $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_{\Delta t}$.

In this new formulation of elasticity theory, the convex stress-potential Φ assumes the role of primal potential while the complementary elastic strain-potential is introduced by EULER–LEGENDRE transform.[4]

To better illustrate the new model, let us sketch its application to the formulation of the thermoelastic constitutive model.

There, the incremental elastic state $\dot{\mathbf{e}}$, induced by stress and temperature increments, is defined as output of a bilinear constitutive operator $\mathbf{C}(\boldsymbol{\sigma}, \theta)$, nonlinearly dependent on stress and temperature states and acting linearly on stress and temperature increments $(\dot{\boldsymbol{\sigma}}, \dot{\theta})$ as input.

The incremental law is depicted in diagram (10), where $\boldsymbol{\eta}$ denotes the entropy field and $\dot{\boldsymbol{\eta}}$ the entropy increment:



(10)

By integrability, the constitutive operator can be expressed as second derivative of GIBBS potential Φ:

$$\mathbf{C}(\boldsymbol{\sigma}, \theta) = d^2\Phi(\boldsymbol{\sigma}, \theta). \tag{11}$$

Analogous models are adopted for other material behaviours.

Along this line of thought, the newly proposed stress-driven nonlocal elastic law is formulated in the context of a geometrically linearized theory at a placement $\boldsymbol{\Omega}$, under the assumption of a compliance operator $\mathbf{C}$ independent of the stress state. The law is described by the integral convolution[5]

$$\Delta\mathbf{e}(\mathbf{x}) = \int_{\boldsymbol{\Omega}} \phi_\lambda\big(\mathbf{x} - \boldsymbol{\xi}\big) \cdot (\mathbf{C} \cdot \Delta\boldsymbol{\sigma})\big(\boldsymbol{\xi}\big) \, d\boldsymbol{\Omega}_{\boldsymbol{\xi}}\,, \tag{12}$$

where $\Delta\mathbf{e}$ and $\Delta\boldsymbol{\sigma}$ are finite increments of elastic and stress states, possibly from a natural configuration where both are assumed to vanish.

It is to be underlined that the new integral convolution law in Eq. (12) is by no means the inverse of ERINGEN's law (Eq. (1)).

[4] This is to be compared with the terminology of *complementary elastic potential* commonly adopted in infinitesimal elasticity for the stress potential.

[5] Here $\mathbf{C} = \mathbf{E}^{-1}$ is the compliance of the local elasticity theory. The symbol $d\boldsymbol{\Omega}_{\boldsymbol{\xi}}$ emphasises that integration is performed with respect to the variable $\boldsymbol{\xi} \in \boldsymbol{\Omega}$. In the nonlinear theory, the elastic strain is the increment of referential elastic states between two body placements. In the linearized theory the elastic strain is expressed simply by the increment $\boldsymbol{\varepsilon}_{el} := \Delta\mathbf{e}$.

By linearity of the convolution operator, the basic properties of the local elastic model translate into analogous properties of the nonlocal elastic model, provided that the nonlocal stress-potential is defined as convolution of the local stress potential with the averaging kernel and all properties are expressed in terms of fields over $\mathbf{\Omega}$ rather than in terms of their local values.

According to the stress-driven nonlocal model of elasticity, in applications to geometrically linearized Bernoulli–Euler nanobeams, the key point consists in adopting a nonlocal elastic relation in which the bending interaction field is the input variable while the elastic curvature field is defined as output of an integral convolution law between the bending interaction field and an averaging kernel.

The new approach will be investigated in detail along the following lines.

By imposing linear kinematic boundary constraints, the variational condition of equilibrium defines an affine subspace of equilibrated bending fields.

In beam's theory, and hence in beams assemblies, this affine subspace is finite dimensional and can be described by detecting a particular equilibrated bending field and a finite basis of self-equilibrated ones.

In the geometrically linearized theory, once the elastic curvature has been expressed by means of the integral nonlocal constitutive law, kinematic compatibility is imposed by means of variational conditions expressing orthogonality, in the mean square sense, to a basis of self-equilibrated bending fields.

The resulting system of linear equations is well-posed and provides the unique compatible elastic curvature field apt to ensure existence of a displacement field conforming to the kinematic boundary constraints. Evaluation of the displacement field is readily performed by integrating the curvature twice.

The widely discussed issue of equivalence between integral and differential formulations of nonlocal constitutive laws and the explanation of paradoxical outcomes of nonlocal analyses of simple beams, based on the strain-driven model, are then addressed.

It is shown that, assuming a special kernel, equivalence between integral and differential forms of the nonlocal constitutive law holds under the imposition of constitutive boundary conditions deduced from the integral law and needed to evaluate integration constants able to ensure fulfilment of the integral law. This differential problem provides an alternative tool for evaluating the nonlocal elastic curvature field.

Ill-posedness of the elastostatic problem governed by the strain-driven nonlocal integral law is assessed and it is shown that the claimed paradoxical outcomes of that theory are generated by management of solutions of elastostatic problems that in fact do not admit solution, for any positive value of the nonlocal parameter.

On the contrary, well-posedness of elastostatic problems based on the stress-driven nonlocal model is assured. Absence of conflicting conditions is evidenced by the solution of simple beam problems.

2. The stress-driven integral constitutive law

To illustrate the proposed nonlocal model of elastic beams, in view of applications to the analysis of nanobeams adopted for realisation of actuators and sensors, we consider its specialisation to plane and straight simple beams.

In the sequel a dot $\cdot$ denotes linear dependence, and the crochét $\langle \cdot, \cdot \rangle$ is the duality pairing.

The beam length is the difference of end-point abscissae $L = b - a > 0$ and $\lambda > 0$ is the nonlocal parameter with $L_c = \lambda \cdot L$ characteristic length measuring the nonlocality effect.

As usual, $\boldsymbol{\delta}(x)$ denotes the Dirac delta distribution at $0 \in \Re$ (unit impulse) formally defined by

$$\int_a^b f(y) \cdot \boldsymbol{\delta}(x - y)\, dy = f(x)\,, \tag{13}$$

for any continuous test function $f \in \mathcal{C}^0(a, b)$.

The standard local elastic curvature is $C \cdot M$, where $M \in \mathcal{L}^2(a, b)$ is the bending interaction and C is the positive elastic bending compliance.

In the geometrically linearized nonlocal theory, the elastic curvature field along a straight beam of length $L = b - a > 0$ is dependent on the whole square integrable bending field on $[a, b]$ denoted by $M \in \mathcal{L}^2(a, b)$.

The elastic curvature is in fact defined by convolution between the local elastic curvature $C \cdot M$ induced by the bending interaction and a scalar averaging kernel ϕ_λ having the physical dimension of reciprocal of a length. The law is sketched in diagram (14) below, where C is the flexural compliance and $\star$ denotes the convolution:

$$\xrightarrow{\;M\;} \bullet \;\; \phi_\lambda \star C \cdot () \;\; \bullet \xrightarrow{\;\chi_{el}\;} \tag{14}$$

and is expressed by

$$\chi_{el}(x) = \;(\phi_\lambda \star (C \cdot M))(x) := \int_a^b \phi_\lambda(x - y) \cdot (C \cdot M)(y)\, dy, \tag{15}$$

with M and χ_{el} finite increments of bending interaction and elastic curvature, possibly from a natural configuration where both vanish.

The kernel fulfils symmetry, positivity and limit impulsivity:

$$\begin{cases} i) & \phi_\lambda(x-y) = \phi_\lambda(y-x) \geq 0\,, \\ ii) & \lim\limits_{\lambda\to 0} \phi_\lambda(x,\lambda) = \boldsymbol{\delta}(x). \end{cases} \tag{16}$$

Property $i)$ assures that the quadratic form

$$\int_a^b \int_a^b \phi_\lambda(x-y)\cdot\langle (C\cdot M)(y), M(x)\rangle\, dy\, dx \geq 0\,, \tag{17}$$

is vanishing only if $M=0$ identically. The distributional expression of the formal definition of limit impulsivity in item $ii)$ of Eq. (16) writes

$$\lim_{\lambda\to 0} \int_{-\infty}^{+\infty} \phi_\lambda(x-y)\cdot(C\cdot M)(y)\, dy = (C\cdot M)(x). \tag{18}$$

As apparent from the definitions, the kernel has the physical dimension of inverse of a length and when $\lambda \to 0$ it tends to the Dirac impulse at the origin. Consequently the nonlocal constitutive law tends to reproduce the local one, and so goes for the solution of the elastostatic problem, except for boundary effects due to boundedness of the structural domain $[a,\, b]$ (Romano & Barretta, 2017).

3. Nonlocal elastostatic problem

Let us denote by $\mathcal{H} = H^0(a,b) = \mathcal{L}^2(a,b)$ the linear Hilbert space of square integrable fields on $[a,\, b]$ with the inner product

$$(u,v) = \int_a^b u\cdot v\, dx\,, \quad \forall\, u,v\in\mathcal{H}. \tag{19}$$

The Sobolev space $V = H^2(a,b) \subset H^0(a,b)$ is the linear subspace of those with square integrable first and second generalised derivatives, so that being $v, v', v'' \in \mathcal{H}$ the inner product is given by

$$((u,v)) = \int_a^b (u\cdot v + u'\cdot v' + u''\cdot v'')\, dx, \quad \forall\, u,v\in V, \tag{20}$$

where the apex $'$ denotes derivative along the beam axis x.

Functions in V have well-defined boundary values of zeroth and first order derivatives.

We may thus consider a prescribed displacement field $w\in V$ and the closed affine manifold $\mathcal{L}\subset V$ of admissible displacements

$$\mathcal{L} = w + \mathcal{L}_o \subset V\,, \tag{21}$$

fulfilling kinematic boundary conditions imposed on displacement fields and on their first derivatives, at the end points a, $b\in\Re$ of the beam.

The linear space of conforming displacements $\mathcal{L}_o\subset V$ is parallel to $\mathcal{L}$ and made of those fulfilling the corresponding homogeneous kinematic boundary conditions.

Force systems acting on the beam are in the dual space V^* of V. Constraint's reactions are force systems vanishing on conforming displacements. The reaction's space is then $\mathcal{R} := \mathcal{L}_o^\perp \subset V^*$.

Effective loadings are in the Hilbert space $\mathcal{L}_o^*$ dual to $\mathcal{L}_o$ and are required to obey the equilibrium condition of vanishing virtual work for any conforming and rigid virtual displacement

$$f\in\mathcal{L}_o^*\cap \ker(\chi)^\perp \iff \langle f, \delta v\rangle = 0\,, \quad \forall\, \delta v\in\mathcal{L}_o \,:\, \chi_{\delta v} = 0\,, \tag{22}$$

where χ is the geometric curvature operator defined by

$$\chi_{\delta v} := \delta v''. \tag{23}$$

The linear space $\mathcal{L}_o^*$ can be identified with the quotient space $V^*/\mathcal{R}$ of force systems in V^* equivalent modulo reaction systems in $\mathcal{R}$.

Boundary kinematic conditions are characterised, among other possible kinematic conditions, by the property that indefinitely continuously differentiable displacements with compact support in the open interval $(a,\, b)$ (that is vanishing in a boundary layer) are conforming, i.e. $\mathcal{C}_o^\infty(a,b)\subset\mathcal{L}_o$.

This is the key property allowing for localisation of variational conditions to get equivalent differential equations and boundary conditions, since the linear subspace $\mathcal{C}_o^\infty(a,b)$ is dense in $H^0(a,b) = \mathcal{L}^2(a,b)$.

The nonlocal elastic problem is then characterised by the following items.

1. Equilibrium under a prescribed loading $f\in\mathcal{L}_o^*$ is imposed on bending fields $M\in H$ by the virtual work variational condition

$$\int_a^b \langle M(x),\, \chi_{\delta v}(x)\rangle\, dx = \langle f, \delta v\rangle\,, \tag{24}$$

for all conforming virtual displacements $\delta v \in \mathcal{L}_o$ and associated geometric virtual curvature $\chi_{\delta v}$.
Bending fields $M \in \mathcal{H}$ in equilibrium with a prescribed loading $f \in \mathcal{L}_o^*$ belong to a linear variety

$$\Sigma_f = M_f + \Sigma_o , \tag{25}$$

described by a particular bending field $M_f \in \mathcal{H}$ fulfilling the equilibrium (Eq. (24)) with the loading $f \in \mathcal{L}_o^*$ and by the subspace Σ_o of all self-equilibrated bending fields M_o, characterised by the variational condition

$$\int_a^b \langle M_o(x), \chi_{\delta v}(x) \rangle \, dx = 0 , \quad \forall \, \delta v \in \mathcal{L}_o. \tag{26}$$

2. Kinematic compatibility is expressed by the requirement that the sum of the nonlocal elastic curvature $\chi_{el} \in \mathcal{H}$ and of the anelastic curvature $\chi_o \in \mathcal{H}$ (thermal, plastic, etc.) must be equal to the geometric curvature

$$\chi_{el} + \chi_o = \chi_u. \tag{27}$$

Here $u \in \mathcal{L}$ is an admissible displacement and

$$\chi_u := u'' , \tag{28}$$

is the associated geometric curvature. To deal with linear problems it is expedient to write, according to Eq. (21)

$$u = v + w , \quad v \in \mathcal{L}_o , \quad w \in V. \tag{29}$$

so that

$$\chi_{el} + \chi_o = \chi_u = \chi_v + \chi_w. \tag{30}$$

Kinematic compatibility (Eq. (27)) can then be implicitly expressed by the variational requirement of mean square orthogonality of the conforming curvature, defined by

$$\chi_{el} + \chi_o - \chi_w \in \mathcal{H} , \tag{31}$$

to all self-equilibrated bending fields $\delta M_o \in \Sigma_o$.

$$\int_a^b \langle (\chi_{el} + \chi_o - \chi_w)(x), \delta M_o(x) \rangle \, dx = 0. \tag{32}$$

This condition is in fact necessary and sufficient for the existence of a conforming displacement field $v \in \mathcal{L}_o$ such that

$$\chi_v = \chi_{el} + \chi_o - \chi_w \in \mathcal{H}. \tag{33}$$

Substituting the expression for χ_{el} given by nonlocal elastic law (Eq. (15)), the variational condition of kinematic compatibility (Eq. (32)) writes

$$\int_a^b \int_a^b \phi_\lambda(x-y) \cdot \langle (C \cdot M)(y), \delta M_o(x) \rangle \, dy \, dx = \int_a^b \langle \chi_w - \chi_o, \delta M_o(x) \rangle \, dx , \tag{34}$$

for all $\delta M_o \in \Sigma_o$. Imposing equilibrium, we have that

$$M = M_f + M_o \in \Sigma_f , \qquad M_o \in \Sigma_o. \tag{35}$$

Setting

$$\chi_f(x, \lambda) := \int_a^b \phi_\lambda(x-y) \cdot (C \cdot M_f)(y) \, dy , \tag{36}$$

we define the curvature data χ_d by

$$\chi_d := \chi_w - \chi_o - \chi_f \in \mathcal{H}. \tag{37}$$

Elastic equilibrium is then expressed in terms of self-equilibrated trial bending fields $M_o \in \Sigma_o$ by the variational condition

$$\int_a^b \int_a^b \phi_\lambda(x-y) \cdot \langle (C \cdot M_o)(y), \delta M_o(x) \rangle \, dy \, dx = \int_a^b \langle \chi_d(x, \lambda), \delta M_o(x) \rangle \, dx , \tag{38}$$

for all $\delta M_o \in \Sigma_o$.

The elastostatic problem formulated by Eq. (38) is linear and well-posed. It admits a unique solution $M_o \in \Sigma_o$ for any data $\chi_d \in \mathcal{H}$, that is for any $\{ f, w, \chi_o \} \in \mathcal{L}_o^* \times V \times \mathcal{H}$ and the solution depends in a continuous way on the data.

The bending field is evaluated by Eq. (35) as $M = M_f + M_o \in \Sigma_f$. The nonlocal elastic curvature χ_{el} evaluated by Eq. (15) fulfils the condition of kinematic compatibility (Eq. (32)) and hence a double integration yields the unique[6] admissible displacement $u \in \mathcal{L}$ at solution.

[6] Rigid displacement fields are assumed to be non-conforming, that is not allowed by kinematic constraints, as mandatory in computational treatments.

4. Computational method

In one-dimensional structures, such as assemblies of Bernoulli–Euler beams, the linear subspace of self-equilibrated bending interactions is of finite dimension, being the kernel of a system of ordinary differential equations with homogeneous boundary conditions.

This finite dimension n is the *degree of statical indeterminacy* of the structural assembly.

The program described in §3 can be effectively carried out by detecting a particular equilibrated bending field $M_f \in \mathcal{H}$ and a finite basis

$$\{ M_k \,,\ 1 \le k \le n \} \,, \tag{39}$$

of the linear subspace Σ_o of all self-equilibrated bending fields. Any equilibrated bending field can then be represented as an affine combination

$$M = M_f + \sum_{k=1}^{n} p_k \cdot M_k. \tag{40}$$

By the constitutive relation (Eq. (15)), the elastic curvature is described as

$$\chi_{el} = \chi_f + \sum_{k=1}^{n} p_k \cdot \chi_k \,, \tag{41}$$

where

$$\begin{aligned} \chi_f(x) &:= \int_a^b \phi_\lambda(x-y) \cdot (C \cdot M_f)(y)\, dy \,, \\ \chi_k(x) &:= \int_a^b \phi_\lambda(x-y) \cdot (C \cdot M_k)(y)\, dy. \end{aligned} \tag{42}$$

The parameters $\mathbf{p} = \{ p_k \,,\ 1 \le k \le n \}$ are detected by imposing kinematic compatibility through (Eq. (32)) to get the linear system

$$\mathbf{A} \cdot \mathbf{p} = \mathbf{b} \iff A_{ik}\, p_k = b_i \,, \quad 1 \le i, k \le n \,, \tag{43}$$

where, according to Eq. (38), we have

$$\begin{aligned} A_{ik} &:= \int_a^b \langle M_i(x), \chi_k(x) \rangle \, dx \\ &= \int_a^b \int_a^b \phi_\lambda(x-y) \cdot \langle (C \cdot M_i)(y), M_k(x) \rangle \, dx\, dy \,, \\ b_i &:= \int_a^b \langle M_i(x), \chi_d(x) \rangle \, dx. \end{aligned} \tag{44}$$

By symmetry and positive definiteness of the matrix $\mathbf{A}$, the linear system (Eq. (43)) admits a unique solution $\mathbf{p}$ for any $\mathbf{b}$. Substituting in Eq. (40) we get the bending field $M \in \Sigma_f$. Integrating the differential equation (Eq. (33)), with homogeneous kinematic boundary constraints and with χ_{el} given by Eq. (41), the conforming solution $v \in \mathcal{L}_o$ is found. The admissible displacement field solution of the nonlocal elastic problem is got by adding the prescribed displacement field $w \in V$.

5. Averaging kernels

Averaging kernels usually adopted in one-dimensional formulations, with $\lambda > 0$ nonlocal parameter and $L_c = \lambda \cdot L$ characteristic length, are the following:

$$\phi_\lambda(x) := \frac{1}{2\, L_c} \exp\Big(- \frac{|x|}{L_c} \Big) \,, \tag{45}$$

$$\psi_\lambda(x) := \frac{1}{L_c \sqrt{2\, \pi}} \exp\Big(- \frac{x^2}{2\, L_c^2} \Big). \tag{46}$$

Both expressions in Eqs. (45) and (46) fulfil the properties in Eq. (16).

The kernel (Eq. (46)) is the normal distribution with zero mean and standard deviation equal to λ. By setting $\lambda = t$ it can be also described as the Green function associated with the diffusion differential equation with the Dirac impulse $\delta(x)$ as initial condition (Eringen, 1983):

$$\begin{cases} \dfrac{d\psi}{dt}(x,t) - \psi''(x,t) = 0 \,, \\ \lim\limits_{t \to 0} \psi(x,t) = \delta(x). \end{cases} \tag{47}$$

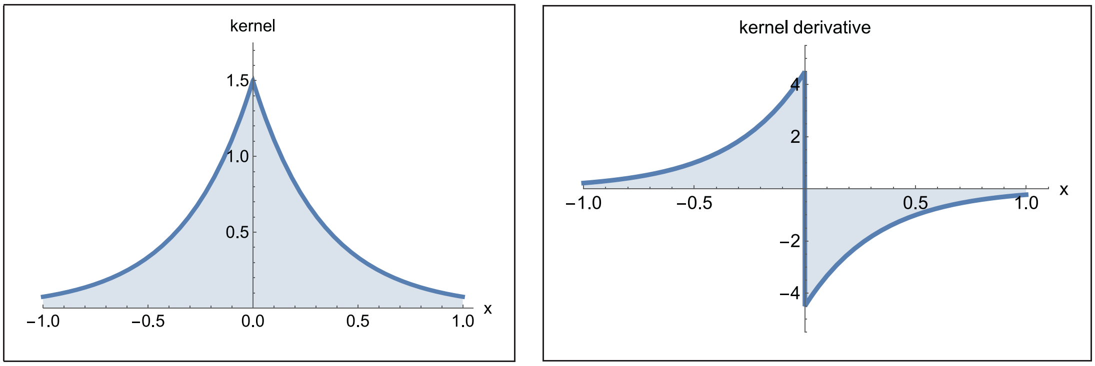


Fig. 1. Special kernel, $\lambda = 1/3$: Left ϕ_λ – Right ϕ'_λ.

The numerical results got by adopting the kernels ϕ_λ and ψ_λ are technically indistinguishable in the local limit $\lambda \to 0$, but the kernel ϕ_λ has peculiar properties, as illustrated below.

6. Special kernel and Green's function

A peculiar result is consequent to the choice of the kernel ϕ_λ of Eq. (45) which will therefore be referred to as the *special kernel*, see Fig. 1. Symbolic and numerical computations and graphic visualisation have been carried out by the software Mathematica authored by Stephen Wolfram.

The kernel ϕ_λ is described by the rule

$$\phi_\lambda(x) = \begin{cases} \omega_\lambda(x)\,, & x < 0\,, \\ \omega_\lambda(-x)\,, & x > 0\,, \end{cases} \tag{48}$$

where

$$\omega_\lambda(x) := \frac{1}{2\,L_c}\,\exp\left(\frac{x}{L_c}\right). \tag{49}$$

The first and second derivatives of the map (Eq. (48)) evaluate to

$$\phi'_\lambda(x) = \begin{cases} \dfrac{1}{L_c}\cdot\omega_\lambda(x) = \dfrac{1}{L_c}\cdot\phi_\lambda(x)\,, & x < 0\,, \\ -\dfrac{1}{L_c}\cdot\omega_\lambda(-x) = -\dfrac{1}{L_c}\cdot\phi_\lambda(x)\,, & x > 0\,, \end{cases} \tag{50}$$

$$\phi''_\lambda(x) = \begin{cases} \dfrac{1}{L_c^2}\cdot\omega'_\lambda(x) = \dfrac{1}{L_c^2}\cdot\phi_\lambda(x)\,, & x < 0\,, \\ \dfrac{1}{L_c^2}\cdot\omega'_\lambda(-x) = \dfrac{1}{L_c^2}\cdot\phi_\lambda(x)\,, & x > 0\,, \end{cases} \tag{51}$$

with

$$\phi_\lambda(0) = \omega_\lambda(0) = \frac{1}{2\,L_c}. \tag{52}$$

The jump discontinuity in the first derivative of ϕ_λ at $x = 0$ is given by

$$[[\phi'_\lambda(0)]] = -\frac{1}{L_c^2}. \tag{53}$$

Hence from Eq. (51) we get

$$\phi''_\lambda(x) = \frac{1}{L_c^2}(\phi_\lambda(x) - \boldsymbol{\delta}(x)). \tag{54}$$

The results in Eqs. (50) and (51) lead to the following characterisation of the special kernel as a Green's function.

Proposition 6.1. *The special kernel ϕ_λ is the Green function, solution of the differential problem Eq. (54) with the boundary conditions at $a < 0 < b$ given by*

$$\begin{cases} \phi'_\lambda(a) = \frac{1}{L_c}\cdot\phi_\lambda(a), \\ -\phi'_\lambda(b) = \frac{1}{L_c}\cdot\phi_\lambda(b). \end{cases} \tag{55}$$

The differential problem admits a unique solution, since the associated homogeneous problem, with a vanishing impulse, admits only the trivial solution.

In nonlocal elasticity, on the contrary, the convolution integral is extended over the bounded domain of definition of the input field, so that constitutive boundary conditions are naturally induced as an essential part of the constitutive differential problem whose solution yields the relevant GREEN's function, as explicated by Eqs. (54) and (55).

7. Differential formulation and constitutive boundary conditions

The convolution of the special kernel ϕ_λ with the local elastic curvature $C \cdot M$ will be referred to as the *special integral law*.

Proposition 7.1. *The output of the special integral constitutive law*

$$\chi_{el}(x) = \int_a^b \phi_\lambda(x-y) \cdot (C \cdot M)(y)\, dy\,, \tag{56}$$

provides the unique solution of the constitutive differential equation:

$$\frac{\chi_{el}(x)}{L_c^2} - \chi_{el}''(x) = \frac{C \cdot M}{L_c^2}(x)\,, \tag{57}$$

with the homogeneous constitutive boundary conditions

$$\begin{cases} \chi_{el}'(a) = \dfrac{1}{L_c} \cdot \chi_{el}(a)\,, \\ -\chi_{el}'(b) = \dfrac{1}{L_c} \cdot \chi_{el}(b). \end{cases} \tag{58}$$

Proof. Splitting the integral in Eq. (56) according to the partition

$$[a, b] = [a, x] \cap (x, b]\,, \tag{59}$$

we may set

$$\chi_{el}(x) = \chi_1(x) + \chi_2(x)\,, \tag{60}$$

with

$$\begin{cases} \chi_1(x) := \displaystyle\int_a^x \phi_\lambda(x-y) \cdot (C \cdot M)(y)\, dy \\ \qquad\quad = \displaystyle\int_a^x \frac{1}{2\,L_c} \exp\left(\frac{y-x}{L_c}\right) \cdot (C \cdot M)(y)\, dy\,, \\ \chi_2(x) := \displaystyle\int_x^b \phi_\lambda(x-y) \cdot (C \cdot M)(y)\, dy \\ \qquad\quad = \displaystyle\int_x^b \frac{1}{2\,L_c} \exp\left(\frac{x-y}{L_c}\right) \cdot (C \cdot M)(y)\, dy. \end{cases} \tag{61}$$

Taking the first derivative of Eq. (61) we get

$$\begin{cases} \chi_1'(x) = \dfrac{1}{2\,L_c}\, \chi_{el}(x) - \dfrac{1}{L_c}\, \chi_1(x)\,, \\ \chi_2'(x) = -\dfrac{1}{2\,L_c}\, \chi_{el}(x) + \dfrac{1}{L_c}\, \chi_2(x)\,, \end{cases} \tag{62}$$

so that the first derivative of the nonlocal curvature is given by

$$\chi_{el}'(x) = \frac{1}{L_c} \cdot \Big(\chi_2(x) - \chi_1(x)\Big). \tag{63}$$

A further derivation gives (Eq. (57)). Evaluating the first derivative (Eq. (63)) at the boundary points and observing that

$$\begin{cases} \chi_2(a) = \chi_{el}(a)\,, \quad \chi_1(a) = 0\,, \\ \chi_1(b) = \chi_{el}(b)\,, \quad \chi_2(b) = 0\,, \end{cases} \tag{64}$$

yields the constitutive boundary conditions (Eq. (58)). Uniqueness of solution follows since it easily proven that the corresponding homogeneous differential problem ($\chi_{el} = 0$) with the homogeneous constitutive boundary conditions admits only the trivial solution. The proof of Proposition 7.1 could also be got by taking the convolution of Eq. (54) with the elastic curvature χ_{el} and observing that the constitutive boundary conditions in Eq. (55) are homogeneous, so that, by linearity, they are preserved by the convolution. □

As a direct consequence of Proposition 7.1, the curvature fields χ_f and χ_k in Eq. (42) can also be evaluated by solving the differential problems got by setting $M = M_f$ and $M = M_k$ in Eq. (57), with the boundary conditions (Eq. (58)).

Remark 7.1 (Differential and boundary conditions)**.** The problem of nonlocal elastic equilibrium can be expressed as a differential equation in the unknown transversal displacement $u\colon [a, b] \mapsto \Re$. The differential conditions of kinematic compatibility and equilibrium are imposed by setting $\chi_{el} = u''$ and $M'' = q$, with $q\colon [a, b] \mapsto \Re$ intensity of the transversal loading. Under a uniform bending stiffness, differentiating twice the constitutive differential expression (Eq. (57)) gives the sixth order equation:

$$\frac{u^{\mathrm{IV}}}{L_c^2} - u^{\mathrm{VI}} = \frac{C \cdot q}{L_c^2}\,, \tag{65}$$

In addition to boundary conditions expressing kinematic and natural constraints, two more conditions are to be imposed to get a unique solution of the nonlocal elastic problem. These are provided by the constitutive boundary conditions (Eq. (58)). The problem is thus closed and existence and uniqueness of a solution are assured. In the limit $\lambda \to 0$, the differential expression in Eq. (65) tends to the standard fourth order one of local elasticity where constitutive boundary conditions are not needed.

8. Comparison with the strain-driven integral model

Let us now compare the new stress-driven model expressed by Eq. (56) with the strain-driven model adopted in literature, in the wake of the original formulation in Eringen (1983). In that model the bending field M was defined in terms of the elastic curvature χ_{el} by the convolution

$$M(x) = \Big(\phi_\lambda \star (K \cdot \chi_{el})\Big)(x)\,, \tag{66}$$

where $K = C^{-1} = I_E$ is the standard local elastic bending stiffness with I_E second moment of the field E of Euler elastic moduli on the beam cross section, in the bending direction.

The curvature-driven law of Eq. (2) is depicted in the diagram

$$\xrightarrow{\;\chi_{el}\;} \bullet \;\; \phi_\lambda \star K \cdot () \;\; \bullet \xrightarrow{\;M\;} \tag{67}$$

which is to be compared with the diagram (14).

We see that the strain-driven convolution of Eq. (66) provides an implicit definition of the input elastic curvature, as solutions of the Fredholm integral equation for given output bending interaction. The difficulty is that this integral equation may have no solution at all for bending interactions fulfilling the equilibrium condition, as revealed by the next result.

By adopting in Eq. (2) the special kernel (Eq. (45)), a procedure formally analogous to the one adopted in Proposition 7.1 leads to the following result.

Proposition 8.1. *The output of the nonlocal integral constitutive law*

$$M(x) = \int_a^b \phi_\lambda(x-y) \cdot (K \cdot \chi_{el})(y)\, dy\,, \tag{68}$$

with the special kernel Eq. (45), provides the unique solution of the constitutive differential problem:

$$\frac{M(x)}{L_c^2} - M''(x) = \frac{K \cdot \chi_{el}}{L_c^2}(x)\,, \tag{69}$$

with the homogeneous constitutive boundary conditions

$$\begin{cases} M'(a) = \dfrac{1}{L_c} \cdot M(a)\,, \\ -M'(b) = \dfrac{1}{L_c} \cdot M(b)\,. \end{cases} \tag{70}$$

The differential equation Eq. (69) is still widely adopted in treatments of nonlocal elastic beams, without mention of constitutive boundary conditions.[7] Necessity of these boundary conditions was evidenced in Polyanin and Manzhirov (2008) and discussed in Benvenuti and Simone (2013). Failure of the strain-driven nonlocal elastic law was finally clarified by proving non-existence of solutions for all elastic equilibrium problems of engineering interest (Romano et al., 2017).

A decisive difference emerges when the strain-driven integral model is confronted with the new stress-driven integral model illustrated in Section 2.

[7] In Gopalakrishnan and Narendar (2013, Eq. (4.26)) Eq. (69) is quoted as a *simplified nonlocal constitutive relation* and, dealing with wave propagation, conditions of decay at infinity are assumed.

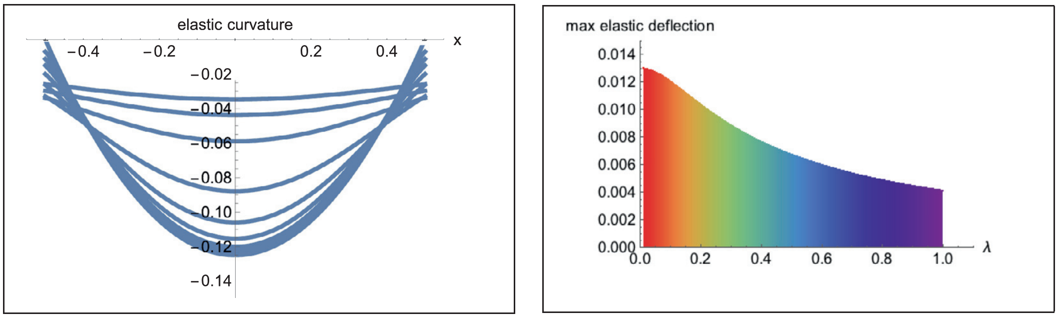


Fig. 2. Simply supported beam (Section 9.1). Elastic curvature – maximum deflection.

1. The differential and boundary equilibrium conditions on the bending interaction field M are generally incompatible with the constitutive law output of the integral convolution Eq. (2). In fact kinematic boundary equilibrium conditions are in contrast with the constitutive boundary conditions of Eq. (70). This fact renders the model unsuitable for the formulation of well-posed nonlocal elastic problems (Romano & Barretta, 2016; Romano et al., 2017).
2. On the contrary, in the new stress-driven model no conflict occurs between the constitutive requirement imposed by the integral convolution (Eq. (56))and the condition of kinematic compatibility, either in the explicit formulation $\chi_{el} = u''$ (Eq. (27)) or in the equivalent implicit variational formulation of Eq. (32).

This difference is the key motivation for replacing the strain-driven model with the stress-driven model.

8.1. Constitutive vs. kinematic boundary conditions

The constitutive boundary conditions (Eq. (70)) imposed on the bending field according to the strain-driven model, being intrinsic to the integral constitutive law, have no relation with the boundary conditions imposed by equilibrium. Accordingly, the Green's function providing the averaging kernel of the convolution is independent of kinematic and static boundary conditions imposed on the structural model, contrary to statements in literature (Challamel et al., 2014). This basic property is in line with the physical requirement of reproducibility[8] of a structural problem in continuum mechanics. Reproducibility implies that constitutive properties should concern only the material which the body is made of and cannot depend on the imposed kinematical constraints. In this respect it is to be underlined that in the new model the constitutive boundary conditions (Eq. (58)) are imposed on the elastic curvature field χ_{el} while kinematic boundary constraints act on the displacement field and its first derivative, and static boundary constraints are deducted by duality from the variational condition of equilibrium, as illustrated in Section 3.

9. Examples

Three simple beam problems with the Bernoulli–Euler kinematics have been investigated to illustrate effectiveness of the new nonlocal theory:

1. A simply supported beam under uniform load.
2. A cantilever under end-point load.
3. A doubly clamped beam under uniform load.

Solutions are got by straightforward application of the theory described in Section 4. The evaluations of the involved convolutions and the graphic output have been carried out with Wolfram's software Mathematica.

In all examples the length of the beam is $L = 1$ and the nonlocal parameter λ ranges in the list of values

$$\{.0001, .02, .05, .07, .10, .15, .25, .50, .75, 1.00\}.$$

9.1. Simply supported beam under uniform load

The bending field is parabolic and the corresponding nonlocal elastic curvature field and mid-point deflection are depicted in Fig. 2.

The nonlocal curvature is parabolic for $\lambda = 0$ (i.e. in the local case) and becomes more and more smaller and uniform as the nonlocal parameter λ increases. The progressive reduction of the mid-span deflection versus the increase of λ is apparent from Fig. 2.

[8] The axiom of reproducibility states that general laws of continuum physics, expressed with reference to a given body placement, must be applicable as well to any sub-boby placement. A well-known instance is the Euler–Cauchy principle of sectioning, concerning equilibrium of every part of a continuous body (Romano, 2014).

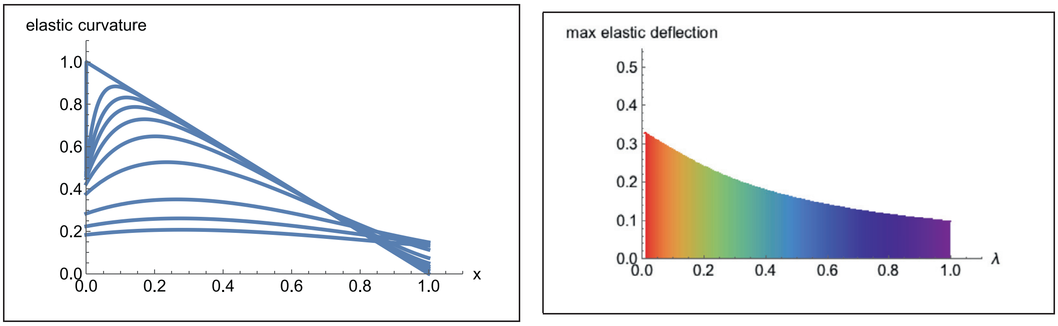


Fig. 3. Cantilever beam (Section 9.2). Elastic curvature – maximum deflection.

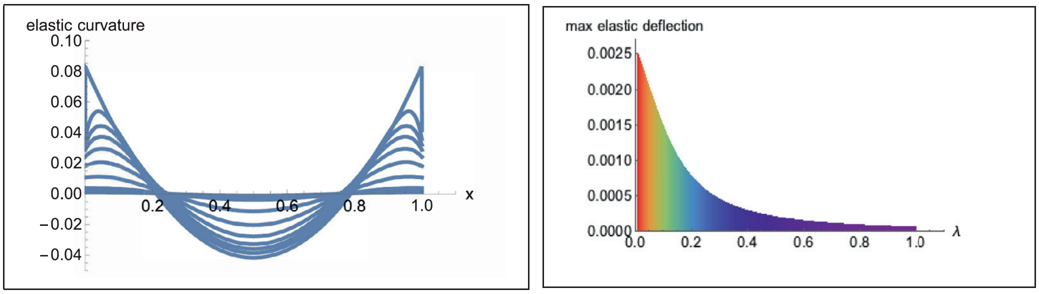


Fig. 4. Clamped beam (Section 9.3). Elastic curvature – maximum deflection.

9.2. Cantilever under end-point load

The bending field is linear and the corresponding nonlocal elastic curvature field and end-point deflection are depicted in Fig. 3. The nonlocal curvature is linear for $\lambda = 0$ (i.e. in the local case) and becomes lower and uniform as the nonlocal parameter λ increases, as shown in Fig. 3.

9.3. Doubly clamped beam under uniform load

For a doubly clamped beam the stiffening effect due to the increase of the nonlocal parameter is still greater, as depicted in Fig. 4.

10. Concluding remarks

The new theory of nonlocal elasticity, illustrated above with special reference to elastostatics of simple beams, is a natural outcome of the revised formulation of elasticity introduced in Romano and Barretta (2013), Romano et al. (2014a) and Romano et al. (2014b) where the input state variable in the elastic constitutive law is the field of natural stress-state while the elastic-state is the output. This means that, in the geometrically linearized Bernoulli–Euler beam model, the elastic curvature field is a pointwise linear function of the bending interaction field.

In the nonlocal elastic law, the elastic curvature field is expressed by an integral convolution between the local elastic response to the bending interaction and an averaging kernel.

A careful investigation of the strain-driven approach to nonlocal elasticity revealed that there are basic points deserving attention. These are summarised below.

1. The convolution in Eq. (2), which was intended to be an adaptation to Bernoulli–Euler beam model of those proposed for 2D and 3D continua in Eringen (1983), involves an intrinsic difficulty since the elastic curvature field χ_{el}, appearing under the integral sign, should in fact be defined in terms of the output bending interaction field M. The expression in Eq. (2) is therefore an integral equation to be satisfied by the elastic curvature field. Solvability of this problem holds if and only if the bending field meets the constitutive boundary conditions (Eq. (70)).
2. The requirements of the constitutive boundary conditions (Eq. (70)) are contrasted by the concurring equilibrium conditions on the bending field, so that in general the nonlocal elastostatic problem does not admit solution, for any value $\lambda > 0$ of the nonlocal parameter.

A careful check of equilibrium reveals the failure of beam problems treated in literature according to the strain-driven nonlocal elastic law.

The new theory, based on the constitutive law (Eq. (15)) is free from these difficulties and generates well-posed nonlocal elastic problems which admit a unique solution for any prescribed data, loading, imposed distortions and constraint displacements.

We underline that the differential equation stemming from the new approach is identical to the one exposed in Aifantis (2009, 2011) in the context of a strain-gradient model of elasticity. A significant innovation is that the required additional boundary conditions, are directly and univocally provided by the new stress-driven model, as illustrated in Remark 7.1.

Numerical evaluations witness the effectiveness of the new model of nonlocal elastic behaviour in solving simple beam problems of applicative interest.

Increasing values of the nonlocal parameter λ correspond to stiffer responses, an effect due to the kernel normalisation leading to the basic impulsivity property in Eq. (16). An increase of λ lowers down the peak value of the scalar kernel and enlarges its standard deviation from the mean point. Reduction of the peak is prevalent and hence the elastic compliance is accordingly reduced.

This general feature implies that nanobeams, investigated by the new nonlocal elastic model, are stiffer that the standard local ones.

The theory based on the stress-driven nonlocal constitutive law described by the convolution in Eq. (15) proposes itself as an effective substitute to the analogous strain-driven model. A new road is thus opened to bypass ill-posed structural problems.

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