Acta Mech
https://doi.org/10.1007/s00707-017-2050-x

ORIGINAL PAPER

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On the role of control windows in continuum dynamics

Received: 28 May 2017

Abstract The general role of control windows is investigated by an assessment of the fundamentals of continuum dynamics in the spacetime context. The momentum rate of change of a continuous body is evaluated with reference to a control window travelling in the spacetime trajectory manifold. The resulting formula involves the rate of change of the momentum in the control window and the rate of outflow of the momentum across the control window boundary. An effective tool for investigating challenging dynamical problems is thus available, and some interesting examples, taken from early and recent contributions, are illustrated. Among these, motions of chains falling under the action of gravity, object of several investigations in the past two centuries, are revisited. Around the middle of the nineteenth century, the problem was inserted as a daring task in the Mathematical Tripos at Cambridge. The choice of a special control window, named the skeleton, in motion with the solid case, provides the proper methodology for the formulation of solid–fluid interaction problems, with applications to turbines, jets, rockets and sprinklers. As a further significant application of travelling control windows, a variational formulation of conservation of mass is developed and shown to yield the notion of mass flow vector field, with applications in problems of fluid flow through a porous solid.

1 Introduction

A modern geometric account of fundamentals of continuum dynamics is introduced in the general context of Euclid spacetime, so that a general framework for the proper treatment of the problems at hand is available.

Adoption of a description from the point of view of a control window, travelling in the trajectory manifold, is shown to provide a convenient evaluation of the rate of change of the projected momentum along the motion. This alternative is especially useful in hydrodynamics and in the analysis of special systems, here exemplified by the classical tricky problems of falling chains.

The formulation of the dynamics of falling chains has in fact been considered as a challenging task and after about a century and a half of history is presently still actively discussed. Investigations do range from the early paper [1] till the recent treatment in [2]. "Falling chains" were also reported as daring exercises in many tracts of dynamics [3–7] most often with brief but improper or even Galilei non-invariant formulations.

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Published online: 23 December 2017

Incorrect formulations and unreliable experimental procedures are still proposed here and there in the literature [8–10].

Our approach is based on classical dynamics by formulating the equation of motion as seen from a control window travelling with a suitable movement in the trajectory manifold.

Valuable treatments facing problems involving variable mass systems were contributed in [2,11–13].

The peculiar feature of our approach is that it stands on standard continuum dynamics based on the axiom of conservation of mass, needed to assure equivalence between the Euler and d'Alembert laws of motion. The adoption of a travelling control window leads then to an alternative expression of the equation of motion, respectful of Galilei's principle of relativity.

The adoption of a fixed control window (also called control volume) to evaluate the thrust exerted on a solid case by an interacting fluid in relative motion is critically discussed and compared with the effective treatment recently contributed in [14].

This new approach provides the expression for the thrust on the basis of suitable simplifying assumptions referring to systems where the solid–fluid complex is possibly of variable total mass, but with a small rate of mass variation and a large rate of momentum variation, such as rockets, jet motors, turbines and lawn sprinklers.

The resulting expression of the thrust fulfils Galilei's relativity principle, validates the von Buquoy–Meshchersky formula of particle mechanics and provides its extension to classical continuum mechanics. This result is a distinctive feature with respect to proposals of introducing von Buquoy–Meshchersky formula as a substitute for Newton's second law in the dynamics of point particles with variable mass [15–18].

For the reader's convenience, in Appendix A essential definitions and basic notions of differential geometry, referred to in the mathematical treatment, are briefly recalled.

2 Kinematics in spacetime

The theory of dynamics is best developed in the general framework of a 4D manifold of events $\mathbf{e} \in \mathcal{E}$ and of the relevant tangent bundle $T\mathcal{E}$ with projection[1] $\boldsymbol{\tau}_{\mathcal{E}} : T\mathcal{E} \mapsto \mathcal{E}$ which assigns to each tangent vector $\mathbf{d}_{\mathcal{E}} \in T\mathcal{E}$ its base point in $\mathcal{E}$.

Each observer performs a double foliation of the 4D events manifold $\mathcal{E}$ into complementary 3D *space slices* $\mathcal{S}$ of *isochronous* events (with a same corresponding time instant) and 1D *timelines* of *isotopic* events (with a same corresponding space location).

Timelines do not intersect one another, and each timeline intersects a space slice just at one point. Analogously, space slices do not intersect one another and each space slice intersects a timeline just at one point.

Each *timeline* is parametrized by time in such a way that a *time projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ assigns the same time instant $t_{\mathcal{E}}(\mathbf{e}) \in \mathcal{Z}$ to each event in a *space slice*, that is

$$t_{\mathcal{E}}(\overline{\mathbf{e}}) = t_{\mathcal{E}}(\mathbf{e}), \quad \forall\, \overline{\mathbf{e}} \in \mathcal{S}. \tag{1}$$

Velocities of *timelines* define the field of *time arrows* $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$.[2]

The tangent space $T_{\mathbf{e}}\mathcal{E}$ at any event $\mathbf{e} \in \mathcal{E}$ is split into a complementary pair of a 3D time-vertical subspace $V_{\mathbf{e}}\mathcal{E}$ (tangent to a space slice) and a 1D time-horizontal subspace $H_{\mathbf{e}}\mathcal{E}$ (tangent to a timeline) generated by the time arrow $\mathbf{Z}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$.

The time projection $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ and the time arrow $\mathbf{Z}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$ are assumed to be *tuned* so that

$$\langle dt_{\mathcal{E}}, \mathbf{Z} \rangle = 1 \circ t_{\mathcal{E}}. \tag{2}$$

The symbol $\langle , \rangle$ denotes the pairing between dual fields and the dot $\cdot$ indicates linear dependence.

In the tangent bundle $T\mathcal{E}$, the *time-vertical* subbundle $V\mathcal{E}$ (*time-horizontal subbundle* $H\mathcal{E}$) is the disjoint union of all time-vertical (time-horizontal) subspaces. They are respectively called *spatial bundle* and *time bundle*.

In the familiar Euclid setting of classical mechanics, the space slices and the *time projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ are the same for all observers (universality of time).

A reference frame $\{ \mathbf{d}_i \,;\, i = 0, 1, 2, 3 \}$ for the event manifold is *adapted* if $\mathbf{d}_0 = \mathbf{Z}$ and $\mathbf{d}_i \in V\mathcal{E}$, $i = 1, 2, 3$.

[1] A *submersion* is a map whose differentials are surjective, a *projection* is a surjective submersion.
[2] *Zeit* is the German word for *Time*.

Definition 1 *(Trajectory)* The *trajectory manifold* is the geometric object investigated in mechanics, characterised by an embedding[3] $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$ into the event manifold $\mathcal{E}$ such that the image $\mathcal{T}_{\mathcal{E}} := \mathbf{i}(\mathcal{T})$ is a submanifold.

Definition 2 *(Motion)* The *motion* along the trajectory

$$\{\boldsymbol{\varphi}^{\mathcal{T}}_{\alpha} : \mathcal{T} \mapsto \mathcal{T},\ \alpha \in \mathcal{Z}\} \tag{3}$$

is a simultaneity preserving one-parameter family of diffeomorphisms, called movements, fulfilling the composition rule

$$\boldsymbol{\varphi}^{\mathcal{T}}_{\alpha} \circ \boldsymbol{\varphi}^{\mathcal{T}}_{\beta} = \boldsymbol{\varphi}^{\mathcal{T}}_{(\alpha+\beta)} \tag{4}$$

for any pair of time lapses $\alpha, \beta \in \mathcal{Z}$. Each $\boldsymbol{\varphi}^{\mathcal{T}}_{\alpha} : \mathcal{T} \mapsto \mathcal{T}$ is a *movement*.

The trajectory will alternatively be considered as a $(1+n)$D manifold $\mathcal{T}$ by itself or as a submanifold $\mathcal{T}_{\mathcal{E}} = \mathbf{i}(\mathcal{T}) \subset \mathcal{E}$ of the event manifold.

Then, a coordinate system is adopted on $\mathcal{T}$ while an adapted 4D spacetime coordinate system in $\mathcal{E}$ is adopted on $\mathcal{T}_{\mathcal{E}}$.

The trajectory inherits from the events manifold the time projection $t_{\mathcal{T}} := t_{\mathcal{E}} \circ \mathbf{i} : \mathcal{T} \mapsto \mathcal{Z}$ which defines the *time bundle* denoted by $V\mathcal{T}$ and called the *material bundle*.

The immersion $V\mathcal{T}_{\mathcal{E}} := \mathbf{i}{\uparrow}(V\mathcal{T})$ is also named *material bundle*, and a fibre of simultaneous events $\boldsymbol{\Omega} \subset \mathcal{T}_{\mathcal{E}}$ is called a *body placement*. The spatial slice including the placement $\boldsymbol{\Omega}$ is denoted by $\mathcal{S}_{\boldsymbol{\Omega}}$.

The spacetime movement $\boldsymbol{\varphi}_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ and the trajectory movement $\boldsymbol{\varphi}^{\mathcal{T}}_{\alpha} : \mathcal{T} \mapsto \mathcal{T}$ are related by the commutative diagram

$$\begin{tikzcd}
\mathcal{T}_{\mathcal{E}} \arrow[r, "\boldsymbol{\varphi}_{\alpha}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{T}_{\mathcal{E}} \arrow[dd, bend left=60, "t_{\mathcal{E}}"] \\
\mathcal{T} \arrow[u, "\mathbf{i}"] \arrow[r, "\boldsymbol{\varphi}^{\mathcal{T}}_{\alpha}"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[d, "t_{\mathcal{T}}"] \\
\mathcal{Z} \arrow[r, "\theta_{\alpha}"] & \mathcal{Z}
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{cases} t_{\mathcal{E}} \circ \boldsymbol{\varphi}_{\alpha} = \theta_{\alpha} \circ t_{\mathcal{E}}, \\ t_{\mathcal{T}} \circ \boldsymbol{\varphi}^{\mathcal{T}}_{\alpha} = \theta_{\alpha} \circ t_{\mathcal{T}}, \end{cases} \tag{5}$$

where the time translation $\theta_{\alpha} : \mathcal{Z} \mapsto \mathcal{Z}$ is defined by

$$\theta_{\alpha}(t) := t + \alpha, \quad t, \alpha \in \mathcal{Z}. \tag{6}$$

Trajectory and motion are assumed to be evaluated by an inertial observer, with any two such observers related by a Galilei frame transformation.

Definition 3 *(Material particles and body manifold)* The physical notion of *material particle* corresponds in the geometric view to a time-parametrized curve of events in the trajectory, related by the motion as follows

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{T} \ : \ \mathbf{e}_2 = \boldsymbol{\varphi}^{\mathcal{T}}_{\alpha}(\mathbf{e}_1). \tag{7}$$

Accordingly, we will say that a geometrical object is defined *along* (not *at*) a material particle. Events belonging to a *material particle* form a class of equivalence, and the quotient manifold so induced in the trajectory is the *body manifold*.

The spacetime *velocity* of the motion is defined by the derivative

$$\mathbf{v}_{\boldsymbol{\varphi}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto T\mathcal{T}_{\mathcal{E}}. \tag{8}$$

Taking the time derivative of (5), we have

$$\partial_{\alpha=0}\,(t_{\mathcal{E}} \circ \boldsymbol{\varphi}_{\alpha}) = \langle dt_{\mathcal{E}}, \mathbf{v}_{\boldsymbol{\varphi}} \rangle = (\partial_{\alpha=0}\, \theta_{\alpha}) \circ t_{\mathcal{E}} = 1 \circ t_{\mathcal{E}}. \tag{9}$$

[3] An *immersion* is a map whose differentials are injective. An *embedding* is an injective immersion whose co-restriction is continuous with the inverse.

Comparing with Eq. (2), we get the decomposition into space and time components

$$\boxed{\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} + \mathbf{Z},} \tag{10}$$

with $\langle dt_{\mathcal{E}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} \rangle = 0$.

Due to the spacetime split performed by an observer, a spacetime motion can be decomposed into the chain of a space motion and of a time shift,

$$\boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_\alpha^{\mathcal{S}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{Z}} = \boldsymbol{\varphi}_\alpha^{\mathcal{Z}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{S}}, \tag{11}$$

so that we have

$$\begin{aligned} \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} &:= \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{S}}, \\ \mathbf{Z} &:= \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{Z}}, \end{aligned} \tag{12}$$

as sketched in the commutative diagram (13):

$$\begin{tikzcd} \cdot \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}"] & \bullet \\ \bullet \arrow[u, dashed, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"] \arrow[ur, squiggly, "\boldsymbol{\varphi}_\alpha"] \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}"'] & \cdot \arrow[u, dashed, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"'] \end{tikzcd} \tag{13}$$

3 Mass conservation

A basic axiomatic statement in dynamics is conservation of mass.

The mass of a 3D continuous body is represented, in each placement $\boldsymbol{\Omega}$, by a special *volume form* $\mathbf{m} : \boldsymbol{\Omega} \mapsto \mathrm{VOL}(T\boldsymbol{\Omega})$ called *mass form*.

This is a field of alternating trilinear and nonsingular functions which evaluate the "weighted volume" of any positively oriented parallelepiped $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ in the space tangent at each $\mathbf{x} \in \boldsymbol{\Omega}$. In the Euclid space, the metric tensor $\mathbf{g}$ induces in each spatial slice a *metric volume form* $\boldsymbol{\mu}_{\mathbf{g}} : \boldsymbol{\Omega} \mapsto \mathrm{VOL}(T\boldsymbol{\Omega})$ such that positively oriented unit cubes do have unit *metric volume*.

As is well known, all volume forms are proportional. The scalar *mass density* ρ rescales the metric volume form to give the corresponding mass form, according to the relation $\mathbf{m} = \rho \cdot \boldsymbol{\mu}_{\mathbf{g}}$.

Definition 4 *(Mass conservation)* The axiom of mass conservation along the motion is expressed by each one of the equivalent local properties

$$\begin{cases} i) \quad \boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{m} = \mathbf{m}, \\ ii) \quad \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{m}) := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{m}) = \mathbf{0}, \end{cases} \tag{14}$$

or by the following integral conditions to be fulfilled by any part of $\mathcal{P} \subset C$:

$$\begin{cases} iii) \quad \displaystyle\int_{\boldsymbol{\varphi}_\alpha(\mathcal{P})} \mathbf{m} = \int_{\mathcal{P}} \boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{m} = \int_{\mathcal{P}} \mathbf{m}, \\ iv) \quad \partial_{\alpha=0} \displaystyle\int_{\boldsymbol{\varphi}_\alpha(\mathcal{P})} \mathbf{m} = \int_{\mathcal{P}} \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{m}) = \mathbf{0}. \end{cases} \tag{15}$$

The physical meaning of property $i)$ in Eq. (14) can be described as follows. Let $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be the sides of any parallelepiped in the tanget space $T_{\mathbf{x}}\boldsymbol{\Omega}$ and $\{\boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{d}_1, \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{d}_2, \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{d}_3\}$ the sides of the parallelepiped in $T_{\boldsymbol{\varphi}_\alpha(\mathbf{x})}\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})$, transformed by the motion. Then property $i)$ states that the mass of the transformed parallelepiped, given by the pullback:

$$(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{m})(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) := \mathbf{m}(\boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{d}_1, \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{d}_2, \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{d}_3), \tag{16}$$

is equal to the mass

$$\mathbf{m}(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) \tag{17}$$

of the original one.

As will be shown in Sect. 4, Eq. (31), the mass conservation property expressed by Eq. (14), is an essential requirement in order to deduce d'Alembert's law of motion in terms of the acceleration field and to ensure in this way the fulfilment of the following basic axiom of dynamics.

Proposition 1 (Galilei principle of relativity) *Motions in Euclid spacetime whose relative velocity field is constant, according to parallel transport by translation, are governed by the same law of dynamics.*

4 Continuum dynamics

Definition 5 *(Virtual motions)* A synchronous virtual motion from a placement $\boldsymbol{\Omega}$ is a one-parameter family $\delta\boldsymbol{\varphi}_\lambda : \boldsymbol{\Omega} \mapsto \mathcal{S}_{\boldsymbol{\Omega}}$ of virtual movements, i.e. time-preserving morphisms fulfilling the commutative diagram:

$$\begin{tikzcd} \boldsymbol{\Omega} \arrow[r, "\delta\boldsymbol{\varphi}_\lambda"] \arrow[d, "t_\lambda"'] & \mathcal{S}_{\boldsymbol{\Omega}} \arrow[d, "t_\lambda"] \\ \mathcal{Z} \arrow[r, "\theta_\lambda"] & \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad t_\lambda \circ \delta\boldsymbol{\varphi}_\lambda = \theta_\lambda \circ t_\lambda . \tag{18}$$

Here, $\lambda \in \delta\mathcal{Z}$ is the pseudo-time evolving along a 1D oriented manifold $\delta\mathcal{Z}$ of pseudo-instants. The translation $\delta\theta_\lambda : \delta\mathcal{Z} \mapsto \delta\mathcal{Z}$ is defined by

$$\theta_\lambda(\delta t) := \lambda + \delta t, \quad \lambda, \delta t \in \delta\mathcal{Z}. \tag{19}$$

Virtual movements $\delta\boldsymbol{\varphi}_\lambda : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$ are diffeomorphisms on their range. The associated virtual velocity on $\boldsymbol{\Omega}$ is the spatial vector field

$$\delta\mathbf{v}_{\mathcal{S}} := \partial_{\alpha=0}\, \delta\boldsymbol{\varphi}_\lambda : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}} . \tag{20}$$

Definition 6 *(External force)* An *external force* $\mathbf{f}_{\mathrm{EXT}}$ on a placement $\boldsymbol{\Omega}$ is represented by a one-form on the space of virtual velocity fields $\delta\mathbf{v}_{\mathcal{S}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$. It can be expressed in term of bulk densities $\mathbf{b}$ and surficial densities $\mathbf{t}$ per unit volume:

$$\langle \mathbf{f}_{\mathrm{EXT}}, \delta\mathbf{v}_{\mathcal{S}} \rangle_{\boldsymbol{\Omega}} := \int_{\boldsymbol{\Omega}} \langle \mathbf{b}, \delta\mathbf{v}_{\mathcal{S}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{g}} + \int_{\partial\boldsymbol{\Omega}} \langle \mathbf{t}, \delta\mathbf{v}_{\mathcal{S}} \rangle \cdot \partial\boldsymbol{\mu}_{\mathbf{g}}, \tag{21}$$

where $\boldsymbol{\mu}_{\mathbf{g}}$ is the volume form induced by the spatial metric field $\mathbf{g}$. The area form on the boundary $\partial\boldsymbol{\Omega}$ is

$$\partial\boldsymbol{\mu}_{\mathbf{g}} := \boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{n}, \tag{22}$$

with $\mathbf{n}$ normal versor.

Definition 7 *(Internal force)* The *internal force* $\mathbf{f}_{\mathrm{INT}}(\boldsymbol{\sigma})$, associated with a stress field $\boldsymbol{\sigma}$, is the one-form on a body placement $\boldsymbol{\Omega}$ defined by the duality pairing

$$\langle \mathbf{f}_{\mathrm{INT}}(\boldsymbol{\sigma}), \delta\mathbf{v}_{\mathcal{S}} \rangle_{\boldsymbol{\Omega}} := \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}_{\mathcal{S}}) \rangle \cdot \mathbf{m}, \tag{23}$$

where $\boldsymbol{\varepsilon}(\delta\mathbf{v}_{\mathcal{S}}) := \mathrm{sym}(\nabla\delta\mathbf{v}_{\mathcal{S}})$ is the stretching associated with the virtual velocity field $\delta\mathbf{v}_{\mathcal{S}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$.

Definition 8 *(Dynamical force)* The *dynamical force* $\mathbf{f}_{\mathrm{DYN}}$ is the difference between external and internal forces:

$$\mathbf{f}_{\mathrm{DYN}} := \mathbf{f}_{\mathrm{EXT}}(\mathbf{b}\,, \mathbf{t}) - \mathbf{f}_{\mathrm{INT}}(\boldsymbol{\sigma}). \tag{24}$$

The Euler law of continuum dynamics states that for piecewise smooth motions, the time rate of variation of the virtual power of the projected kinetic momentum[4] is equal to the virtual power of the dynamical force system acting on the body, at each placement in the event manifold along the trajectory, as expressed by the next proposition.

[4] The adjective *projected* refers to the inner product between the spatial velocity and the parallel transported virtual velocity field, by means of the spatial metric.

Proposition 2 (Euler law of continuum dynamics) *Denoting* $\Uparrow$ *the parallel transport by* $\mathbf{g}$ *the metric tensor in the spatial bundle* $V\mathcal{E}$ *according to the Levi-Civita connection associated with the spatial metric tensor,*[5] Euler *law of motion writes, for regular motions*

$$\boxed{\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m} = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle_{\boldsymbol{\Omega}}.} \tag{25}$$

The virtual velocity field $\delta\hat{\mathbf{v}}_{\mathcal{S}}$ *is the spatial vector field generated by parallel transport along the trajectory:*

$$\delta\hat{\mathbf{v}}_{\mathcal{S}} \circ \boldsymbol{\varphi}_\alpha := \boldsymbol{\varphi}_\alpha \Uparrow \delta\mathbf{v}_{\mathcal{S}}, \tag{26}$$

and, at singularities of the kinetic momentum (i.e. at collisions)

$$\boxed{\left[\left[\int_{\boldsymbol{\Omega}} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}\right]\right] = \langle \mathbf{f}_{\mathrm{SING}}, \delta\mathbf{v}_{\mathcal{S}} \rangle_{\boldsymbol{\Omega}},} \tag{27}$$

where the jump $[[\bullet]]$ *is the difference between the limit from the right and the limit from the left, at points of discontinuity. The singular force* $\mathbf{f}_{\mathrm{SING}}$ *is named an* impulse*, in mechanics.*

Lemma 1 (Linear dependence on virtual velocity fields) *The Euler law of dynamics is well posed since the time rate of increase in projected momentum depends on a linear way on virtual velocity fields* $\delta\mathbf{v}_{\mathcal{S}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$.

Proof The rate of variation of the projected kinetic momentum in Eq. (25) may be rewritten by applying the Jacobi pullback integral transformation and the definition of the Lie derivative along a flow:

$$\begin{aligned} \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m} &= \int_{\boldsymbol{\Omega}} \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \downarrow \Big(\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}\Big) \\ &= \int_{\boldsymbol{\Omega}} \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}} \Big((\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}\Big). \end{aligned} \tag{28}$$

Moreover, the Leibniz rule for the Lie-derivative gives

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}} \Big((\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}\Big) &= \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}} \Big(\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}})\Big) \cdot \mathbf{m} \\ &\quad + \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{m}). \end{aligned} \tag{29}$$

By construction of the field $\delta\hat{\mathbf{v}}_{\mathcal{S}}$ in Eq. (26), it follows that $\nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}(\delta\hat{\mathbf{v}}_{\mathcal{S}}) = \mathbf{0}$. Since parallel transport and push of scalar fields are coincident, applying the Leibniz rule for the parallel derivative we get

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}} \Big(\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}})\Big) &= \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}} \Big(\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\hat{\mathbf{v}}_{\mathcal{S}})\Big) \\ &= \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{g})(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \delta\mathbf{v}_{\mathcal{S}}) + \mathbf{g}(\nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}), \delta\mathbf{v}_{\mathcal{S}}). \end{aligned} \tag{30}$$

Substituting into Eq. (29), we get the result. □

Proposition 3 (d'Alembert law of continuum dynamics) *By conservation of mass, the Euler differential law Eq.* (25) *is equivalent to the following:*

$$\boxed{\int_{\boldsymbol{\Omega}} \mathbf{g}(\mathbf{a}_{\boldsymbol{\varphi}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m} = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle_{\boldsymbol{\Omega}},} \tag{31}$$

where $\mathbf{a}_{\boldsymbol{\varphi}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$ *is the spatial acceleration field, defined as parallel time rate of variation of the spatial velocity along the motion*[6]

$$\boxed{\mathbf{a}_{\boldsymbol{\varphi}} := \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) = \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \Downarrow \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}).} \tag{32}$$

[5] In classical dynamics the parallel transport is the familiar path-independent operation of translation in the Euclid space.

[6] In 3D treatments the acceleration is often defined by $\mathbf{a}_{\boldsymbol{\varphi}} = \nabla_{\mathbf{Z}}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) + \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}})$, see e.g. [19]. This split formula is not applicable to lower dimensional trajectories such as the ones pertaining to bullets, wires and membranes.

Proof Conservation of mass, expressed by the condition in Eq. (14), imposes that $\mathcal{L}_{\mathbf{v}_{\varphi}}(\mathbf{m}) = \mathbf{0}$. Moreover $\nabla_{\mathbf{v}_{\varphi}}(\mathbf{g}) = \mathbf{0}$ since the connection is the Levi-Civita, so that, by Eqs. (29) and (30), the Euler law translates into the d'Alembert law Eq. (31). □

The fulfilment of Galilei's principle of relativity is evident from the expression in Eq. (32), with dynamical forces Galilei invariant by assumption. The result underlines the essential role played by conservation of mass in a proper formulation of dynamics.

Proposition 4 (Mechanical power balance) *Along a regular motion, the law of dynamics implies balance of mechanical power, stating that the power of dynamical force is equal to the rate of variation of kinetic energy:*

$$\boxed{\langle \mathbf{f}_{\mathrm{DYN}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} \rangle_{\boldsymbol{\Omega}} = \partial_{\alpha=0}\ \tfrac{1}{2} \int_{\boldsymbol{\varphi}_{\alpha}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m}.} \tag{33}$$

Proof Conservation of mass Eq. (14) gives $\mathcal{L}_{\mathbf{v}_{\varphi}}(\mathbf{m}) = \mathbf{0}$. Then, since parallel transport and push are coincident for scalar fields, we have

$$\begin{aligned}
\partial_{\alpha=0}\ \tfrac{1}{2} \int_{\boldsymbol{\varphi}_{\alpha}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m} &= \int_{\boldsymbol{\Omega}} \partial_{\alpha=0}\, \boldsymbol{\varphi}_{\alpha}\!\downarrow \tfrac{1}{2}\Big(\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m}\Big) \\
&= \int_{\boldsymbol{\Omega}} \mathcal{L}_{\mathbf{v}_{\varphi}}\Big(\tfrac{1}{2}\,(\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m}\Big) = \int_{\boldsymbol{\Omega}} \mathcal{L}_{\mathbf{v}_{\varphi}}\Big(\tfrac{1}{2}\,\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}})\Big) \cdot \mathbf{m} \\
&= \int_{\boldsymbol{\Omega}} \nabla_{\mathbf{v}_{\varphi}}\Big(\tfrac{1}{2}\,\mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}})\Big) \cdot \mathbf{m} = \int_{\boldsymbol{\Omega}} \mathbf{g}(\nabla_{\mathbf{v}_{\varphi}}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}), \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m} \\
&= \int_{\boldsymbol{\Omega}} \mathbf{g}(\mathbf{a}_{\boldsymbol{\varphi}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m}.
\end{aligned} \tag{34}$$

Setting

$$\delta \mathbf{v}_{\mathcal{S}} = \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}} \tag{35}$$

in the d'Alembert law (31), the equality in Eq. (33) follows. □

If the dynamical force is an exact differential with time-independent potential $\psi : \boldsymbol{\Omega} \mapsto \mathrm{FUN}(T\mathcal{E}_{\boldsymbol{\Omega}})$, we have that

$$\langle d\psi, \mathbf{Z} \rangle_{\boldsymbol{\Omega}} = 0, \tag{36}$$

and hence

$$\langle \mathbf{f}_{\mathrm{DYN}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} \rangle_{\boldsymbol{\Omega}} = \langle d\psi, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} \rangle_{\boldsymbol{\Omega}} = \langle d\psi, \mathbf{v}_{\boldsymbol{\varphi}} \rangle_{\boldsymbol{\Omega}} = \partial_{\alpha=0}\ (\psi \circ \boldsymbol{\varphi}_{\alpha}). \tag{37}$$

The mechanical power balance Eq. (33) can then be expressed as conservation of total energy, sum of potential and kinetic energies, along the motion:

$$\partial_{\alpha=0} \left((\psi \circ \boldsymbol{\varphi}_{\alpha}) + \tfrac{1}{2} \int_{\boldsymbol{\varphi}_{\alpha}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}) \cdot \mathbf{m} \right) = 0. \tag{38}$$

When the dynamical force vanishes, we get conservation of kinetic energy along the motion.

5 The role of control windows

The investigation of the motion $\boldsymbol{\varphi}_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ of a body along the dynamical trajectory $\mathcal{T}_{\mathcal{E}}$ may happen to be a challenging task.

A convenient choice may then consist in observing the body as it appears from a *control window* $C \subset \boldsymbol{\Omega}$, travelling along its own trajectory $\mathcal{T}_C \subseteq \mathcal{T}_{\mathcal{E}}$ which is a submanifold of the body trajectory $\mathcal{T}_{\mathcal{E}}$.

The control windows is said to undergo a spacetime *travel* in the body trajectory.

This nomenclature points out that the travel map $\boldsymbol{\xi}_{\alpha} : \mathcal{T}_C \mapsto \mathcal{T}_C$ is designed in the body trajectory by the investigator in just a convenient manner for the description of the body motion.

The travel map shares with the body motion the property of simultaneity preservation, as described by the commutative diagram:

$$\begin{tikzcd} \mathcal{T}_C \arrow[r, "\boldsymbol{\xi}_\alpha"] \arrow[d, "t_{\mathcal{E}}"'] & \mathcal{T}_C \arrow[d, "t_{\mathcal{E}}"] \\ \mathcal{Z} \arrow[r, "\theta_\alpha"] & \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad t_{\mathcal{E}} \circ \boldsymbol{\xi}_\alpha = \theta_\alpha \circ t_{\mathcal{E}}, \tag{39}$$

where $\theta_\alpha : \mathcal{Z} \mapsto \mathcal{Z}$ is the time translation defined in Eq. (19). The spacetime velocity field along the travel $\boldsymbol{\xi}_\alpha : \mathcal{T}_C \mapsto \mathcal{T}_C$ is $\mathbf{v}_{\boldsymbol{\xi}} := \partial_{\alpha=0}\, \boldsymbol{\xi}_\alpha = \mathbf{v}_{\boldsymbol{\xi}}^{\mathcal{S}} + \mathbf{Z}$, with $\langle dt_{\mathcal{E}}, \mathbf{u}_{\mathcal{S}} \rangle = 0$.

The description of the dynamics of a body from the standpoint of a control window undergoing an arbitrary travel in the trajectory manifold is based on the comparison between the rates of change of the projected momentum in the part of the body belonging to the control window, respectively, according to the motion and to the travel, as described hereafter.

1. Along the motion $\boldsymbol{\varphi}_\alpha : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$, the rate of change is given by:

$$\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(C)} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \boldsymbol{\varphi}_\alpha \Uparrow \delta \mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}, \tag{40}$$

which is the r.h.s. of Euler's law (25) with $\boldsymbol{\Omega} = C$.

2. Along the travel $\boldsymbol{\xi}_\alpha : \mathcal{T}_C \mapsto \mathcal{T}_C$, the rate of change is instead given by:

$$\partial_{\alpha=0} \int_{\boldsymbol{\xi}_\alpha(C)} \mathbf{g}(\mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}}, \boldsymbol{\varphi}_\alpha \Uparrow \delta \mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}. \tag{41}$$

The outcome of the comparison is provided by the following general result.

Lemma 2 (Window travel vs body motion) *Let $\boldsymbol{\mu} : \mathcal{T}_{\mathcal{E}} \mapsto \mathrm{VOL}(V\mathcal{T}_{\mathcal{E}})$ be a material volume form over the trajectory $\mathcal{T}_{\mathcal{E}}$. The gap between the rates of variation of the global volume of a window $C \subset \boldsymbol{\Omega}$, along the motion $\boldsymbol{\varphi}_\alpha : \mathcal{T}_C \mapsto \mathcal{T}_C$ and along the travel $\boldsymbol{\xi}_\alpha : \mathcal{T}_C \mapsto \mathcal{T}_C$, is equal to the volumetric flux, through the boundary of the window ∂C, of the relative velocity*

$$\mathbf{v}_{\mathrm{REL}} := \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} - \mathbf{v}_{\boldsymbol{\xi}}^{\mathcal{S}}, \tag{42}$$

of the body in motion with respect to the travelling control window:

$$\boxed{\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(C)} \boldsymbol{\mu} = \partial_{\alpha=0} \int_{\boldsymbol{\xi}_\alpha(C)} \boldsymbol{\mu} + \oint_{\partial C} \boldsymbol{\mu} \cdot \mathbf{v}_{\mathrm{REL}}.} \tag{43}$$

Proof By Jacobi transformation and the Leibniz rule, the integral at the l.h.s. of Eq. (43) writes

$$\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(C)} \boldsymbol{\mu} = \int_C \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha \downarrow \boldsymbol{\mu}) = \int_C \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\boldsymbol{\mu}). \tag{44}$$

Similarly, the first integral at the r.h.s. of Eq. (43) writes

$$\partial_{\alpha=0} \int_{\boldsymbol{\xi}_\alpha(C)} \boldsymbol{\mu} = \int_C \partial_{\alpha=0} (\boldsymbol{\xi}_\alpha \downarrow \boldsymbol{\mu}) = \int_C \mathcal{L}_{\mathbf{v}_{\boldsymbol{\xi}}}(\boldsymbol{\mu}). \tag{45}$$

The relative velocity of the motion with respect to the travelling control window is given by

$$\mathbf{v}_{\mathrm{REL}} := \mathbf{v}_{\boldsymbol{\varphi}} - \mathbf{v}_{\boldsymbol{\xi}} = \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} + \mathbf{Z} - \mathbf{v}_{\boldsymbol{\xi}}^{\mathcal{S}} - \mathbf{Z} = \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} - \mathbf{v}_{\boldsymbol{\xi}}^{\mathcal{S}}. \tag{46}$$

By linearity of the Lie-derivative, applying the extrusion formula Eq. (A.20), observing that $d\boldsymbol{\mu} = \mathbf{0}$, and the Stokes formula Eq. (A.11), we get

$$\begin{aligned} \int_C \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\boldsymbol{\mu}) - \int_C \mathcal{L}_{\mathbf{v}_{\boldsymbol{\xi}}}(\boldsymbol{\mu}) &= \int_C \mathcal{L}_{\mathbf{v}_{\mathrm{REL}}}(\boldsymbol{\mu}) \\ &= \int_C d\big(\boldsymbol{\mu} \cdot \mathbf{v}_{\mathrm{REL}}\big) = \oint_{\partial C} \boldsymbol{\mu} \cdot \mathbf{v}_{\mathrm{REL}}, \end{aligned} \tag{47}$$

and hence Eq. (43) follows. $\square$

Proposition 5 (Control window description of the law of motion) *The rate of variation of the projected kinetic momentum along the dynamical trajectory of a body can be evaluated by adding two contributions:*

1. *the rate of increase in the projected kinetic momentum of the body in a control window travelling in the trajectory,*
2. *the projected kinetic momentum outflowing from its boundary.*

The law of motion Eq. (25) *pertaining to* $C \subset \boldsymbol{\Omega}$ *can then be written as*

$$\boxed{\begin{aligned}\langle \mathbf{f}_{\mathrm{DYN}}, \delta \mathbf{v}_{\mathcal{S}} \rangle_C = \partial_{\alpha=0} \int_{\boldsymbol{\xi}_\alpha(C)} \mathbf{g}(\mathbf{v}^{\mathcal{S}}_{\boldsymbol{\varphi}}, \delta \hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m} \\ + \oint_{\partial C} \mathbf{g}(\mathbf{v}^{\mathcal{S}}_{\boldsymbol{\varphi}}, \delta \mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m} \cdot \mathbf{v}_{\mathrm{REL}}.\end{aligned}} \tag{48}$$

Proof As described in Prop. 2, the Euler law of dynamics, Eq. (25) evaluates the rate of variation of the projected kinetic momentum by extending the virtual velocity field $\delta \mathbf{v}_{\mathcal{S}} : C \mapsto TC$ to the whole trajectory $\mathcal{T}_{\mathcal{E}}$:

$$\delta \hat{\mathbf{v}}_{\mathcal{S}} \circ \boldsymbol{\varphi}_\alpha := \boldsymbol{\varphi}_\alpha \Uparrow \delta \mathbf{v}_{\mathcal{S}}. \tag{49}$$

In this manner, the material volume form

$$\boldsymbol{\mu} := \mathbf{g}(\mathbf{v}^{\mathcal{S}}_{\boldsymbol{\varphi}}, \delta \hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}, \tag{50}$$

is well defined on the whole trajectory $\mathcal{T}_{\mathcal{E}}$ and can be integrated on the window placements $\boldsymbol{\xi}_\alpha(C) \in \mathcal{T}_C$. Substituting the expression Eq. (50) in the geometric formula Eq. (43) of Lemma 2, we get the result in Eq. (48). □

It is to be underlined that neither of the single two terms at the r.h.s. of Eq. (48) can be interpreted as a force since Galilei's principle of relativity stated in Proposition 1 is not fulfilled. This observation is the starting point for introducing the discussion in Sect. 8.

In Sects. 6.1.2 and 6.2, the formula in Eq. (48) will be applied to discuss tricky problems of dynamics involving falling chains.

6 Tricky problems of dynamics

Let us now analyse some seemingly simple but tricky problems of dynamics usually considered akin to impact problems or to variable mass problems.

Our approach is based on a direct application of Euler's law of dynamics by relying on the evaluation of the momentum rate by means of a travelling control window, as described in Sect. 5.

This route provides an effective tool for the correct formulation of some dynamical problems involving discontinuous motions.

6.1 The hanging heaped chain

The falling chain is a tricky problem in dynamics, reported and investigated in the treatise by Peter Guthrie Tait and William John Steele, as an instance of continuous series of indefinitely small impacts [3, p. 332].

The same problem was also addressed by Arthur Cayley in [1], with no reference to [3]. His treatment begins:

"*There are a class of dynamical problems which, so far as I am aware, have not been considered in a general manner. The problems referred to (which might be designated as continuous impact problems) are those in which the system is continually taking into connexion with itself particles of infinitesimal mass... For instance, a problem of the sort arises when a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table.*"

In Cayley's paper, the equation of motion for the falling chain was given as in Eq. (51) below, without explicit derivation from the laws of classical dynamics,[7] as observed also in [2]. The same equation is also reported by Arnold Sommerfeld, still without motivation, in [20, p. 257].

According to Edward John Routh, problems of this kind were already well known in Cambridge about the year 1850.

They were inserted by the coach William Hopkins as challenges in Mathematical Tripos [21, p. 80] [22]. Cayley himself was a pupil of Hopkins and Senior Wrangler in 1842.

6.1.1 Tait, Steele and Cayley treatments

The case study treated in [3] and in [1] is a falling heavy chain heaped near the edge of a table and hanging down.

Let $x(t)$ be the length of the hanging falling chain, ρ its mass per unit length and g the acceleration of gravity.

Denoting by a superimposed dot the time derivative, the differential equation of dynamics, according to [1], writes

$$(\rho x\dot{x})^{\cdot} = g\,\rho\,x. \tag{51}$$

Since ρ is positive and time independent, multiplying both sides by $x\dot{x}/\rho$, we get

$$x\dot{x}\,(x\dot{x})^{\cdot} = \tfrac{1}{2}((x\dot{x})^2)^{\cdot} = g\,x^2\dot{x}. \tag{52}$$

Integrating in time and assuming that $x(0)\dot{x}(0) = 0$, we then obtain

$$\tfrac{1}{2}(x\dot{x})^2 = \int_0^t g\,x^2\dot{x}\,d\tau = \int_0^x g\,y^2\,dy = \tfrac{1}{3}g\,x^3, \tag{53}$$

and hence

$$\tfrac{1}{2}\dot{x}^2 = \tfrac{1}{3}g\,x. \tag{54}$$

Taking the time derivative, we eventually get the unexpected result reported in [1,3]

$$\dot{x}\ddot{x} = \tfrac{1}{3}g\,\dot{x} \implies \ddot{x} = \tfrac{1}{3}g, \quad (\dot{x} \neq 0). \tag{55}$$

According to Eq. (55), the chain falls downward with an acceleration which is only one-third of the gravity. Equation (51) is, however, unphysical since the law

$$(\rho x\dot{x})^{\cdot} = g\,\rho x \iff \rho x\,\ddot{x} = g\,\rho x - \rho\dot{x}^2 \tag{56}$$

does not obey Galilei's relativity principle [23], due to presence of the speed in the term $\rho\dot{x}^2$.

6.1.2 Analysis from a control window

In fact, Eq. (51) has no ground in classical dynamics since it outcomes from

1. either considering a system with variable total mass $\rho\,x(t)$ (the hanging part of the chain),
2. or a system with constant mass (the whole chain) undergoing a motion with a discontinuous velocity field.

In both instances, the balance of mechanical power is not necessarily fulfilled, as expressed by the statement of Proposition 4.

To see this, let us write the Euler law of dynamics for the case at hand, but according to the formulation in Eq. (48) which is most useful in fluid dynamics.

In fact, the problem of the free falling hanging heaped chain does share with the motion of fluids, the peculiarity that the description of dynamics is much simplified by looking at the motion through a suitable control window travelling in the investigated trajectory.

In the trajectory of the falling chain, there is a discontinuity in the velocity field, since the chain, heaped at rest on the table's edge, starts to fall with a nonzero velocity equal to the velocity of the hanging chain that pulls it downwards.

The involved dynamics is conveniently described by considering a vertical control window that includes a piece of hanging chain. To exemplify the reasoning, we consider two computations leading to the same result.

[7] In [1, p. 506] it is said: "*The general equation of dynamics applied to the case in hand will be...*".

1. If the control window has a fixed length L and the hanging chain crosses the whole control window and emerges from the lower end, the law of dynamics, expressed by imposing equality between external force and rate of increase in kinetic momentum, according to Eq. (48), writes
$$(\rho L\dot{x})^{\cdot} - \rho\dot{x}^2 + \rho\dot{x}^2 = g\,\rho L \iff \rho L\,\ddot{x} = g\,\rho L \implies \ddot{x} = g. \tag{57}$$
2. If the control window has a variable length x with the upper end fixed with respect to the table and the lower end coinciding with the terminal part of the hanging chain of variable length x, Eq. (48) gives
$$(\rho x\dot{x})^{\cdot} - \rho\dot{x}^2 = g\,\rho x \iff \rho x\,\ddot{x} = g\,\rho x \implies \ddot{x} = g. \tag{58}$$
In both cases, the expected result is that the chain falls as it was completely free from the heaped tail on the table, with Galilei's relativity principle fulfilled. The equality $\ddot{x} = g$ agrees with a vanishing axial interaction along the chain.

6.2 Free falling folded chain

A similar problem arises in considering an ideal hanging chain with one end fixed and the other end bent upwards and then left free to fall down.

Denoting by L the chain length, by x the length of the vertically hanging part of the chain and by y the vertical distance from the fixed end to the free falling end, the falling part will have a length equal to $L - x$ and the following relation is fulfilled:
$$L - x + y = x \iff L + y = 2\,x, \tag{59}$$
so that
$$\dot{y} = 2\,\dot{x}. \tag{60}$$
The falling chain moves downwards with a uniform speed $\dot{y}$, and the axial interaction along the falling chain is vanishing.

We may consider two distinct *control windows*. One is moving with the same velocity of the chain, while the other one is moving vertically with a non-uniform velocity field. More precisely:

1. The first one is a tubular *control window* of unit length which is falling with the same speed of the chain. Then, the equation of motion Eq. (48) gives
$$\big(\rho\cdot\dot{y}\big)^{\cdot} = \rho\cdot g \implies \ddot{y} = g. \tag{61}$$
2. The second one is a tubular control window containing the falling end and extending down till the sharp bent. The upper end of the control window moves downward with speed $\dot{y}$, while the lower end moves downward with speed $\dot{x}$.
Hence, through the lower end of the control window there will be an outward flow of kinetic momentum that, by Eq. (60), is equal to
$$(\dot{y} - \dot{x})\cdot\rho\,\dot{y} = \dot{x}\cdot\rho\,\dot{y}. \tag{62}$$
From Eq. (48), we get
$$\Big(\rho\,(L - x)\cdot\dot{y}\Big)^{\cdot} + \dot{x}\cdot\rho\,\dot{y} = (L - x)\cdot\rho\,g, \tag{63}$$
which evaluates to
$$(L - x)\cdot\ddot{y} - \dot{x}\cdot\rho\,\dot{y} + \dot{x}\cdot\rho\,\dot{y} = (L - x)\cdot g \implies \ddot{y} = g. \tag{64}$$

In both cases, the falling part of the chain accelerates as it was free.

Our conclusion agrees with the one drawn by Irschik in [2] by means of a different approach.

Contrary to claims in the literature [24], we see that there is no "paradox of the free falling folded chain" since there is no ground to pretend that the free part of the chain, when released, will fall faster than a free falling body.

Critical remarks about the analysis of the motion of the hanging folded chain by means of energy conservation or by an application of Lagrange equations were expressed in [25].

Their conclusion, that the system undergoes a non-conservative motion with plastic dissipation at the sharp fold of the chain (the so-called Carnot energy loss), is based on an incorrect appeal to balance of mechanical power for systems which either have variable total mass or undergo a velocity jump during the motion. In both cases, conservation of energy does not apply, as directly verifiable on the basis of Proposition 4. Also, mistaken is the momentum balance there exposed in terms of axial interaction.

7 Thrust due to solid–fluid interaction

Following the treatment developed in [14], the thrusting force exerted on a solid case by an interacting fluid in relative motion can be estimated by a simple formula provided by the following peculiar reasoning.

The strategy is based on the simplifying assumption that motion of the solid case can be extended to the trajectory of a mobile control window $\boldsymbol{\Omega}_{\mathrm{SKE}}$, called the *skeleton*, travelling in the fluid trajectory.

The skeleton boundary is composed of the surface of interaction between the fluid and the solid case $\boldsymbol{\Omega}_{\mathrm{SOL}}$ and of the surfaces through which the fluid is allowed to enter or leave the skeleton.

The geometry of the skeleton is assumed, so that feasibility of the resulting computations in a sufficiently approximate manner is assured.

It is then possible to consider the relative motion of the fluid with respect to the skeleton, according to the chain decomposition

$$\boxed{\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} = \boldsymbol{\varphi}_\alpha^{\mathrm{REL}} \circ \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}}, \quad \text{with} \begin{cases} t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha^{\mathrm{REL}} = t_{\mathcal{E}}, \\ t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}} = \theta_\alpha \circ t_{\mathcal{E}}. \end{cases}} \tag{65}$$

where $\theta_\alpha(t) := t + \alpha$ is the time translation.

The upper equality in parenthesis means that the relative motion is spatial (a motion at constant time). The relative velocity $\mathbf{v}_{\mathrm{SKE}}^{\mathrm{FLU}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{REL}}$ of the fluid with respect to the skeleton is then a spatial vector field.

Setting $\mathbf{v}^{\mathrm{SKE}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}}$, the relevant velocity fields are related by

$$\boxed{\mathbf{v}^{\mathrm{FLU}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} = \mathbf{v}_{\mathrm{SKE}}^{\mathrm{FLU}} + \mathbf{v}^{\mathrm{SKE}}.} \tag{66}$$

In writing Euler's law Eq. (25), the rate of variation of the kinetic momentum of the system can be split as sum of the contributions of the solid case and of the fluid-filled skeleton, as expressed by

$$\boxed{\begin{aligned} \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle &= \int_{\boldsymbol{\Omega}_{\mathrm{SOL}}} \mathbf{g}(\mathbf{a}_{\boldsymbol{\varphi}}^{\mathrm{SOL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{SOL}} \\ &+ \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{FLU}}, \boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} {\Uparrow} \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}}, \end{aligned}} \tag{67}$$

Here, conservation of mass of the solid case was resorted to in order to apply Proposition 3.

Dynamical problems, in which the solid case interacts through the skeleton with a fluid that undergoes large rates of variation of kinetic momentum but small rates of mass variation, are in the focus of the present analysis.

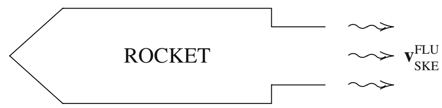
It can be shown [14] that the chain decomposition in Eq. (65) and the physically reasonable assumption that mass is conserved in the skeleton-like motion of the fluid, i.e. that $\mathcal{L}_{\mathbf{v}^{\mathrm{SKE}}}(\mathbf{m}) = \mathbf{0}$, together lead to the formula:

$$\boxed{\begin{aligned} &\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{FLU}}, \boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} {\Uparrow} \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}} \\ &\quad = \int_{\boldsymbol{\Omega}_{\mathrm{SKE}}} \mathbf{g}(\mathbf{a}_{\boldsymbol{\varphi}}^{\mathrm{SKE}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}} \\ &\quad + \int_{\boldsymbol{\Sigma}_{\mathrm{OUT}}^{\mathrm{IN}}} \mathbf{g}(\mathbf{v}_{\mathrm{SKE}}^{\mathrm{FLU}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\mathrm{FLU}} \cdot \mathbf{v}_{\mathrm{SKE}}^{\mathrm{FLU}}). \end{aligned}} \tag{68}$$

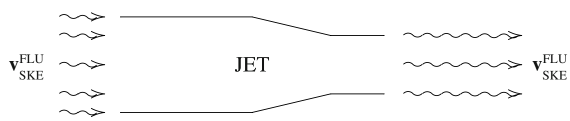
Here,

$$\boldsymbol{\Sigma}_{\mathrm{OUT}}^{\mathrm{IN}} \subset \partial\boldsymbol{\Omega}_{\mathrm{SKE}} \tag{69}$$

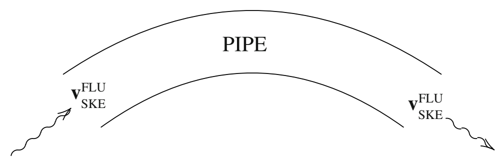
is the skeleton bounding surface where the mass outflow term $\mathbf{m}_{\mathrm{FLU}} \cdot \mathbf{v}_{\mathrm{SKE}}^{\mathrm{FLU}}$ is allowed to be non-vanishing, as for instance the nozzle exit surface for the exhaust gas in a rocket or the cross sections at the extremities of an elbow in a pressure pipe, schematically depicted in the diagrams (70), (71) and (72).



(70)



(71)



(72)

The surface integral provides the continuum mechanics extension and validation of the von Buquoy–Meshchersky formula for the thrust.

In standard treatments, this formula is introduced as a substitute for Newton's second law and assumed to be a new principle for the dynamics of particles with variable mass. The new assessment properly includes those important dynamical problems where variability of mass does not occur or is uninfluent for the evaluation of the thrust so that the original von Buquoy–Meshchersky formula is not applicable. A critical discussion on the notion of point particles with variable mass can be found in [14].

8 A warning concerning control windows

The formulation of Euler's law of dynamics provided by Eq. (48) of Proposition 5 is often restricted to control windows undergoing rigid body motions; see for instance [26, § 1.3, p. 6], [6, Eq. 12.89, p. 485], [19, § 15, Eq. 1, p. 108], [27] and [28, Ch. 23, p. 168].

In some treatments, only fixed control windows are considered and the resulting formula is claimed to yield an evaluation of the thrust exerted on a solid case by a fluid in relative motion.

The formula there exposed can be derived as a special case of Eq. (48) by setting

$$\boldsymbol{\xi}_\alpha = \boldsymbol{\varphi}_\alpha^{\mathcal{Z}}, \quad \text{so that} \quad \mathbf{v}_{\boldsymbol{\xi}}^{\mathcal{S}} = \mathbf{0}, \quad \mathbf{v}_{\boldsymbol{\varphi}}^{\mathcal{S}} = \mathbf{v}_{\mathrm{REL}}. \tag{73}$$

The law of motion, as seen from a fixed control window, writes

$$\boxed{\begin{aligned} \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle_C = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}(C)} \mathbf{g}(\mathbf{v}_{\mathrm{REL}}, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m} \\ + \oint_{\partial C} \mathbf{g}(\mathbf{v}_{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m} \cdot \mathbf{v}_{\mathrm{REL}}. \end{aligned}} \tag{74}$$

The negative of the boundary term in Eq. (74) is interpreted in [19,28] as the thrusting force caused by the outflow of momentum of the fluid through the boundary of the fixed control window. To eliminate the first term at the r.h.s. of Eq. (74), the fluid was there assumed to undergo a steady motion. This procedure for the evaluation of the thrusting force leads to a seemingly correct result. This conclusion is, however, deceitful since Eq. (74) involves the relative velocity $\mathbf{v}_{\text{REL}}$ of the body with respect to the fixed control window instead of the relative velocity $\mathbf{v}_{\text{SKE}}^{\text{FLU}}$ of the fluid with respect to the skeleton, as illustrated in Sect. 7.

There are three good reasons, which we list in order of importance, to underline the absolute need for considering moving control windows.

1. A control window that is fixed with respect to an observer will not be such with respect to other observers, and this means that the claimed expression for the thrust so derived is observer dependent, even with respect to Galilei frame transformations, and therefore they cannot be interpreted as a force.
2. The assumption of a fixed control window rules out any possibility of dealing with technically important problems such as the evaluation of the thrusting force exerted by the fluid on solid cases of rockets, get engines, turbines or lawn sprinklers and the like; see Sect. 7.
3. At last, the assumption of a fixed control window eliminates the possibility of considering deformable windows travelling along the trajectory, a convenient choice, in some instances, for the formulation of the equation of motion, as illustrated by the examples in Sect. 6.2.

To avoid unphysical interpretations of the law expressed by Eq. (48), we call for attention to the following points.

a. The formulation of the equation of motion provided by Eq. (48) does not pertain to the travel of the spatial control window C in a trajectory manifold $\mathcal{T}_{\mathcal{E}}$, which can be conceived by the investigator in any convenient way. Rather, it concerns the motion of the body which is tracking the dynamical trajectory $\mathcal{T}_{\mathcal{E}}$ in the spacetime manifold.
b. Proposition 5 provides an alternative way (and possibly a more convenient one) of computing the rate of variation of the projected momentum of the body under investigation, along its own dynamical trajectory.

As detailedly exposed in [14] and shown here in Eq. (68), the proper expression for the thrust exerted on a solid case by an interacting fluid in relative motion is given by the Galilei invariant expression:

$$-\int_{\boldsymbol{\Sigma}_{\text{OUT}}^{\text{IN}}} \mathbf{g}(\mathbf{v}_{\text{SKE}}^{\text{FLU}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\text{FLU}} \cdot \mathbf{v}_{\text{SKE}}^{\text{FLU}}). \tag{75}$$

The assessment of this formula requires suitable approximations and a peculiar reasoning, as recalled in Sect. 7.

The correct formula for the thrust exposed in Eq. (75) is deceptively similar to the boundary term in Eq. (74). Note, however, that the relative velocities there appearing have different meanings and that accelerations of the solid case and of the fluid filling the skeleton do not appear in Eq. (74).

The quoted similarity is an undesirable consequence of the adoption of a fixed control window and of the assumption of a steady motion of the fluid. The similarity disappears as soon as the absolute velocity $\mathbf{v}_{\mathcal{S}}$ is put in place of the relative velocity $\mathbf{v}_{\text{REL}}$ in Eq. (74) to recover the law of motion expressed by Eq. (48) whence it is apparent that neither of the single terms at the r.h.s. is Galilei invariant and therefore neither can be interpreted as a force.

9 Mass flow

An interesting application of the control window point of view concerns the axiom of mass conservation pertaining to a travelling control window $C \subset \boldsymbol{\Omega}$. Mass conservation is a local property to be fulfilled by any part of $\mathcal{P}_C \subset C$.

According to item $iv)$ of Eq. (14), substituting $\mathbf{m}$ for $\boldsymbol{\mu}$ in the geometric formula Eq. (43), mass conservation may be written as:

$$\partial_{\alpha=0} \int_{\boldsymbol{\xi}_\alpha(\mathcal{P}_C)} \mathbf{m} + \oint_{\partial\mathcal{P}_C} \mathbf{m} \cdot \mathbf{v}_{\text{REL}} = 0. \tag{76}$$

Condition Eq. (76) is conveniently expressed by the following variational condition on a given control window C:

$$\boxed{\int_C \delta\lambda \cdot \mathcal{L}_{\mathbf{v}_{\boldsymbol{\xi}}}(\mathbf{m}) + \oint_{\partial C} \delta\lambda \cdot \mathbf{m} \cdot \mathbf{v}_{\text{REL}} = 0,} \tag{77}$$

for any scalar test field $\delta\lambda : C \mapsto \Re$, piecewise constant in C.

A similar procedure was adopted in [29] to express the first principle of thermodynamics as a variational principle.

The Lagrange multipliers theorem [29,30] provides the following existence result, analogous to the one concerning stress fields in continuum mechanics.

Proposition 6 (Mass flow vector field) *Conservation of mass Eq.* (77) *entails the existence in the control window* C *of a mass flow vector field* $\mathbf{q} : C \mapsto TC$ *fulfilling the variational condition*

$$\int_C \delta\lambda \cdot \mathcal{L}_{\mathbf{v}_\xi}(\mathbf{m}) + \oint_{\partial C} \delta\lambda \cdot \mathbf{m} \cdot \mathbf{v}_{\text{REL}} = \int_C \mathbf{g}(\mathbf{q}, \nabla\delta\lambda) \cdot \mathbf{m}, \tag{78}$$

where $\delta\lambda : C \mapsto \Re$ *is a square integrable scalar test field with piecewise square integrable derivatives. The corresponding local mass balance equations are*

$$\begin{cases} -d(\mathbf{m} \cdot \mathbf{q}) = \mathcal{L}_{\mathbf{v}_\xi}(\mathbf{m}), & \textit{in } C\text{, bulk mass source,} \\ \quad \mathbf{m} \cdot \mathbf{q} = \mathbf{m} \cdot \mathbf{v}_{\text{REL}}, & \textit{on } \partial C\text{, boundary mass flux.} \end{cases} \tag{79}$$

In view of applications to Darcy-type permeability problems, to be briefly discussed in Sect. 9.2, it is convenient that the test fields $\delta\lambda : C \mapsto \Re$ be named *virtual pressures*.

By virtue of the existence result in Proposition 6, the formulations in Eqs. (76), (77), (78) are equivalent one another.

The flux of the mass flow velocity field through a surface in the control window yields the mass crossing that surface per unit time.

If the surface is the boundary of a domain, the flux of the mass flow vector field yields the mass coming into the domain per unit time.

If the control window is dragged along the trajectory by the motion of the body, we have that $\mathbf{v}_\varphi - \mathbf{v}_\xi = \mathbf{v}_{\text{REL}} = \mathbf{0}$ and the local mass balance equations become

$$\begin{cases} -d(\mathbf{m} \cdot \mathbf{q}) = \mathcal{L}_{\mathbf{v}_\varphi}(\mathbf{m}) = \mathbf{0}, & \text{in } C\,, \\ \quad \mathbf{m} \cdot \mathbf{q} = \mathbf{0}, & \text{on } \partial C\,. \end{cases} \tag{80}$$

The flux across any closed surface then vanishes and the divergence theorem implies that $\mathcal{L}_{\mathbf{v}_\varphi}(\mathbf{m}) = \mathbf{0}$. The rate form of the mass conservation principle is thus recovered.

9.1 Fluid flow through a porous medium

As an application of the analysis developed above, let us consider a two-phase medium composed of a fluid phase and of a porous solid skeleton in which a fixed control window is drawn.[8]

Let us assume that the fluid has a stationary flow through the porous skeleton under prescribed boundary conditions on the pressure field.

We consider the affine manifold $\mathcal{L}_{adm}(C)$ of admissible pressure fields fulfilling the non-homogeneous boundary conditions and the linear subspace $\mathcal{L}_o(C)$ of pressure fields conforming the related homogeneous boundary conditions.

Due to the assumption that the control window is fixed with respect to the porous skeleton, we have that $\mathbf{v}_\xi^S = \mathbf{0}$.

Moreover, by stationarity of the fluid flow, the partial time derivative $\mathcal{L}_{\mathbf{Z}}(\mathbf{m})$ vanishes too. Consequently, the rate term $\mathcal{L}_{\mathbf{v}_\xi}(\mathbf{m})$ vanishes since

$$\mathcal{L}_{\mathbf{v}_\xi}(\mathbf{m}) = \mathcal{L}_{\mathbf{v}_\xi^S}(\mathbf{m}) + \mathcal{L}_{\mathbf{Z}}(\mathbf{m}) = \mathbf{0}. \tag{81}$$

The mass conservation principle, stated in Eq. (78) as a principle of virtual pressures, yields then the variational *mass balance law*:

$$\boxed{\oint_{\partial C} \delta\lambda \cdot \mathbf{m} \cdot \mathbf{v}_{\text{REL}} = \int_C \mathbf{g}(\mathbf{q}, \nabla\delta\lambda) \cdot \mathbf{m},} \tag{82}$$

for all conforming virtual pressure fields $\delta\lambda \in \mathcal{L}_o(C)$.

[8] Turbulent kinetic models and flows of incompressible Newtonian fluids, adopted in oceanography and marine engineering, have recently been studied in [31].

9.2 Variational principle for mass flow

Let us denote by λ_0 a pressure field in the fluid in static equilibrium.

By assuming a DARCY-type permeability law, the *fluid mass flow* is related to the gradient of the overpressure $\lambda - \lambda_0$ by the gradient law

$$\boxed{\mathbf{q} = \nabla\Psi(\nabla(\lambda - \lambda_0)),} \tag{83}$$

governed by a scalar convex potential

$$\Psi : \boldsymbol{\Omega} \mapsto \mathrm{FUN}(T\boldsymbol{\Omega}), \tag{84}$$

depending on the gradient of the overpressure $\nabla(\lambda-\lambda_0)$ and designed for describing the nonlinear permeability properties of the medium.

The evaluation of the pressure in the permeating fluid can be performed by means of the variational principle, stating that for all $\delta\lambda \in \mathcal{L}_o(C)$

$$\oint_{\partial C} \delta\lambda \cdot \mathbf{m} \cdot \mathbf{v}_{\mathrm{REL}} = \int_C \mathbf{g}(\nabla\Psi(\nabla(\lambda - \lambda_0)), \nabla\delta\lambda) \cdot \mathbf{m}. \tag{85}$$

By introducing the functional F defined by

$$F(\lambda) := \int_C \Psi(\nabla(\lambda - \lambda_0)) \cdot \boldsymbol{\mu}_{\mathbf{g}} - \oint_{\partial C} \lambda \cdot \mathbf{m} \cdot \mathbf{v}_{\mathrm{REL}}, \tag{86}$$

the principle can be written as a stationarity condition at $\lambda \in \mathcal{L}_{adm}(C)$:

$$\langle dF(\lambda), \delta\lambda \rangle = 0, \quad \forall\, \delta\lambda \in \mathcal{L}_o(C), \tag{87}$$

which is analogous to the stationarity condition for the total potential energy of an elastic structure as a functional of the (small) displacement field.

10 Concluding remarks

The role of control windows in the description of the rate of change of the projected kinetic momentum is investigated by a direct geometric approach based on standard notions and results of differential geometry.

The following issues have been dealt with in detail.

1. A direct application of the travelling control window point of view is convenient and effective when motions with discontinuous velocity fields are investigated. A simple answer can thus be given to otherwise troublesome dynamical problems, such as classical challenging problems involving falling chains.
2. Evaluation of the thrust exerted by a fluid on a solid case is shown to require a properly conceived control window travelling in the fluid trajectory while keeping contact with the solid case interaction boundary. The direct application of the formula for the variation of kinetic momentum as seen by a fixed control window, improperly adopted in treatments of variable mass systems, is shown to lead to an alleged equation of motion in terms of forces that are not Galilei invariant. A warning concerning the adoption of fixed control windows should contribute to overcome improper treatments in the literature and to propose relevant amendments.
3. The mass flow of a fluid in motion through a porous solid can be effectively investigated by a description involving a control window fixed to the solid. This formulation leads to the notion of mass flow vector field. A variational principle related to a Darcy-type permeability law is also established.

All issues are here treated with a geometric approach and new results are obtained. Item (1) is mainly of theoretical relevance and was discussed by many illustrious scholars. Item (2) has an important field of applications in mechanics to investigate the dynamics of systems with possibly variable total mass, but characterised by a small rate of mass variation and a large rate of momentum variation. Such are rockets, jets, turbines, lawn sprinklers and similar systems. Item (3) yields a constructive treatment of fluid flows through porous solids starting from the sole principle of conservation of mass.

A Mathematical notes

All notions listed in this section are illustrated in detail in [30,32]. To a manifold $\mathbf{M}$, there corresponds a tangent bundle $T\mathbf{M}$ whose fibre at $\mathbf{x} \in \mathbf{M}$ is the linear space of velocities of curves through that point. A tangent vector field $\mathbf{u} : \mathbf{M} \mapsto T\mathbf{M}$ is characterised by the property that $\mathbf{u}(\mathbf{x}) \in T_{\mathbf{x}}\mathbf{M}$, which may be expressed by stating that the projection $\boldsymbol{\tau}_{\mathbf{M}} : T\mathbf{M} \mapsto \mathbf{M}$ on the base manifold is a right inverse of the vector field, i.e. that $\boldsymbol{\tau}_{\mathbf{M}} \circ \mathbf{u} : \mathbf{M} \mapsto \mathbf{M}$ is the identity. The flow $\mathbf{Fl}^{\mathbf{u}}_{\alpha} : \mathbf{M} \mapsto \mathbf{M}$ is generated by solutions of the differential equation $\mathbf{u} = \partial_{\alpha=0}\, \mathbf{Fl}^{\mathbf{u}}_{\alpha}$. The push forward of a tangent vector field $\mathbf{w} : \mathbf{M} \mapsto T\mathbf{M}$ along the flow $\mathbf{Fl}^{\mathbf{u}}_{\alpha} : \mathbf{M} \mapsto \mathbf{M}$ is defined by the tangent functor[9]

$$(\mathbf{Fl}^{\mathbf{u}}_{\alpha}\!\uparrow\!\mathbf{w}) \circ \mathbf{Fl}^{\mathbf{u}}_{\alpha} := T\mathbf{Fl}^{\mathbf{u}}_{\alpha} \cdot \mathbf{w}, \tag{A.1}$$

where $T\mathbf{Fl}^{\mathbf{u}}_{\alpha} : T\mathbf{M} \mapsto T\mathbf{M}$ is the tangent map.

Here and in the following, a circle $\circ$ means composition of maps and an interposed dot $\cdot$ denotes linear dependence on subsequent arguments. The pullback is defined by $\mathbf{Fl}^{\mathbf{u}}_{\alpha}\!\downarrow := \mathbf{Fl}^{\mathbf{u}}_{-\alpha}\!\uparrow$, and the Lie derivative $\mathcal{L}_{\mathbf{u}}(\mathbf{w}) \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ of a tangent vector field $\mathbf{w} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ along a tangent vector field $\mathbf{u} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ is the derivative of the pullback by the relevant flow:

$$\mathcal{L}_{\mathbf{u}}(\mathbf{w}) := \partial_{\alpha=0}\, (\mathbf{Fl}^{\mathbf{u}}_{\alpha}\!\downarrow\!\mathbf{w}) = \partial_{\alpha=0}\, T\mathbf{Fl}^{\mathbf{u}}_{-\alpha} \circ \mathbf{w} \circ \mathbf{Fl}^{\mathbf{u}}_{\alpha}. \tag{A.2}$$

Push–pull of scalar fields is just a change of base points, and hence the Lie derivative coincides with the directional derivative. Push–pull of tensors is defined by invariance.

Adopting the notation $\mathbf{u} f := \mathcal{L}_{\mathbf{u}} f$, with $f \in \mathrm{C}^1(\mathbf{M} \mapsto \Re)$ any scalar field, the *commutator* of tangent vector fields $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ is the skew-symmetric tangent vector-valued differential operator defined by

$$[\mathbf{u}\,,\mathbf{w}] f := (\mathbf{u}\mathbf{w} - \mathbf{w}\mathbf{u}) f = (\mathcal{L}_{\mathbf{u}}\mathcal{L}_{\mathbf{w}} - \mathcal{L}_{\mathbf{w}}\mathcal{L}_{\mathbf{u}}) f. \tag{A.3}$$

A basic theorem concerning Lie derivatives states that $\mathcal{L}_{\mathbf{u}}(\mathbf{w}) = [\mathbf{u}\,,\mathbf{w}]$, and hence the commutator of tangent vector fields is called the Lie bracket. Moreover, for any injective morphism $\boldsymbol{\phi} \in \mathrm{C}^1(\mathbf{M} \mapsto \mathbf{N})$, the following push naturality property $[\boldsymbol{\phi}\!\uparrow\!\mathbf{v}\,, \boldsymbol{\phi}\!\uparrow\!\mathbf{u}] = \boldsymbol{\phi}\!\uparrow\![\mathbf{v}\,,\mathbf{u}]$ holds.

A linear connection ∇ in a manifold $\mathbf{M}$ fulfils the characteristic properties of a point derivation,

$$\begin{aligned} \nabla_{\alpha_1\mathbf{w}_1+\alpha_2\mathbf{w}_2}(\mathbf{u}) &= \alpha_1 \nabla_{\mathbf{w}_1}(\mathbf{u}) + \alpha_2 \nabla_{\mathbf{w}_2}(\mathbf{u}), \\ \nabla_{\mathbf{w}}(\alpha_1(\mathbf{u}_1) + \alpha_2(\mathbf{u}_2)) &= \alpha_1 \nabla_{\mathbf{w}}(\mathbf{u}_1) + \alpha_2 \nabla_{\mathbf{w}}(\mathbf{u}_2), \\ \nabla_{\mathbf{w}}(f\mathbf{u}) &= f\, \nabla_{\mathbf{w}}(\mathbf{u}) + (\nabla_{\mathbf{w}} f)(\mathbf{u}), \end{aligned} \tag{A.4}$$

with $\alpha_1, \alpha_2 : \mathbf{M} \mapsto \Re$ scalar fields and $\mathbf{u}, \mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2 : \mathbf{M} \mapsto T\mathbf{M}$ tangent vector fields. In terms of parallel transport $\Uparrow$ ($\Downarrow$ denotes the backward parallel transport) along a curve $\mathbf{c} : \Re \mapsto \mathbf{M}$ with $\mathbf{w}_{\mathbf{x}} = \partial_{\alpha=0}\, \mathbf{c}(\alpha)$ and $\mathbf{x} = \mathbf{c}(0)$, the parallel derivative of a vector field $\mathbf{w} : \mathbf{M} \mapsto T\mathbf{M}$ according to a connection is defined by

$$\nabla_{\mathbf{w}}(\mathbf{u}) := \partial_{\alpha=0}\, \mathbf{c}(\alpha)\!\Downarrow\!(\mathbf{u} \circ \mathbf{c})(\alpha). \tag{A.5}$$

Parallel transport vector fields $(\mathbf{u}\circ\mathbf{c})(\alpha) = \mathbf{c}(\alpha)\!\Uparrow\!\mathbf{u}_{\mathbf{x}}$ are characterised by a null parallel derivative, because

$$\nabla_{\mathbf{w}_{\mathbf{x}}}(\mathbf{u}) := \partial_{\alpha=0}\, \mathbf{c}(\alpha)\!\Downarrow\!(\mathbf{u} \circ \mathbf{c})(\alpha) = \partial_{\alpha=0}\, (\mathbf{c}(\alpha)\!\Downarrow \circ\, \mathbf{c}(\alpha)\!\Uparrow)\mathbf{u}_{\mathbf{x}} = \partial_{\alpha=0}\, \mathbf{u}_{\mathbf{x}} = 0.$$

The parallel transport of a tensor field is defined by invariance, and the parallel derivative fulfils a Leibniz rule, which for a covector field $\boldsymbol{\alpha} : \mathbf{M} \mapsto T^*\mathbf{M}$ writes

$$\langle \nabla_{\mathbf{w}}(\boldsymbol{\alpha}), \mathbf{u} \rangle = \nabla_{\mathbf{w}} \langle \boldsymbol{\alpha}, \mathbf{u} \rangle - \langle \boldsymbol{\alpha}, \nabla_{\mathbf{w}}(\mathbf{u}) \rangle. \tag{A.6}$$

In a Riemann manifold $(\mathbf{M}\,,\mathbf{g})$, a linear connection ∇ is *metric preserving* if the metric is invariant under parallel transport

$$\mathbf{g}_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}}\,, \mathbf{v}_{\mathbf{x}}) = \mathbf{g}_{\mathbf{c}(\alpha)}(\mathbf{c}(\alpha)\!\Uparrow\!\mathbf{u}_{\mathbf{x}}\,, \mathbf{c}(\alpha)\!\Uparrow\!\mathbf{v}_{\mathbf{x}}), \tag{A.7}$$

so that its parallel derivative vanishes: $\nabla\mathbf{g} = 0$.

[9] Applying the tangent functor T to a map $\boldsymbol{\phi} : \mathbf{M} \mapsto \mathbf{N}$ between manifolds, the outcome is the tangent map $T\boldsymbol{\phi} : T\mathbf{M} \mapsto T\mathbf{N}$ which associates, in a linear way, with the velocity of a curve at a given point, the velocity of the image curve at the image point.

The curvature of the connection is the operator CURV which maps tensorially a tangent vector field $\mathbf{s} : \mathbf{M} \mapsto T\mathbf{M}$ to a tangent vector-valued two-form[10] $\text{CURV}(\mathbf{s})$ defined by

$$\text{CURV}(\mathbf{s})(\mathbf{u}\,,\mathbf{w}) := ([\nabla_{\mathbf{u}}\,,\nabla_{\mathbf{w}}] - \nabla_{[\mathbf{u}\,,\mathbf{w}]})(\mathbf{s}), \tag{A.8}$$

and the torsion TORS is the tangent vector-valued two-form defined by

$$\text{TORS}(\mathbf{u}\,,\mathbf{w}) := \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} - [\mathbf{u}\,,\mathbf{w}]. \tag{A.9}$$

Mixed tensor fields $\text{TORS}(\mathbf{u})$ and $\text{CURV}(\mathbf{s}\,,\mathbf{u})$ are defined by the identities

$$\begin{aligned} \text{TORS}(\mathbf{u})\cdot\mathbf{w} &:= \text{TORS}(\mathbf{u}\,,\mathbf{w}) = -\text{TORS}(\mathbf{w}\,,\mathbf{u}),\\ \text{CURV}(\mathbf{s}\,,\mathbf{u})\cdot\mathbf{w} &:= \text{CURV}(\mathbf{s})(\mathbf{u}\,,\mathbf{w}) = -\text{CURV}(\mathbf{s})(\mathbf{w}\,,\mathbf{u}). \end{aligned} \tag{A.10}$$

A connection with vanishing torsion is named *torsion free* or *symmetric*, and a connection with vanishing curvature is said to be *curvature free* or *flat*. The Levi-Civita connection is the unique one that is metric and symmetrical.

The modern way to introduce integral transformations is to consider maximal forms, that is forms of order equal to the manifold dimension, as geometric objects to be integrated over a (orientable) manifold and to resort to the notion of exterior differential of a form [30,33].

In a m-dimensional manifold $\mathbf{M}$, let $\boldsymbol{\Gamma}$ be any n-dimensional submanifold ($m \geq n$) with $(n-1)$-dimensional boundary manifold $\partial\boldsymbol{\Gamma}$.

The classical Stokes formula, in its modern formulation by Volterra, characterises the exterior derivative of a $(n-1)$-form $\boldsymbol{\omega} : \mathbf{M} \mapsto \text{ALT}^{n-1}(T\mathbf{M})$, defined as the n-form $d\boldsymbol{\omega} : \mathbf{M} \mapsto \text{ALT}^{n}(T\mathbf{M})$ fulfilling the identity

$$\int_{\boldsymbol{\Gamma}} d\boldsymbol{\omega} = \int_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega}. \tag{A.11}$$

Because $\partial\partial\boldsymbol{\Gamma} = 0$ for any manifold $\boldsymbol{\Gamma}$, also $dd\boldsymbol{\omega} = 0$ for any form $\boldsymbol{\omega}$ [30,33].

The exterior derivative of differential forms commutes with the pullback by an injective immersion $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$ between manifolds $\mathbf{M}$ and $\mathbf{N}$:

$$d \circ \boldsymbol{\chi}\!\downarrow\, = \boldsymbol{\chi}\!\downarrow \circ\, d, \tag{A.12}$$

a result inferred from Stokes and integral transformation formulae,

$$\begin{aligned} \int_{\boldsymbol{\Gamma}} d(\boldsymbol{\chi}\!\downarrow\!\boldsymbol{\omega}) &= \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\chi}\!\downarrow\!\boldsymbol{\omega} = \oint_{\boldsymbol{\chi}(\partial\boldsymbol{\Gamma})} \boldsymbol{\omega}\\ &= \oint_{\partial\boldsymbol{\chi}(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \int_{\boldsymbol{\chi}(\boldsymbol{\Gamma})} d\boldsymbol{\omega} = \int_{\boldsymbol{\Gamma}} \boldsymbol{\chi}\!\downarrow\!(d\boldsymbol{\omega}). \end{aligned} \tag{A.13}$$

Then, for $\mathbf{v} := \partial_{\alpha=0}\,\boldsymbol{\chi}_\alpha$ we infer that

$$\mathcal{L}_{\mathbf{v}}\,(d\boldsymbol{\omega}) = d\,\mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}). \tag{A.14}$$

The *geometric homotopy formula* relates the boundary chain generated by the extrusion of a manifold $\boldsymbol{\Gamma}$ and of its boundary $\partial\boldsymbol{\Gamma}$, as follows:

$$\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\alpha)) = \boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma}) - \boldsymbol{\Gamma} - J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\alpha),$$

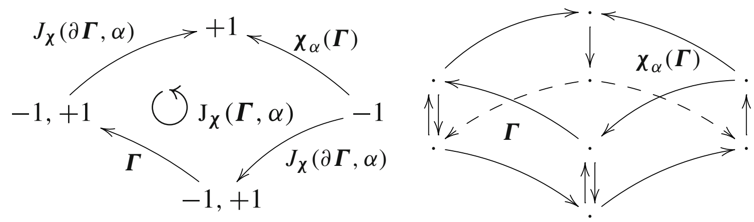
with $\alpha \in \mathcal{Z}$ extrusion parameter and $\boldsymbol{\chi} : \boldsymbol{\Gamma} \times \mathcal{Z} \mapsto \mathbf{M}$ extrusion map fulfilling the commutative diagram

$$\begin{array}{ccc} \boldsymbol{\Gamma} \times \mathcal{Z} & \xrightarrow{\ \boldsymbol{\chi}_\alpha\ } & \mathbf{M} \\ {\scriptstyle \boldsymbol{\pi}_{\mathcal{Z}}}\big\downarrow & & \big\downarrow{\scriptstyle t_{\mathcal{Z}}} \\ \mathcal{Z} & \xrightarrow{\ \boldsymbol{\theta}_\alpha\ } & \mathcal{Z} \end{array} \quad \iff \quad t_{\mathcal{Z}} \circ \boldsymbol{\chi}_\alpha = \boldsymbol{\theta}_\alpha \circ \boldsymbol{\pi}_{\mathcal{Z}}, \tag{A.15}$$

with $\boldsymbol{\theta}_\alpha : \mathcal{Z} \mapsto \mathcal{Z}$ the translation defined by $\boldsymbol{\theta}_\alpha(\beta) := \alpha + \beta$ for $\alpha, \beta \in \mathcal{Z}$.

[10] Tensoriality of a linear map, acting on vector fields and generating a vector field, means that point values of the image field depend only on the values of the source fields at the same point. An (exterior) form is a vector-valued, tensorial, alternating multilinear map.

The signs in the formula are motivated as follows. The orientation of the $(n+1)$-dimensional flow tube $J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \alpha)$ induces an orientation on its boundary $\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \alpha))$. In the boundary chain, composed by the elements $\boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma})$, $\boldsymbol{\Gamma}$ and $J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \alpha)$, each one with the induced orientation, the element $\boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma})$ has orientation opposed to the orientation of $\boldsymbol{\chi}_0(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma}$ and $J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \alpha)$, as depicted in the diagrams (A.16), for $\dim \boldsymbol{\Gamma} = 1$ and $\dim \boldsymbol{\Gamma} = 2$.



(A.16)

Let $\boldsymbol{\omega}$ be an n-form defined on the $(n+1)$-manifold $J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \alpha)$ spanned by extrusion of the n-manifold $\boldsymbol{\Gamma}$, so that the geometric homotopy formula gives

$$\int_{\boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \oint_{\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\alpha))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\alpha)} \boldsymbol{\omega} + \int_{\boldsymbol{\Gamma}} \boldsymbol{\omega}. \tag{A.17}$$

Differentiation with respect to the extrusion time yields

$$\partial_{\alpha=0} \int_{\boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \partial_{\alpha=0} \left(\oint_{\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\alpha))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\alpha)} \boldsymbol{\omega} \right). \tag{A.18}$$

Then, denoting by $\mathbf{v} := \partial_{\alpha=0}\, \boldsymbol{\chi}_\alpha$ the velocity field of the extrusion, applying the Stokes formula and taking into account that by the Fubini theorem [32]

$$\begin{aligned} \partial_{\alpha=0} \int_{J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\alpha)} d\boldsymbol{\omega} &= \int_{\boldsymbol{\Gamma}} (d\boldsymbol{\omega}) \cdot \mathbf{v}, \\ \partial_{\alpha=0} \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\alpha)} \boldsymbol{\omega} &= \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v}, \end{aligned} \tag{A.19}$$

we get the integral *extrusion formula*

$$\boxed{\partial_{\alpha=0} \int_{\boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \int_{\boldsymbol{\Gamma}} (d\boldsymbol{\omega}) \cdot \mathbf{v} + \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v}.} \tag{A.20}$$

On the other hand, taking the time rate of the integral transformation formula leads to the Lie-Reynolds formula

$$\partial_{\alpha=0} \int_{\boldsymbol{\chi}_\alpha(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \partial_{\alpha=0} \int_{\boldsymbol{\Gamma}} (\boldsymbol{\chi}_\alpha {\downarrow} \boldsymbol{\omega}) = \int_{\boldsymbol{\Gamma}} \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}). \tag{A.21}$$

Comparing the expressions in Eq. (A.21) and in Eq. (A.20) and applying the Stokes formula to get the transformation

$$\oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v} = \int_{\boldsymbol{\Gamma}} d(\boldsymbol{\omega} \cdot \mathbf{v}), \tag{A.22}$$

a standard localisation yield the *differential homotopy formula* expressing the Lie derivative of a k-form in terms of exterior derivatives:

$$\boxed{\mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}) = d(\boldsymbol{\omega} \cdot \mathbf{v}) + (d\boldsymbol{\omega}) \cdot \mathbf{v}.} \tag{A.23}$$

References

1. Cayley, A.: On a class of dynamical problems. Proc. R. Soc. Lond. **VIII**, 506–511 (1857)
2. Irschik, H.: The Cayley variational principle for continuous-impact problems: a continuum mechanics bases version in the presence of a singular surface. J. Theor. Appl. Mech. **50**(3), 717–727 (2012)
3. Tait, P.G., Steele, W.J.: A Treatise on Dynamics of a Particle, 4th edn, pp. 250–251. MacMillan & Co, London (1856)
4. Hamel, G.: Theoretische Mechanik. Grundlehren der Mathematischen Wissenschaften. Bd LVII. Springer, Berlin (1949)
5. Gantmacher, F.R.: Lectures in Analytical Mechanics. Mir publishers, Moscow (1970)
6. Meirovitch, L.: Methods of Analytical Dynamics. McGraw-Hill, New York (1970)
7. Rosenberg, R.M.: Analytical Dynamics of Discrete Systems. Mathematical Concepts in Science and Engineering, vol. 4, 3rd print. Plenum Press, New York (1991)
8. Huang, Z.: The equilibrium equations and constitutive equations of the growing deformable body in the framework of continuum theory. Int. J. Non-Linear Mech. **39**, 951–962 (2004)
9. de Sousa, C.A., Gordo P.M., Costa P.: Falling chains as variable mass systems: theoretical model and experimental analysis. arXiv:1110.6035 [pdf] (2011)
10. Gardi, E.: Lecture 6: Momentum and variable-mass problems. The University of Edinburgh, School of Physics and Astronomy. http://www2.ph.ed.ac.uk/~egardi/MfP3-Dynamics/Dynamics_lecture_6.pdf (2014)
11. Irschik, H., Holl, H.J.: Mechanics of variable-mass systems-part 1: balance of mass and linear momentum. Appl. Mech. Rev. **57**(2), 145–160 (2004)
12. Irschik, H., Belyaev, A.K.: Dynamics of Mechanical Systems with Variable Mass. CISM Courses and Lectures, vol. 557. Springer, Wien (2014)
13. Irschik, H., Casetta, L., Pesce, C.P.: A generalization of Noether's theorem for a non-material volume. Z. Angew. Math. Mech. **96**(6), 696–706 (2016)
14. Romano, G., Barretta, R., Diaco, M.: Solid–fluid interaction: a continuum mechanics assessment. Acta Mech. **2283**, 851–869 (2017)
15. von Buquoy, G.: Analytische Bestimmung des Gesetzes der virtuellen Geschwindigkeiten in mechanischer und statischer Hinsicht. Leipzig: bei Breitkopf und Härtel (1812). https://doi.org/10.3931/e-rara-14843
16. von Buquoy, G.: Exposition d'un nouveau principe général de dynamique, dont le principe des vitesses virtuelles n'est qu'un cas particulier. Courcier, Paris (1815)
17. Meshchersky, I.V.: The dynamics of a point of variable mass. Dissertation, St Petersburg Mathematical Society (1897)
18. Meshcherky, I.V.: Works on the Mechanics of Bodies with Variable Mass [in Russian], with an Introduction by A.A. Kosmodemyansky, Moscow, Leningrad: G.I.T.T.L. (1949)
19. Gurtin, M.E.: An Introduction to Continuum Mechanics. Academic, San Diego (1981)
20. Sommerfeld, A.: Mechanics—Lectures on Theoretical Physics, vol. I. Academic, New York (1952)
21. Routh, E.J.: A Treatise on Dynamics of a Particle. Cambridge University Press, Cambridge (1898)
22. Wong, C.W., Youn, S.H., Yasui, K.: The falling chain of Hopkins, Tait, Steele and Cayley. Eur. J. Phys. **28**, 385–400 (2007)
23. Galilei, G.: Discorsi e Dimostrazioni Matematiche intorno a due nuove Scienze Attenenti alla Mecanica e & i Movimenti Locali, del signor Galileo Galilei Linceo, Filosofo e Matematico primario del Sereniffimo Gran Duca di Tofcana. Con una Appendice del centro di gravità d'alcuni solidi. Leida, Appreffo gli Elfevirii M.D.C.XXXVIII (1638)
24. Schagerl, M., Steindl, A., Steiner, W., Troger, H.: On the paradox of the free falling folded chain. Acta Mech. **125**, 155–168 (1997)
25. Steiner, W., Troger, H.: On the equations of motion of the folded inextensible string. Z. Angew. Mech. Phys. **46**, 960–970 (1995)
26. Gantmakher, F.R., Levin, L.M.: The Flight of Uncontrolled Rockets. Pergamon Press, Oxford (1964)
27. Ziegler, F.: Mechanics of Solids and Fluids. Springer, Berlin (1995). Transl. from Technische Mechanik der festen und flüssigen Körper. Springer, Berlin (1985)
28. Gurtin, M.E., Fried, E., Anand, L.: The Mechanics and Thermodynamics of Continua. Cambridge University Press, Cambridge (2010)
29. Romano, G., Diaco, M., Barretta, R.: Variational formulation of the first principle of continuum thermodynamics. Contin. Mech. Thermodyn. **22**(3), 177–187 (2010)
30. Romano, G.: Continuum Mechanics on Manifolds. Lecture Notes (2007–2017). University of Naples Federico II, Italy. http://wpage.unina.it/romano
31. Gaudiello, A., Guibé, O., Murat, F.: Homogenization of the brush problem with a source term in L1. Arch. Ration. Mech. Anal. **225**, 1–64 (2017)
32. Abraham, R., Marsden, J.E., Ratiu, T.S.: Manifolds, Tensor Analysis, and Applications. Springer, New York (2002)
33. Dieudonné, J.: Treatise on Analysis. vol. I–IV, Academic Press, New York (1969–1974)