DEPARTMENT OF STRUCTURE

POSTCRITICAL BEHAVIOUR OF ELASTIC STRUCTURES

Part I: Incremental Uniqueness and Stability

by

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1. - Introduction

The investigation of the critical and the postcritical bahaviour of elastic structures is presently a classical theme of research.

Indeed, under the increasing demand of structural technology, the interest on the subject has grown enormously in recent years and an extensive literature, including both theoretical and experimental contributions, is available.

The survey article by HUTCHINSON and KOITER [1] on the postbuckling theory and a recent book by THOMPSON and HUNT [3] yield an excellent review on the subyect and include an exhaustive bibliografy.

A general method of analysis of the stability of the equilibrium and of the postbuckling behaviour of continuous elastic structures was first developed by KOITER [3] with an approach based on the expansion of the potential energy functional about a critical point.

More directly inspired to the studies of POINCARE [4] and LYTTLETON [5] in the field of hydrodynamics are the contributions of THOMPSON [6] and his school, which are concerned with discrete structural models.

A general discussion on stability, incremental uniqueness, postcritical behaviour and imperfection sensitivity of elastic structures under conservative loading is presented in this paper.

A unified approach is provided by a systematic perturbative analysis of the nonlinear equilibrium problem and of the associated critical eigenproblem.

Reference is made to discrete structural models but the extension to the continuous case is immediate, from the methodological point of view.

The proposed procedure appears to be simple and open to a direct mechanical interpretation of the various steps of the analysis.

The paper is divided in two parts.

In the first part we give a general description of the structural model under investigation and a detailed discussion on the stability of a given equilibrium state and its connection with the uniqueness of the incremental solution of the nonlinear equilibrium equation in a quasi-static loading process.

The distinction between regular and critical equilibrium states is basic in this respect.

The energy criterion of stability is assumed and its connection with the dynamical definition is briefly discussed.

In the case of critical states, the investigation of the stability is restricted to primary simple critical states, which are of major concern for the applications.

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We explicitly renounce to include the more involved case of multiple critical states, although it is quite interesting in many structural situations and in connection with the problem of the optimal design of structures [2].

The intimate relationship between the stability of a critical equilibrium state and the postcritical behaviour is investigated in the second part of the paper where also the imperfection sensitivity is discussed in detail.

1. SYSTEM DESCRIPTION

Let us consider a discrete model of an elastic structure undergoing a quasi-static deformation process.

The analysis of the kinematical behaviour can be developed in an adequate mathematical framework by esthablishing a one-to-one corrispondence between the set of admissible configurations and a finite dimensional Euclidean point space E .

The points of E and the vectors of the associated translation vector space U will then be respectively called configurations and displacements of the structure.

The mechanical properties of the structure are described by the internal energy functional W(u).

Constitutive requirements call for the special form of W :

$$W(u) = \Phi(D(u)) \tag{2.1}$$

The internal energy, as a function of the displacement vector, will then be composed by the mappings :

$$\varphi = \Phi(\varepsilon) \qquad \text{strain energy} \tag{2.2}$$

$$\varepsilon = D(u) \qquad \text{strain function} \tag{2.3}$$

with $\varepsilon \in \Sigma$, the strain vector space.

The inner products in the displacement and the strain vector spaces will be denoted by :

$$u \cdot v \quad \forall\, u, v \in U \qquad \varepsilon * \sigma \qquad \forall\, \varepsilon, \sigma \in \Sigma \tag{2.4}$$ (o)

In the sequel linear functionals and their representative vectors will be denoted by the same symbol. The same for quadratic functionals and the associated linear operators.

The differentiation symbol d will denote both the FRÉCHET derivative and the gradient of a functional. The differentiation will each time be intended with respect to the vector argument in parenthesis and the functions involved in the

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(o) Notation: vectors of U and Σ will respectively be denoted by latin and greek letters .

analysis will be supposed to be as many times differentiable as needed.

We shall consider a general load, splitten in the sum of a conservative load $\lambda p(u)$, proportional to the load parameter λ, and an arbitrary load f.

If $L(u)$ is the potential of the conservative load $p(u)$, i.e. :

$$p(u) = - dL(u) \tag{2.5}$$

the behaviour of the structure, in a quasi-static conservative loading process, will be described by the total potential energy :

$$E(u,\lambda) = W(u) + \lambda L(u) \tag{2.6}$$

Accordingly any ordered pair (u,λ) will be called a state of the system.

We shall be interested in investigating :

i) The stability of a given equilibrium state (u_o,λ_o) of the structure under the conservative loading $\lambda_o p(u)$.

ii) The existence and uniqueness of the incremental equilibrium response at an equilibrium state (u_o,λ_o) under the action of an arbitrary additional quasi static loading process.

iii) The evolution of the equilibrium configurations of the structure under a quasi static loading process $\lambda(t)\, p(u(t))$ described by a suitable evolution parameter t.

NONLINEAR EQUILIBRIUM EQUATION AND THE STIFFNESS OPERATOR

The equilibrium states (u,λ) are the solutions of the nonlinear equation :

$$d\,E(u,\lambda) = d\,W(u) + \lambda\, d\,L(u) = f \tag{3.1}$$

Defining the stress vector $\sigma \in \Sigma$ as the gradient :

$$\sigma = d\Phi(\varepsilon) \tag{3.2}$$

and the linear equilibrium operator $T(u) : \Sigma \rightarrow U$ as the adjoint of the derivative $d\,D(u) : U \rightarrow \Sigma$, according to the identity :

$$T(u)\,\sigma \cdot v = \sigma * d\,D(u)v \qquad \forall v \in U, \quad \forall \sigma \in \Sigma \tag{3.3}$$

from (2.2) we have :

$$d\,W(u)v = d\,\Phi(\varepsilon)\,d\,D(u)\,v = \sigma * d\,D(u)v = T(u)\,\sigma \cdot v \qquad \forall\, v \in U \tag{3.4}$$

Hence the equilibrium equation (3.1) may be rewritten as :

$$T(u)\sigma = \lambda\, p\,(u) + f \tag{3.5}$$

The linear operator $\tilde{T}(u) = d\,D(u)$ will be called the compatibility operator. Note that from (3.3) and (3.4) we get the virtual work identity :

$$(\lambda p + f) \cdot v = \sigma * e(v) \qquad \forall\, v \in U \tag{3.6}$$

where $e(v) = \tilde{T}(u)v$ is the first-order additional strain at the configuration u due to the additional displacement v .

In the perturbative analysis of the nonlinear equilibrium equation (3.1) and of the associated critical eigenvalue problem, a fundamental role will be played by the stiffness operator :

$$K\,(u,\lambda) = d_u^2 E(u,\lambda) \tag{3.7}$$

An explicit expression of K may be got introducing the operators :

$S = d^2\,\Phi\,(\varepsilon)$	elastic moduli	(3.8)
$K_e = T(u)\,S\,(\epsilon)\,\tilde{T}(u)$	elastic stiffness	(3.9)
$K_L = -\,d\,p(u) = d^2 L(u)$	load stiffness	(3.10)
$K_G = \sigma * d^2\,D(u)$	geometric stiffness	(3.11)

Indeed a simple algebra shows that :

$$K = K_e + K_G + \lambda\, K_L \tag{3.12}$$

All the stiffness operators above are symmetric.

We have then the following spectral representation of the overall stiffness operator K :

$$K = \sum_{i=1}^{n} \tau_i\; e_i \otimes e_i \tag{3.13}$$

where the symbol $\otimes$ denotes the tensor product, defined by the identity :

$$(u \otimes v)\, w = u(v \cdot w) \qquad \forall w \in U \;, \quad \forall u, v \in U \tag{3.14}$$

and $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of U consisting of eigenvectors of K, i.e. :

$$K e_i = \tau_i \, e_i \qquad e_i \cdot e_J = \delta_{iJ} \quad i = 1, 2, \dots, n \tag{3.15}$$

where δ_{iJ} is the Kronecker symbol.

The eigenvalues τ_i are ordered in a non decreasing sequence :

$$\tau_i \le \tau_J \quad \text{if} \quad i < J$$

The mechanical interpretation of the eigenvalues and eigenvectors of the stiffness operator K will be emphasized by the following nomenclature :

$$e_i \qquad \text{principal directions} \tag{3.16}$$

$$\tau_i \qquad \text{principal stiffnesses} \tag{3.17}$$

4. INCREMENTAL UNIQUENESS AND STABILITY

In the discussion of stability we shall assume the following energy definition:

"An equilibrium state (u, λ) is stable if the potential energy functional has a strict minimum at the configuration u".

It can be shown that for discrete elastic structures under conservative loading the definition above implies the stability in the dynamical sense (this is indeed stated by the classical LAGRANGE-DIRICHLET theorem of analytical mechanics).

The converse statement can be proved in whole generality if a strictly positive, velocity dependent dissipation is assumed [7] .

In fact the conjecture that such an assumption could allow for a simple general proof of the converse of the LAGRANGE-DIRICHLET theorem was formulated by KOITER [8]. The argument he gives is however no more than heuristic and he seems not to be aware of the basic difficulties in extending this result to continuous systems.

A discussion of the problem and a result for the continuons case are given in [7].

The problem of the formulation of necessary and sufficient conditions for a strict minimum of the potential energy was first solved by KOITER in his celebrated doctoral thesis in 1945 [3]. A thorough discussion has been given in [9] where also a simple interpretation has been formulated for the expression whose sign is decisive for the existence of a strict minimum at a critical point.

Let us now investigate in detail the connection between the stability of a given equilibrium state and the existence and uniqueness of incremental equilibrium displacements in an arbitrary quasi-static loading process, defined by :

$$\lambda = \lambda(t) \quad , \quad f = f(t)$$

where t is a suitable evolution parameter.

In the sequel a superimposed dot will denote differentiation with respect to t.

We shall distinguish two basic situations :

a) regular states
b) singular (or critical) states

REGULAR STATES

An equilibrium state (u,λ) is said to be regular if the corresponding stiffness operator $K(u,\lambda)$ is regular, i.e. if any of the following equivalent statements holds true :

i) $\det K(u,\lambda) = \prod_{i=1}^{n} \tau_i(u,\lambda) \neq 0$

i) $K\dot{u} = 0 \implies \dot{u} = 0$

i) $K\dot{u} = \dot{f} \implies \dot{u} = K^{-1}\dot{f} = \sum_{i=1}^{n} \frac{1}{\tau_i} (\dot{f}\cdot e_i)\, e_i$

To the statements above we may give the mechanical interpretations :

i) all the principal stiffnesses are different from zero

i) there are no infinitesimally adjacent equilibrium configurations under the same load.

ii) the incremental equilibrium problem admits an unique solution for every increment of load.

Now recalling that : $\tau_1 = \min \{d^2 E(u,\lambda) v^2 \mid \|v\| = 1\}$

we have that the potential energy functional E will have a strict minimum at (u, λ) if $\tau_1 > 0$ and only if $\tau_1 \geq 0$.

Hence, according to the given definition of stability, since at a regular point $\tau_1 \neq 0$, we get the following stability criterion :

A regular equilibrium state is stable if and only if :

i) the stiffness operator is strictly positive

or equivalently ii) all the principal stiffnesses are positive.

The condition for stability may also be formulated as follows :

$$\forall \dot{u}, \dot{f} \quad \text{such that} \quad K\dot{u} = \dot{f} \quad \text{we have} \quad \dot{f}\cdot\dot{u} = K\dot{u}\cdot\dot{u} = K^{-1}\dot{f}\cdot\dot{f} > 0$$

and hence we may give the mechanical interpretation :

"A regular equilibrium state is stable if and only if every increment of load does a positive work due to the incremental equilibrium displacement of the structure".

If the stability condition is met, the unique solution of the incremental equilibrium problem is characterized by the following :

Minimum principle :

The incremental equilibrium problem : $K\dot{u} = \dot{f}$

is equivalent to the minimum problem : $\min \{ \frac{1}{2} K\dot{u}\,\dot{u} - \dot{f}\,\dot{u}\}$

Both problems admit an unique solution $\dot{u}_o$ and the minimum above is equal to :

$$-\frac{1}{2} K\,\dot{u}_o\cdot\dot{u}_o = -\frac{1}{2}\,\dot{f}\cdot\dot{u}_o < 0$$

Until now the attention has been restricted to the incremental response of the system.

A general result does however hold, concerning the existence and uniqueness of the equilibrium path emerging from a regular equilibrium state, under the action of an arbitrary quasi-static loading process.

Indeed a generalized version of the implicit function theorem gives :

THEOREM

Let (u_o, λ_o) be a regular state of the system and assume that the energy gradient:

$$A(u,\lambda) = d_u\, E(u,\lambda)$$

be continuonsly differentiable. Then there exists a function $\hat{u}$ and an open neighbourhood $U_o \times \Lambda_o$ of (u_o, λ_o) such that :

$$A(\hat{u}(\lambda, f), \lambda) = f \qquad \forall\, \lambda \in \Lambda_o \;,\; \forall\, f \in A(U_o \times \Lambda_o) \tag{4.1}$$

where :

$$A(U_o \times \Lambda_o) = \{A(u,\lambda) : u \in U_o \;,\; \lambda \in \Lambda_o\}$$

Moreover if A is analytic and $f(t)$ and $\lambda(t)$ are analytic functions of the evolution parameter t, then the function :

$$u(t) = \hat{u}\,(\lambda(t),\, f(t)) \tag{4.2}$$

is analytic in t.

Hence, considering a quasi-static loading process $\lambda(t)$, $f(t)$ starting at a regular state of the system, we may obtain the various order terms of the expansion:

$$u(t) = \dot{u}\, t + \frac{1}{2}\, \ddot{u}\, t^2 + \ldots\ldots \qquad (u(0) = 0)$$

by means of the subsequent linear problems :

$$\begin{aligned}
K\dot{u} &= \dot{\lambda} p + \dot{f} \\
K\ddot{u} &= \ddot{\lambda} p + \ddot{f} - dK\, \dot{u}^2 \\
K\dddot{u} &= \dddot{\lambda} p + \dddot{f} - d^2K\, \dot{u}^3 - 3dK\, \ddot{u}\, \dot{u} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

all of which admit an unique solution, since K is regular.

CRITICAL STATES

An equilibrium state is said to be critical (or singular) if the stiffness operator $K(u,\lambda)$ is singular, i.e. if any of the following equivalent statements holds true: (o)

i) $\det K(u,\lambda) = \prod_{i=1}^{n} \tau_i(u,\lambda) = 0$

ii) $K\dot{u} = 0 \not\Rightarrow \dot{u} = 0$ viz. $\dim N(K) = m > 0$

iii) the incremental equilibrium problem : $K\dot{u} = \dot{f}$ admits :

a) no solution if $\dot{f} \in N(K) = R(K)^{\perp}$

b) an m-dimensional linear variety L of solutions if $\dot{f} \in R(K)$:

$$L = \dot{u}_0 + N(K)$$

with $\dot{u}_0 \in N^{\perp}(K)$ a special solution of the incremental equilibrium problem.

If we denote by $\{d_i\}$ $i = 1,\dots,n-m$ an orthonormal basis of $R(K)$, i.e. a maximal orthonormal set of principal directions with non vanishing stiffnesses ($\tau_i \neq 0$, $i = 1,\dots, n-m$), the spectral representation of K will be :

$$K = \sum_{i=1}^{n-m} \tau_i \, d_i \otimes d_i$$

Further on let $\{a_k\}$ $k = 1,\dots,m$ be an orthornormal basis of $N(K)$, i.e. a maxi-

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(o) We denote by N(K) and R(K) respectively the null-space and the range of the stiffness operator K, i.e. :

$$N(K) = \{u \in U : K u = 0\}$$

$$R(K) = \{v \in U : \exists u \in U , K u = v\}$$

By a standard result of linear algebra it is: $N(K) = R^{\perp}(K)$, where the symbol $\perp$ denotes the orthogonal complement.

mal orthonormal set of directions with vanishing stiffnesses.

We shall use the nomenclature :

$a \in N(K)$	critical direction	(buckling mode)
$N(K)$	critical subspace	

In the case iii) b we may then give the following explicit expression for the solutions of the incremental equilibrium problem :

$$\dot{u} = \dot{u}_0 + \sum_{i=1}^{m} \dot{\rho}_i a_i \qquad ; \qquad \dot{u}_0 = \sum_{i=1}^{n-m} \frac{1}{\tau_i} (\dot{f} \cdot d_i)\, d_i \tag{4.3}$$

where : $\rho_i(t) = u(t) \cdot a_i$ is the orthogonal projection of the displacement vector $u(t)$ along the critical direction a_i.

Now, observing that the condition $\dot{f} \in N^{\perp}(K)$ in iii) b may equivalently be written :

$$\dot{f} \cdot a = 0 \qquad \forall\, a \in N(K)$$

we have the following mechanical interpretation of the conditions i), ii) and iii) above :

i) There is at least a vanishing principal stiffness

ii) there is at least an infinitesimally adjacent equilibrium configuration under the same load.

iii) a - If the load increment works for a buckling mode there is no solution to the incremental equilibrium problem. As a consequence there will be no incremental solution to the actual nonlinear equilibrium problem and a dynamical analysis must be performed (snapping).

iii) b - If the load increment doesn't work for any buckling mode, the incremental equilibrium problem will admit an infinity of solutions. They can be obtained by adding to a special solution any linear combination of a finite number, m, of linearly independent buckling modes.

By performing some further analysis we may show, however, that only a finite number of solutions of the incremental equilibrium problem are admissible as incremental solutions of the actual nonlinear equilibrium problem.

Indeed the second order perturbation of the non linear equilibrium equation at a critical state :

$$K \ddot{u} = \ddot{f} - d K \dot{u}^2$$

will admit solutions if and only if :

$$\ddot{f} - dK u^2 \in R(K) = N^{\perp}(K)$$

which, taking into account the expression (4.3) of $\dot{u}$, may be written :

$$\dot{\rho}_i \dot{\rho}_J \, d K \; a_i a_J a_k + 2 \dot{\rho}_i \; d K \dot{u}_o a_i a_k + d K \dot{u}_o^2 a_k = \ddot{f} \cdot a_k \tag{4.4}$$

$i, J, k = 1, \ldots, m$

By the Bezout's theorem, this set of m quadratic equations in the m unknowns $\dot{\rho}_i$ $i = 1, \ldots, m$ will admit at most 2^m distinct solutions.

Each solution of (4.4) will be a set of m incremental critical parameters $\dot{\rho}_i$ which, through (4.3), represents an admissible incremental solution of the nonli - near equilibrium problem.

We may then state the following result :

If the incremental equilibrium problem admits an infinity of solutions, only a finite number of them, no greater than 2^m, where m is the maximum number of li - nearly independent buckling modes, will be admissible as incremental solutions of the actual nonlinear equilibrium problem.

In the simplest case where $m = 1$, a bifurcation of the equilibrium path will occour, as a rule.

The investigation of the stability of the equilibrium at a critical state is rather involved in the general case.

We shall discuss the simplest case of a primary simple critical state where only one of the principal stiffnesses vanishes and all the others are positive :

$$\tau_1 = 0 \quad , \quad 0 < \tau_2 \leq \ldots \leq \tau_n$$

To simplify the notations we set

$$\tau_1 = \tau \qquad \text{critical stiffness}$$

$$e_i = e \qquad \text{critical direction}$$

with $\tau(u_o\ \lambda_o) = 0$. This situation occurs in the buckling of elastic structures which exhibit a simple buckling mode, under the critical load.

The stability criterion, in this simplest case, may be formulated as follows [9] :

STABILITY : A primary simple critical equilibrium state (u_o, λ_o) is stable if and only if the critical stiffness τ has a strict minimum $(\tau(u_o,\lambda_o) = 0)$ at the critical state, along the "critical path" defined by :

$$\dot{u}_c(t) = e(u(t)) \qquad \text{with} \quad u_c(0) = u_o\ , \quad e(u_o) = e\ .$$

or explicitly :

$$u_c(t) = u_o + \int_o^t e(u(\theta))\, d\theta \qquad (4.5)$$

The path (4.5) may be equivalently characterized as the path emerging from the critical configuration and whose tangent at any point coincides with the principal direction of lowest stiffness.

We may give the following mechanical interpretation of the stability criterion formulated above :

At a primary simple critical state the system may be infinitesimally displaced along the critical direction without any effort.

Then, according to whether, moving the system further along the path of minimal steffness, the initially vanishing critical stiffness tends to become positive or negative, the equilibrium at the critical state will respectively be stable or unstable.

At a primary simple critical state the stiffness operator is positive and hence the incremental equilibrium problem is equivalent to a minimum problem :

Minimum principle : $$\min \left\{ \frac{1}{2} K \dot{u} \cdot \dot{u} - \dot{f} \cdot \dot{u} \right\}$$

which is solvable if and only if $\dot{f} \in R(K) = N^{\perp}(K)$.

In this case the minimal value is given by :

$$- \frac{1}{2} K \dot{u}_o \cdot \dot{u}_o = - \frac{1}{2} \dot{f} \cdot \dot{u}_o \qquad \text{with} \qquad \dot{u}_o \in L$$

The intimate connection existing between the stability of the equilibrium at a primary simple critical state and the kind of postcritical behaviour of the system, when the load parameter λ is varied, will be investigated in detail in the Part II of this paper.

REFERENCES

[1] Hutchinson, J.W. & Koiter, W.T. — Postbuckling Theory
Appl. Mech. Rev., 23, 1353 (1970)

[2] Thompson, J.M.T. & Hunt, G.W. — *A General Theory of Elastic Stability*
J. Wiley & Sons, 1973.

[3] Koiter, W.T. — On The Stability of Elastic Equilibrium
Dissertation, Delft, Holland, 1945.

[4] Poincaré, H. — Sur l'Equilibre d'Une Masse Fluide Animée d'Un Mouvement de Rotation
Acta Math., 7, 259 (1885).

[5] Lyttleton, R.A. — The Stability of Rotating Liquid Masses
Cambridge University Press, 1953.

[6] Thompson, J.M.T. — Basic Principles in the General Theory of Elastic Stability
J. Mech. Phys. Solids, 11, 13 (1963.

[7] Romano, G. — Instability by the Energy Method *to be published*

[8] Koiter, W.T. — On the Instability of Equilibrium in the Absence of a Minimum of the Potential Energy.
Proc. K. ned. Akad. Wet., B, 66, 265 (1963)

[9] Romano, G. — On the Minimum of the Potential Energy Functional at a Critical Point.
Meccanica, 4 (1974).

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