DEPARTMENT OF STRUCTURE

POSTCRITICAL BEHAVIOUR OF ELASTIC STRUCTURES

Part II: Postcritical analysis and Imperfection Sensitivity

by

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GIOVANNI ROMANO

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1. - Introduction

A detailed discussion on the stability of an equilibrium state of an elastic structure under conservative loading and on the uniqueness of the incremental solution of the corresponding nonlinear equilibrium problem, has been presented in the first part of this paper [1].

A new method of investigation of the postcritical behaviour is presented in this second part: a perturbative analysis of the nonlinear equilibrium equation and of the associated critical eigenproblem yield a systematic information on the subsequent derivatives of the postcritical equilibrium path and on its statibility.

The geometric aspects of the model problem are especially emphasized by the use of a compact notation.

The followed approach is thus formally simple and open to a direct mechanical insight in the single steps of the analysis.

Both stationary (limit) and bifurcation critical states are investigated and the condition under which one or the other does occur is formulated in simple mechanical terms.

These features of the proposed analysis are comparative merits with respect to previous treatments of the subject [2 - 9].

The effect of a slight imperfection on an initially perfect system, which reaches a bifurcation critical state is also investigated.

The well-known imperfection sensitivity of structural systems, whose primary critical state is an unstable bifurcation state, is discussed in detail.

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2. PRIMARY SIMPLE CRITICAL STATE

Let us consider a discrete model of an elastic structure under a conservative load defined by the potential $\lambda L(u)$, where λ is a load multiplier. The behaviour of the structure is then completely described by the potential energy functional :

$$E(u,\lambda) = W(u) + \lambda L(u)$$

with $W(u)$ internal elastic energy.

At a primary simple critical state (u_c, λ_c) the stiffness operator :

$$K(u_c, \lambda_c) = d_u^2 E(u_c, \lambda_c)$$

has the lowest, simple eigenvalue τ vanishing and all the others positive.

The postcritical behaviour may then be effectively investigated by means of a perturbative analysis of the nonlinear equilibrium equation and of the associated critical eigenproblem.

Indeed any equilibrium path $u(t)$, $\lambda(t)$ emerging from the critical state (u_c, λ_c) will identically satisfy the nonlinear equilibrium equation :

$$A\left(u(t), \lambda(t)\right) \equiv 0 \tag{2.1}$$

where $A = d_u E$ and $u(0) = u_c$, $\lambda(0) = \lambda_c$, and the nonlinear critical eigenproblem :

$$K\left(u(t), \lambda(t)\right) e(t) = \tau(t)\, e(t) \tag{2.2}$$

with $e(0) = e$ and $\tau(0) = 0$.

By performing a perturbative analysis of the nonlinear equations (2.1) and (2.2) we may investigate respectively the behaviour of the equilibrium path $u(t)$, $\lambda(t)$ about the critical state (u_c, λ_c) and the stability of the corresponding states.

We shall need the following explicit expansions :

$$K\dot{u} = \dot{\lambda}\, p \tag{2.3}$$

$$K\ddot u = \ddot\lambda\, p - (2\dot\lambda\, K_L\,\dot u + d\,K\,\dot u^2) \tag{2.4}$$

$$K\dddot u + \dddot\lambda\, p - [\,3\ddot\lambda\, K_L\dot u + 3\,\dot\lambda\,(K_L\ddot u + d\,K_L\,\dot u^2) + 3\,d\,K\,\dot u\,\ddot u + d^2\,K\,\dot u^3] \tag{2.5}$$

and

$$K\,e = \tau\,e = 0 \qquad\qquad (\tau = 0) \tag{2.6}$$

$$K\,\dot e = \dot\tau\,e - (\dot\lambda\,K_L\,e + dK\,e\,\dot u) \tag{2.7}$$

$$K\ddot e = \ddot\tau\,e - 2\dot\tau\,\dot e - [\ddot\lambda\,K_L\,e + 2\dot\lambda\,(K_L\,\dot e + d\,K_L\,e\,\dot u) + \\ + 2\,dK\,\dot e\,\ddot u + d\,K\,e\,\ddot u + d^2K\,e\,\dot u^2] \tag{2.8}$$

where the superimposed dot denotes differentiation with respect to a suitable evolu̲tion parameter. All the functions above are intended to be evaluated at the pri - mary simple critical state (u_c, λ_c).

Since K is singular, the linear equations (2.3)-(2.8) will admit solutions if and only if the second members are orthogonal to the critical direction e (compatibility condition).

The solution set will then be a one-dimensional linear variety parallel to the critical direction e.

For the incremental equilibrium equation (2.3) the compatibility condition is given by :

$$\dot\lambda\;p \cdot e = 0 \tag{2.9}$$

and the solution set is the ray :

$$\dot u = \dot\lambda\,u_p + \dot\rho\,e \tag{2.10}$$

where :

$$u_p = \sum_{i=2}^{n} \frac{1}{\tau_i}\,(p\cdot e_i)\;e_i$$

is the unique solution of the linear equation :

$$K\,u = p - (p\cdot e)\,e \qquad \text{under the orthogonality condition} \quad u\cdot e = 0\,.$$

and $\rho(t) = u(t)\cdot e$ is the orthogonal projection of the displacement vector along the critical direction.

Taking into account the expression (2.10) of $\dot u$, the equations (2.4) and (2.7) become :

$$K\ddot{u} = \ddot{\lambda} p - [\dot{\lambda}(dK u_p^2 + 2K_L u_p) + 2\dot{\lambda}\dot{\rho}(dK u_p e + K_L e) + \dot{\rho}^2 \, dK e^2] \tag{2.11}$$

$$K\dot{e} = \dot{\tau} e - [\dot{\lambda}(dK u_p e + K_L e) + \dot{\rho} \, dK e^2] \tag{2.12}$$

and the corresponding compatibility conditions may be written :

$$\ddot{\lambda} \, p \cdot e = \dot{\lambda}^2 h + 2\dot{\lambda}\dot{\rho} \, \tau_p + \dot{\rho}^2 \tau_e \tag{2.13}$$

$$\dot{\tau} = \dot{\lambda} \tau_p + \dot{\rho} \, \tau_e \tag{2.14}$$

where :

$$h = dK u_p^2 e + 2 K_L u_p e \tag{2.15}$$

$$\tau_p = dK u_p e^2 + K_L e^2 \tag{2.16}$$

$$\tau_e = dK e^3 \tag{2.17}$$

In the analysis of the postcritical behaviour we shall distinguish two situations :

a) $p \cdot e \neq 0$ stationary state

b) $p \cdot e = 0$ bifurcation state

according to whether the external load works or doesn't work for the buckling mode .

3. POSTCRITICAL BEHAVIOUR

Let us consider the expansions :

$$\lambda(t) = \lambda_c + \dot{\lambda} t + \frac{1}{2} \ddot{\lambda} t^2 + \ldots \tag{3.1}$$

$$\rho(t) = \dot{\rho} \, t + \frac{1}{2} \ddot{\rho} \, t^2 + \ldots \quad \text{(assuming } u_c = 0\text{)} \tag{3.2}$$

$$\tau(t) = \dot{\tau} \, t + \frac{1}{2} \ddot{\tau} \, t^2 + \ldots \quad (\tau(0) = 0) \tag{3.3}$$

of the load multiplier $\lambda(t)$, the critical displacement $\rho(t) = u(t) \cdot e$ and the critical stiffness $\tau(t)$.

a) *Stationary state* : $p \cdot e \neq 0$

From the compatibility condition (2.9) we have : $$\dot{\lambda} = 0 \tag{3.4}$$

and from (2.10) : $$\dot{u} = \dot{\rho}\, e \tag{3.5}$$

Further on, from the compatibility conditions (2.13) and (2.14) we get :

$$\ddot{\lambda}\,(p \cdot e) = \dot{\rho}^2\, \tau_e \tag{3.6}$$

$$\dot{\tau} = \dot{\rho}\, \tau_e \tag{3.7}$$

Since $\dot{\lambda} = 0$, from (3.1)-(3.3) we obtain the approximate expressions :

$$t^2 \ddot{\lambda} \cong 2\,(\lambda - \lambda_c)$$

$$t\, \dot{\rho} \cong \rho$$

$$t\, \dot{\tau} \cong \tau$$

and hence by (3.6) and (3.7) we have the asymptotic relations :

$$\lambda - \lambda_c = \frac{1}{2}\, \frac{\tau_e}{p \cdot e}\, \rho^2 \tag{3.8}$$

$$\tau = \tau_e\, \rho \tag{3.9}$$

which are sketched in figg. 1 and 2, respectively in the case where $\tau_e < 0$ and $\tau_e > 0$, assuming that the critical vector e is oriented so that $p \cdot e > 0$, i.e there is a positive work of the external load p for the buckling mode e.

As shown in figg. 1 and 2, the equilibrium states corresponding to the pairs (λ, ρ) are stable or unstable depending on whether the associated minimal stiffness, τ, is positive or negative.

According to the stability criterion (sect. 4 of part I of this paper) the critical state itself is unstable in the case of fig 1 and stable in the case of fig. 2.

If $\tau_e = 0$ from (2.13), (2.14) and (3.4) we have that :

$$\dot{\tau} = 0 \tag{3.10}$$

$$\dot{\lambda} = \ddot{\lambda} = 0 \tag{3.11}$$

and the perturbations (2.4) and (2.7) become :

$$K \ddot{u} = - \dot{\rho}^2 \, dK \, e^2 \tag{3.12}$$

$$K \dot{e} = - \dot{\rho} \, dK \, e^2 \tag{3.13}$$

whence we get :

$$\ddot{u} = \dot{\rho}^2 \eta + \ddot{\rho} \, e \tag{3.14}$$

$$\dot{e} = \dot{\rho} \, \eta \qquad (\dot{e} \cdot e = 0 \quad \text{since} \quad \| e(t) \| = 1) \tag{3.15}$$

where η is the unique solution of the linear problem:

$$K \eta = - dK \, e^2$$

under the orthogonality condition $\eta \cdot e = 0$

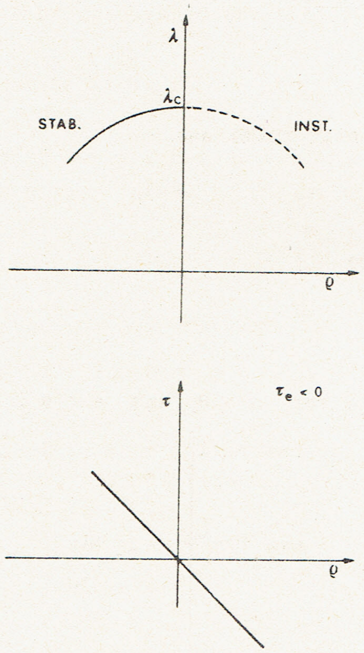


fig. 1

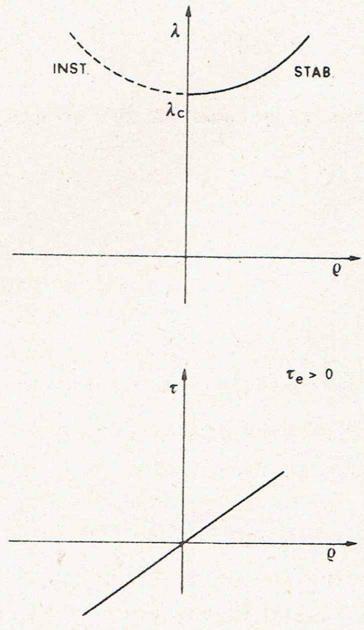


fig. 2

(3.16)

Substituting the expressions (3.14) and (3.15) into (2.5) and (2.8) we get :

$$K \dddot{u} = \dddot{\lambda} \, p - \dot{\rho}^3 \, (d^2K \, e^3 + 3 \, dK \, e \, \eta) \tag{3.17}$$

$$K \ddot{e} = \ddot{\tau} \, e - \dot{\rho}^2 \, (d^2K \, e^3 + 3 \, dK \, e \, \eta) \tag{3.18}$$

The corresponding compatibility conditions yield :

$$\dddot{\lambda}\,(p \cdot e) = \dot{\rho}^3 a \tag{3.19}$$

$$\ddot{\tau} = \dot{\rho}^2 a \tag{3.20}$$

where we have set :

$$a = d^2 K e^4 + 3\, d K e^2 \eta \tag{3.21}$$

Note that, by (3.16) we have also

$$a = d^2 K e^4 - 3 K \eta^2$$

From (3.1)-(3.3) we have the approximate expressions :

$$t^3 \dddot{\lambda} \cong 6(\lambda - \lambda_c)$$

$$t \dot{\rho} \cong \rho$$

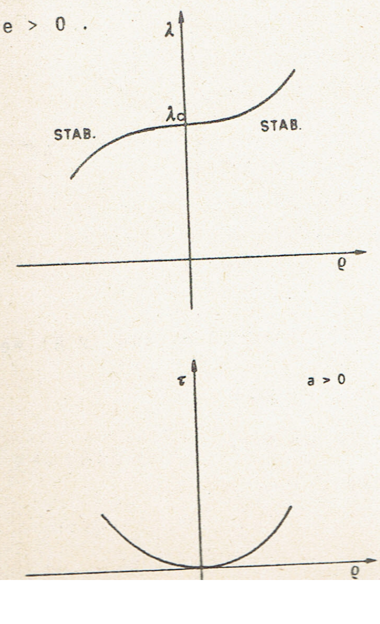
$$t^2 \ddot{\tau} \cong 2\tau$$

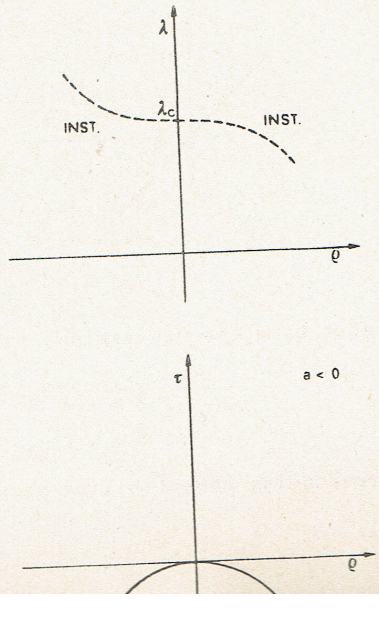
and hence by (3.19) and (3.20) we get the asymptotic relations :

$$\lambda - \lambda_c = \frac{a}{6(p \cdot e)} \rho^3 \tag{3.22}$$

$$\tau = \frac{a}{2} \rho^2 \tag{3.23}$$

which are sketched in figg. 3 and 4, respectively for $a > 0$ and $a < 0$, assuming that $p \cdot e > 0$.





According to the stability criterior the critical state itself is stable if $a > 0$ and unstable if $a < 0$.

If $a = 0$ the analysis must be continued further, by the same procedure.

REMARK

The situations more relevant in the applications are those sketched in fig 1 and 3.

In the case of fig 1 a limit value of the load parameter is reached in corrispondence of the critical state. If the load is increased the system will jump dynamically to a stable equilibrium state (snapping).

In the case of fig 2 the critical state is isolated since all the equilibrium states in a neighbourhood are stable.

b) *Bifurcation state* : $p \cdot e = 0$

In this case the compatibility condition (2.9) is met for every $\dot\lambda$ and the condition (2.13) gives :

$$\dot\lambda^2 h + 2\dot\lambda\dot\rho\tau_p + \dot\rho^2\tau_e = 0 \tag{3.24}$$

Assuming that the discriminant :

$$D = \tau_p^2 - h\tau_e \tag{3.25}$$

of the quadratic form (3.24) is positive(o), we have the solutions :

$$\dot\rho\,\tau_e = \dot\lambda\,(-\tau_p \pm \sqrt{D}) \tag{3.26}$$

or equivalently :

$$\dot\lambda\, h = \dot\rho\,(-\tau_p \pm \sqrt{D}) \tag{3.27}$$

and hence, from (2.14), we get :

$$\dot\tau = \pm\dot\lambda\sqrt{D}$$

From the expansions (3.1)-(3.3) we have then the asymptotic relations :

(o) The case $D \leq 0$ is not relevant in the applications and will not be dealt with here.

$$(\lambda - \lambda_c)(-\tau_p \pm \sqrt{D}) = \rho\,\tau_e \tag{3.28}$$

$$\tau = \pm\,(\lambda - \lambda_c)\sqrt{D} \tag{3.29}$$

which are sketched in fig 5, assuming $\tau_e \neq 0$.

The so called "phenomenon of exchange of stabilities", first pointed out by Poincaré, is apparent.

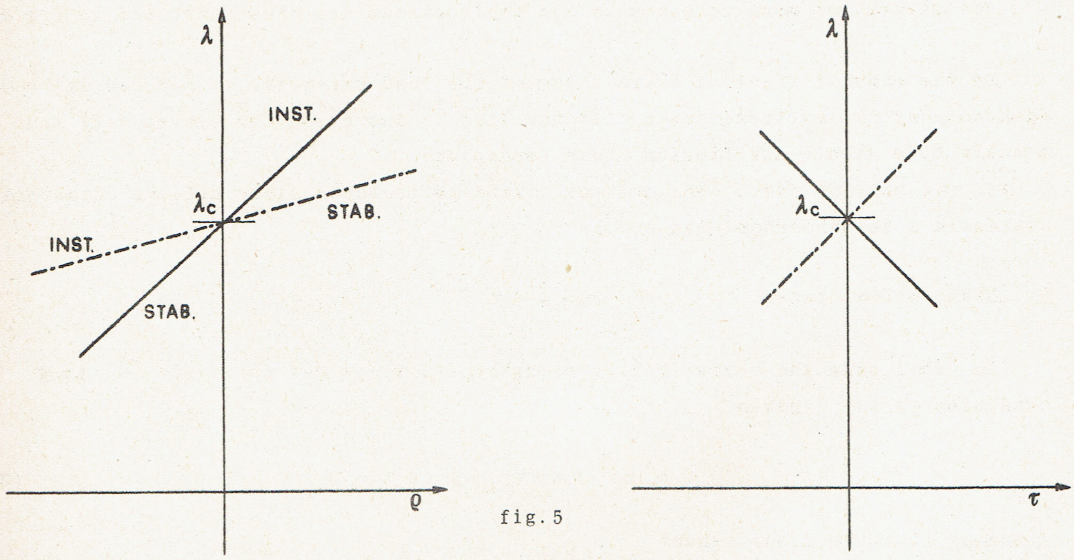


fig. 5

The critical state itself is unstable and will be called an exchanging bifurcation state.

If $\tau_e = 0$, (3.24) becomes :

$$\dot{\lambda}^2 h + 2\dot{\lambda}\dot{\rho}\,\tau_p = 0 \tag{3.30}$$

which has the solutions :

$$\dot{\lambda} = 0 \tag{3.31}$$

$$\dot{\lambda}\,h = -\dot{\rho}\,2\,\tau_p \tag{3.32}$$

and from (2.14) we have :

$$\dot{\tau} = \dot{\lambda}\,\tau_p \tag{3.33}$$

From the solution (3.32) and from (3.33) we get the asymptotic relations :

$$(\lambda - \lambda_c)\; h = - \rho \, 2 \tau_p \tag{3.34}$$

$$\tau = (\lambda - \lambda_c)\; \tau_p \tag{3.35}$$

In corrispondence of the solution (3.31) we have :

$$\dot{\tau} = 0$$

$$\dot{u} = \dot{\rho}\; e$$

and the perturbations (2.4) and (2.7) become :

$$K \ddot{u} = \ddot{\lambda}\; p - \rho^2\, d K e^2 \tag{3.36}$$

$$K \dot{e} = - \dot{\rho}\; d K e^2 \tag{3.37}$$

whence we get :

$$\ddot{u} = \ddot{\lambda}\; u_p + \dot{\rho}^2 \eta + \ddot{\rho}\; e \quad \text{with} \quad u_p \cdot e = \eta \cdot e = 0 \tag{3.38}$$

$$\dot{e} = \dot{\rho}\; \eta \qquad\qquad (\dot{e} \cdot e = 0) \tag{3.39}$$

Substituting in (2.5) and (2.7) we have :

$$K \dddot{u} = \dddot{\lambda}\; p - 3\, \ddot{\lambda}\; \dot{\rho}\; (d K u_p e + K_L e) - \dot{\rho}^3 (d^2 K e^3 + 3\, d K e \eta) \tag{3.40}$$

$$K \ddot{e} = \ddot{\tau}\; e - \ddot{\lambda}\; (d K\; u_p e + K_L e) - \rho^2\; (d^2 K e^3 + 3\, d K e \eta) \tag{3.41}$$

The corresponding compatibility conditions yield :

$$3\, \tau_p\, \ddot{\lambda} = - \dot{\rho}^2\, a \tag{3.42}$$

$$\ddot{\tau} = \ddot{\lambda}\, \tau_p + \dot{\rho}^2\, a \tag{3.43}$$

Hence, by means of the approximate expressions :

$$\ddot{\lambda}\; t^2 \cong 2\, (\lambda - \lambda_c)$$

$$\ddot{\tau}\; t^2 \cong 2\, \tau$$

$$\dot{\rho}\; t \cong \rho$$

We get the asymptotic relations :

$$\lambda - \lambda_c = - \frac{a}{6\tau_p} \rho^2 \tag{3.44}$$

$$\tau = (\lambda - \lambda_c) \, \tau_p + \frac{a}{2} \rho^2 \tag{3.45}$$

By substituting (3.44) into (3.45) we get further :

$$\tau = \frac{a}{3} \rho^2 \tag{3.46}$$

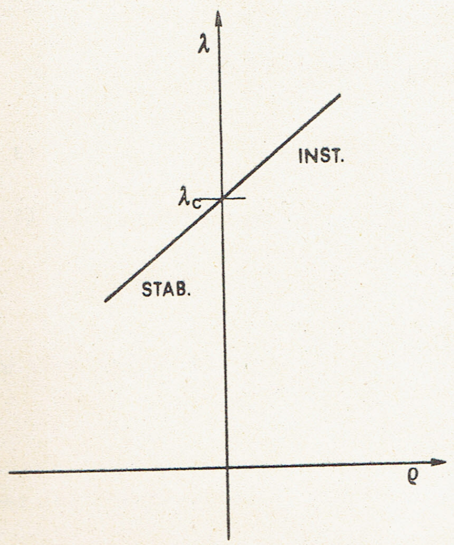
$$\tau = (-2\,\tau_p)(\lambda - \lambda_c) \tag{3.47}$$

which yield respectively the dependence of the minimal stiffness τ upon the critical displacement ρ and the load parameter λ .

Moreover, by setting $\tau = 0$ into (3.45), we obtain the equation of the "stability boundary" in the plane λ , ρ :

$$\lambda - \lambda_c = - \frac{a}{2\,\tau_p} \rho^2 \tag{3.48}$$

The situation is sketched in the figg. 6, 7 and 8, where we have taken $\tau_p < 0$, which corresponds to the usual situation in the structural applications.
The asymptotic relations (3.34) and (3.35) are sketched in fig. 6.



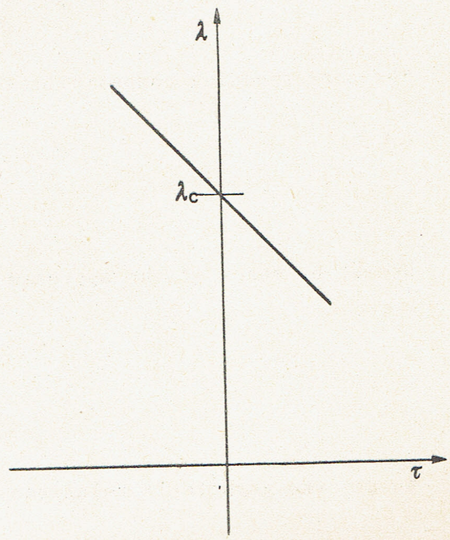


fig. 6

and the asymptotic relations (3.44), (3.46), (3.47) and the stability boundary (dashed curve) are sketched in figg 7 and 8 respectively in the case $a > 0$ and $a<0$.

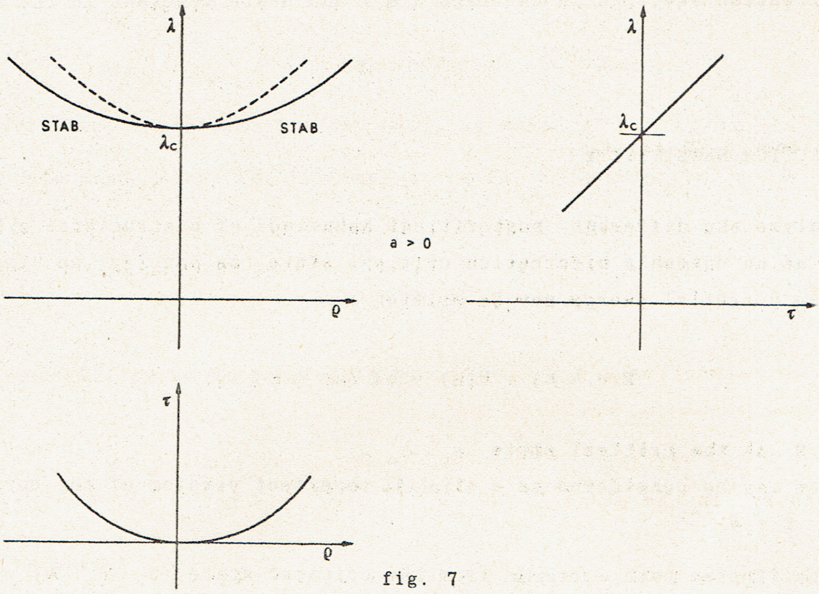


fig. 7

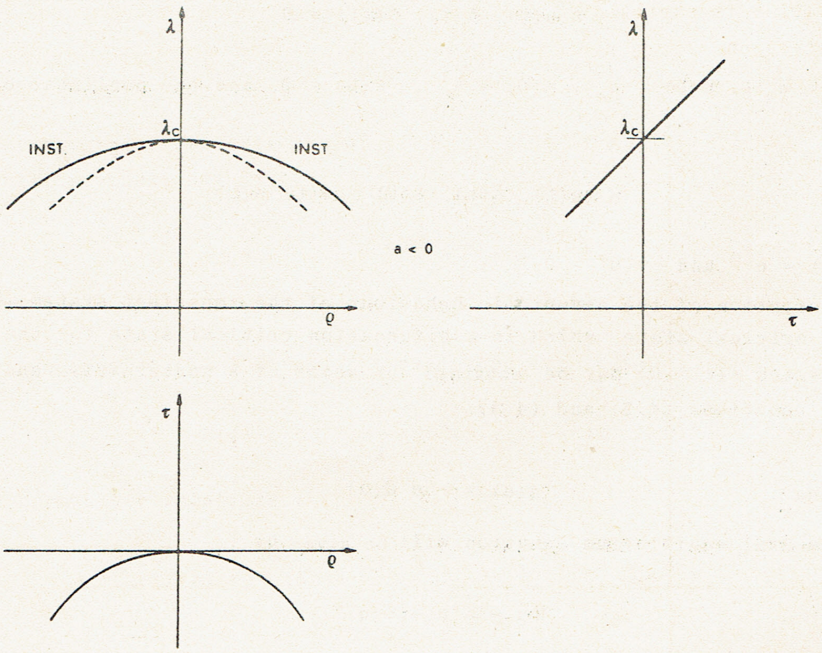


fig. 8

The critical state is stable if $a > 0$ and unstable if $a < 0$.

REMARK

All the situation sketched in the figg 5,6,7 and 8 are relevant in the applications.

4. IMPERFECTION SENSITIVITY

To analyze the different postcritical behaviour of a structural system at a stable or at an unstable bifurcation critical state, we consider an "imperfect" system, whose potential energy may be written :

$$E(u,\lambda,\varepsilon) = W(u) + \lambda L(u) + \varepsilon Q(u) \tag{4.1}$$

with $\varepsilon = 0$ at the critical state u_c, λ_c.
This system may be considered as a slightly imperfect version of the corresponding system with $\varepsilon = 0$.

Any equilibrium path emerging from the critical state $(u_c, \lambda_c, 0)$ will satisfy identically the nonlinear equilibrium equation :

$$A\big(u(t),\ \lambda(t),\ \varepsilon(t)\big) \equiv 0 \tag{4.2}$$

where $A = d_u E$, $u(0) = u_c$, $\lambda(0) = \lambda_c$, $\varepsilon(0) = 0$, and the nonlinear critical eigenproblem :

$$K\big(u(t),\ \lambda(t),\ \varepsilon(t)\big) = e(t)\ e(t) \tag{4.3}$$

with $e(0) = e$ and $\tau(0) = 0$.

A description of the asymptotic behaviour of the imperfect system in a neighbourhood of a critical state, which is a bifurcation critical state for the corresponding perfect system $(\varepsilon = 0)$ may be obtained by means of a perturbative analysis of the nonlinear equations (4.2) and (4.3).

Setting :
$$q(u) = -\ d\ Q(u) \tag{4.4}$$

the incremental equilibrium equation will be given by :

$$K\dot{u} = \dot{\lambda}\ p + \dot{\varepsilon}\ q \tag{4.5}$$

The introduced imperfection is then equivalent to an additional conservative load, with the imperfection parameter ε as the load multiplier.

The qualifying assumption on the imperfection is that :

$$q \cdot e \neq 0 \tag{4.6}$$

i.e. the equivalent load works for the buckling mode.

On the other hand, since the critical state is, by assumption, a bifurcation state, we have :

$$p \cdot e = 0 \tag{4.7}$$

Hence the compatibility condition for (4.5) :

$$\dot{\lambda}(p \cdot e) + \dot{\varepsilon}(q \cdot e) = 0 \tag{4.8}$$

yields :

$$\dot{\varepsilon} = 0$$

Accordingly we have :

$$\dot{u} = \dot{\lambda}\, u_p + \dot{\rho}\, e \tag{4.9}$$

The second order perturbation of (4.2) will then be :

$$K\ddot{u} = \ddot{\lambda}\, p + \ddot{\varepsilon}\, q - (2\dot{\lambda} K_L \dot{u} + d K \dot{u}^2) \tag{4.10}$$

while the corresponding perturbation of (4.3) is exactly (2.7).

The compatibility conditions give :

$$\ddot{\varepsilon}\,(q \cdot e) = \dot{\lambda}^2 h + 2\dot{\lambda}\dot{\rho}\,\tau_p + \dot{\rho}^2\,\tau_e \tag{4.11}$$

$$\dot{\tau} = \dot{\lambda}\,\tau_p + \dot{\rho}\,\tau_e \tag{4.12}$$

From the expansions (3.1)-(3.3) and the analogous for $\varepsilon(t)$:

$$\varepsilon(t) = \dot{\varepsilon}\, t + \frac{1}{2}\ddot{\varepsilon}\, t^2 + \ldots \qquad (\varepsilon(0) = 0) \tag{4.13}$$

we get the asymptotic relations :

$$2\,(q \cdot e)\,\varepsilon = (\lambda - \lambda_c)^2 h + 2\,(\lambda - \lambda_c)\,\rho\,\tau_p + \rho^2\,\tau_e \tag{4.14}$$

$$\tau = (\lambda - \lambda_c)\,\tau_p + \rho\,\tau_e \tag{4.15}$$

The relation (4.14), under the assumption $D = \tau_p^2 - h\tau > 0$, represents an hyperbola in the plane λ, ρ which degenerates in the asymptotes when $\varepsilon = 0$ (perfect system).

The points (λ_M, ρ_M) corresponding to extremal values of λ are characterized by the condition :

$$(\lambda - \lambda_c)\, \tau_p + \rho\, \tau_e = 0 \tag{4.16}$$

which, by (4.15), yields also the stability boundary .

Substituting (4.16) into (4.14) we obtain the relation between λ_M and ε :

$$-2\,(q \cdot e)\, \tau_e\, \varepsilon = (\lambda_M - \lambda_c)^2\, D \tag{4.17}$$

The situation is sketched in fig.9.

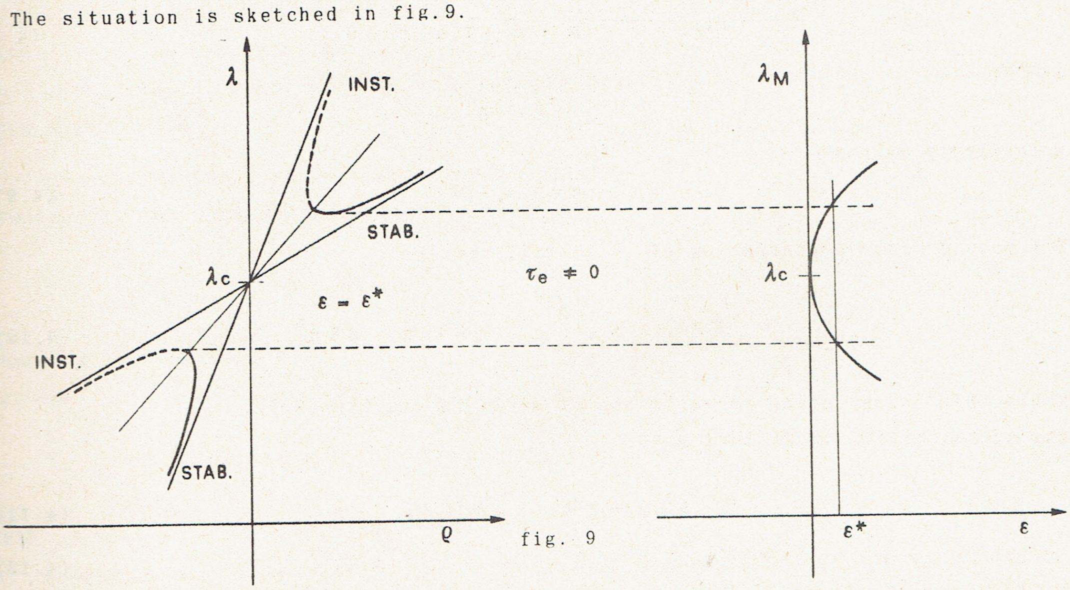


fig. 9

If $\tau_e = 0$ let us look for the asymptotic expression of the stability boundary : $\tau(t) = 0$.

Setting $\dot{\tau} = 0$ and $\tau_e = 0$, from (4.12) we get : $\dot{\lambda} = 0$ and hence from (4.11) : $\ddot{\varepsilon} = 0$.

Moreover from (4.9) we have : $\dot{u} = \dot{\rho}\, e$

The second order perturbation of the nonlinear equilibrium equation will then be:

$$K\ddot{u} = \ddot{\lambda}\, p - \dot{\rho}^2\, d K\, e^2$$

which yields :

$$\ddot{u} = \ddot{\lambda}\, u_p + \dot{\rho}^2\, \eta$$

The third order pertubation of the nonlinear equilibrium equation becomes :

$$K \dddot{u} = \dddot{\lambda}\, p + \dddot{\varepsilon}\, q + 3 \ddot{\lambda}\, \dot{\rho}\, (dK\, u_p\, e + K_L\, e) - \dot{\rho}^3 (d^2 K\, e^3 + 3\, dK\, e\, \eta) \qquad (4.18)$$

and the corresponding perturbation of the nonlinear critical eingenproblem is exactly (3.41).

The compatibility condition give :

$$\dddot{\varepsilon}\, (q \cdot e) = 3\, \ddot{\lambda}\, \dot{\rho}\, \tau_p + a\, \dot{p}^3$$

$$\ddot{\tau} = \ddot{\lambda}\, \tau_p + \dot{\rho}^2\, a$$

which yield the asymptotic relations :

$$6\,(q \cdot e)\, \varepsilon = 6\, \tau_p (\lambda - \lambda_c)\, \rho + a\, \rho^3 \qquad (4.19)$$

$$\tau = 2\, (\lambda - \lambda_c)\, \tau_p + \rho^2\, a \qquad (4.20)$$

The equation of the stability boundary will then be :

$$\lambda - \lambda_c = - \frac{a}{2\tau_p}\, \rho^2 \qquad (4.21)$$

Substituting (4.21) into (4.19) we get the dependence of the extremal load multiplier λ_M upon the imperfection parameter ε :

$$\lambda_M - \lambda_c = - \frac{1}{2\,\tau_p}\, a^{1/3}\, [\, 3\,(q \cdot e)\, \varepsilon\,]^{2/3} \qquad (4.22)$$

The situation is sketched in figg 10 and 11 respectively for $a > 0$ and $a < 0$.

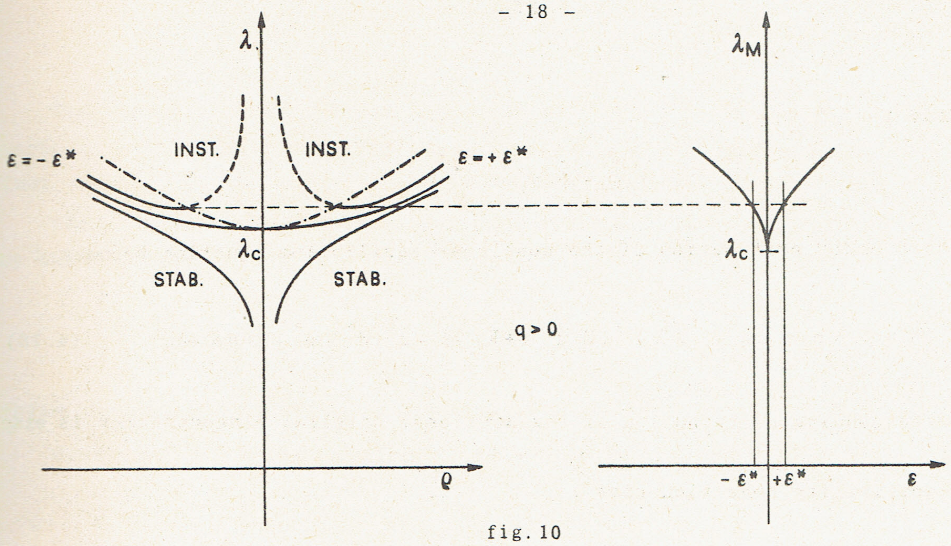


fig. 10

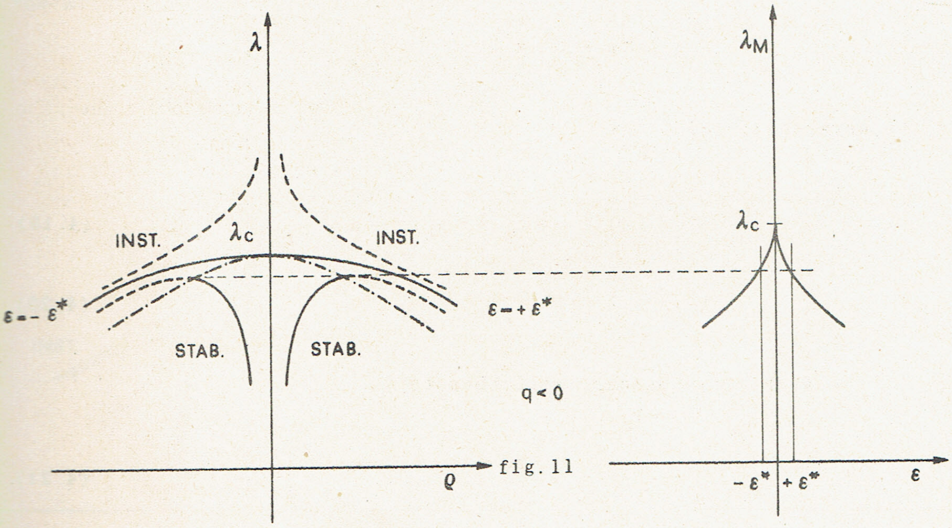


fig. 11

The dashed curve reproduces the stability boundary .

Remark

The basic role played by the postbuckling behaviour of an elastic structure in esthablishing its load carrying capacity, is apparent from the analysis above.

Indeed at a stable bifurcation state (fig.10) the load parameter may be increased above the critical value and the actual carrying capacity of the structure is certainly above the critical load.

At an unstable bifurcation state (fig. 11) we have conversely that the collapse load of the actual, slightly imperfect structure, can be quite below the critical value (imperfection sensitivity).

A yet more drastic reduction of the carrying capacity of the structure with respect to the critical load may occur at an exchanging bifurcation state (fig. 9).

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