A FUNCTIONAL FRAMEWORK
FOR APPLIED CONTINUUM MECHANICS

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We present an abstract formulation of the mechanics of continuous bodies in which the kinematic description is given as basic and the statics is derived by duality. A proper definition of the functional spaces based on finite decompositions of the reference domain allows to include in the theory all the powerful tools usually adopted in the analysis of engineering structures. In this respect general proofs are provided for the Virtual Work Theorem, the abstract CAUCHY's theorem and the theorem of kinematic compatibility. A decomposition formula of special relevance in homogenization theory is derived as a direct application of the previous results.

1 Structural model

Let us consider a bounded domain Ω of an n-dimensional Euclidean space with boundary $\partial\Omega$ and closure $\overline{\Omega} = \Omega \cup \partial\Omega$ and the space $L^2(\Omega;\mathrm{V})$ of square integrable functions in Ω with values in the finite dimensional inner product space V. The LEBESGUE measures in Ω and on $\partial\Omega$ will be denoted by $\mathrm{d}\mu$ and $\mathrm{d}\sigma$.

To provide an abstract definition of a continuous structural model let us preliminarily define some basic mathematical tools.

- The pivot HILBERT spaces (spaces identified with their dual) $\mathcal{H}(\Omega) = L^2(\Omega;\mathrm{W})$ and $H(\Omega) = L^2(\Omega;\mathrm{V})$ of square integrable tensor and vector fields in Ω with inner products $(\!(\cdot , \cdot)\!)$ and $(\cdot , \cdot)$.
- The SOBOLEV space of order $H^m(\Omega;\mathrm{V})$ of vector fields with square integrable distributional derivatives of order up to m (see e.g. [2], [8]):

  $$H^m(\Omega;\mathrm{V}) = \left\{ \mathbf{v} \in L^2(\Omega;\mathrm{V}) \mid D^p\mathbf{v} \in L^2(\Omega;\mathrm{V}), \quad \forall |\,p\,| \le m\,, \quad m \ \text{ integer } \ \ge 0 \right\}$$

  where the derivatives

  $$D^p := \frac{\partial^{|\,p\,|}}{\partial x_1^{p_1} \ldots \partial x_d^{p_d}} \qquad \text{with} \qquad |\,p\,| := \sum_{i=1}^{d} p_i$$

  are taken in the sense of distributions.
- The linear space $\mathbb{D}(\Omega;\mathrm{V}) = \mathrm{C}_o^\infty(\Omega;\mathrm{V})$ of test vector fields which are infinitely differentiable in Ω and have compact support in Ω. The support of a field $\mathbf{u} \in \mathrm{C}(\Omega;\mathrm{V})$, denoted by $\mathrm{supp}(\mathbf{u})$, is the smallest closed subset of Ω outside which the field vanishes.

The space $\mathbb{D}(\Omega;\mathrm{V})$ is endowed with the pseudo-topology induced by the following definition of convergence.

- A sequence $\{\mathbf{u}_n\} \in \mathbb{D}(\Omega;\mathrm{V})$ is said to converge to $\mathbf{u} \in \mathbb{D}(\Omega;\mathrm{V})$ if there exists a compact subset $K \subset \Omega$ such that $\mathrm{supp}(\mathbf{u}_n) \subset K$ and $D^{\mathbf{p}}\mathbf{u}_n \to D^{\mathbf{p}}\mathbf{u}$ uniformly in Ω for any vectorial multiindex $\mathbf{p}$. The vectorial multiindex $\mathbf{p}$ is a list of p scalar multiindices each formed by n non negative integers to denote the order of partial differentiation with respect to the corresponding coordinate. The symbol $|\,\mathbf{p}\,|$ denotes the sum of the integers in $\mathbf{p}$.
- The dual of $\mathbb{D}(\Omega;\mathrm{V})$ is the linear space $\mathbb{D}'(\Omega;\mathrm{V})$ of p-distributions on Ω, formed by the linear functionals which are continuous on $\mathbb{D}(\Omega;\mathrm{V})$. The value of $\mathbb{T} \in \mathbb{D}'(\Omega;\mathrm{V})$ at $\boldsymbol{\varphi} \in \mathbb{D}(\Omega;\mathrm{V})$ is denoted by $\langle\, \mathbb{T}\,,\, \boldsymbol{\varphi}\,\rangle$. The space $\mathbb{D}'(\Omega;\mathrm{V})$ is in turn endowed with the pseudo-topology induced by the following definition of convergence.
- A sequence of distributions $\{\mathbb{T}_n\} \in \mathbb{D}'(\Omega;\mathrm{V})$ is said to converge to a distribution $\mathbb{T} \in \mathbb{D}'(\Omega;\mathrm{V})$ if $\langle\, \mathbb{T}_n\,,\, \boldsymbol{\varphi}\,\rangle \to \langle\, \mathbb{T}\,,\, \boldsymbol{\varphi}\,\rangle$ for any test field $\boldsymbol{\varphi} \in \mathbb{D}(\Omega;\mathrm{V})$.

A continuous structural model is characterized by the definition of a kinematic operator $\mathbf{B}$ which is a linear differential operator of order m. The general form of an mth-order differential operator $\mathbf{B} : H(\Omega) \to \mathbb{D}'(\Omega;\mathrm{W})$ is

$$(\mathbf{B}\,\mathbf{u})(\mathbf{x}) := \sum_{|\,\mathbf{p}\,|\le m} \sum_{i=1}^{n} \mathbf{A}^i_{\mathbf{p}}(\mathbf{x})\; D^{\mathbf{p}}\mathbf{u}_i(\mathbf{x})\,, \quad \mathbf{x} \in \Omega\,,$$

where $\mathbf{A}^i_{\mathbf{p}}(\mathbf{x})$ is a regular field of $n \times n$ matrices in Ω.

Any m-times differentiable kinematic field $\mathbf{u}$ in Ω is transformed by $\mathbf{B}$ into the corresponding strain rate field. The characteristic property of the kinematic operator is that the strain rate field vanishes if and only if the parent kinematic field is rigid. In the linearized theory of structural models, displacement fields are treated as kinematic fields and the corresponding strain rate fields are called linearized strain fields (or for shorthand simply strain fields).

In structural mechanics it is however compelling to consider more general kinematic fields. The motivation is twofold. The first request is a technical one and consists in the need for the complection of the kinematic space, with respect to the mean square norm, to render the nice properties of HILBERT spaces available. The second request is that discontinuous kinematic fields must be allowed to be dealt with in the analysis of structural models. This latter demand stems from the mechanical principle which we refer to as the axiom of reproducibility. The basic idea is the one expressed by the classical principle of sectioning due to EULER and CAUCHY. The axiom of reproducibility states that the kinematic space must include discontinuous fields capable of describing regular relative motions between the elements of an arbitrary finite subdivision of the base domain Ω.

Generalized functions (distributions) are needed since the mathematical modelling leads to analyze vector fields that, being discontinuous, are not differentiable in the classical sense. Here regularity means that the strain rate field associated with a kinematic field must be a distribution representable by a square integrable field in each element of the subdivision.

A continuous structural model is characterized by a distributional differential operator $\mathbb{B} : H(\Omega) \to \mathbb{D}'(\Omega; \mathrm{W})$ which provides the distributional strain rate field $\mathbb{B}\mathbf{v} \in \mathbb{D}'(\Omega; \mathrm{W})$ corresponding to a square integrable kinematic field $\mathbf{v} \in H(\Omega; \mathrm{V})$.

A kinematic field $\mathbf{v} \in H(\Omega)$ is piecewise regular if the corresponding distributional strain rate field $\mathbb{B}\mathbf{v} \in \mathbb{D}'(\Omega; \mathrm{W})$ is piecewise square integrable in Ω. It is then convenient to consider the regular kinematic fields to be defined on a subdivision of Ω into subdomains $\mathcal{P}$ on each of which they belong to $H^m(\mathcal{P}; \mathrm{V})$.

Let us then consider a decompositions $\mathcal{T}(\Omega)$ of Ω into a finite family of non-overlapping subdomains $\Omega_e \subseteq \Omega$ with boundary $\partial\Omega_e$ ($e = 1, \ldots, n$) which realize a covering of $\overline{\Omega}$. The closure $\overline{\Omega}_e = \partial\Omega_e \cup \Omega_e$ is called an element of $\mathcal{T}(\Omega)$ and the following properties are assumed to be fulfilled:

$$\Omega_\alpha \cap \Omega_\beta = \emptyset \quad \text{for} \quad \alpha \neq \beta \qquad \text{and} \quad \bigcup_{e=1}^{n} \overline{\Omega}_e = \overline{\Omega}.$$

A field $\mathbf{v}$ on Ω is said to be piecewise $H^m(\Omega; \mathrm{V})$ if its restriction $\mathbf{v}|_e$ to each element $\overline{\Omega}_e$ of a suitable decomposition $\mathcal{T}(\Omega)$ belong to $H^m(\Omega_e; \mathrm{V})$.

The kinematic space $\mathcal{V}_\Omega \subset H(\Omega)$ of piecewise GREEN-regular kinematic fields on Ω is then defined by assuming that for any $\mathbf{v} \in \mathcal{V}_\Omega$, there exists a decomposition $\mathcal{T}_\mathbf{v}(\Omega)$ such that the distributional strain $(\mathbb{B}\mathbf{v})|_e$, restriction of $\mathbb{B}\mathbf{v} \in \mathbb{D}'(\Omega; \mathrm{W})$ to Ω_e, defined as:

$$(\mathbb{B}\mathbf{v})|_e(\boldsymbol{\phi}) := (\mathbb{B}\mathbf{v})(\boldsymbol{\phi}), \qquad \forall\, \boldsymbol{\phi} \in \mathbb{D}(\Omega; \mathrm{W}) \;:\; \mathrm{supp}\boldsymbol{\phi} \subset \Omega_e,$$

is square integrable over the element Ω_e of $\mathcal{T}_\mathbf{v}(\Omega)$. Then

$$\mathcal{V}_\Omega := \left\{ \mathbf{v} \in H(\Omega) \mid \exists\, \mathcal{T}_\mathbf{v}(\Omega) \;:\; (\mathbb{B}\mathbf{v})|_e \in \mathcal{H}(\Omega_e) \right\}.$$

The regular part of the strain distribution $\mathbb{B}\mathbf{v} \in \mathbb{D}(\Omega; \mathrm{W})$, denoted by $\mathbf{B}\mathbf{v} \in \mathcal{H}(\Omega)$, is defined to be the list of square integrable strain fields $(\mathbb{B}\mathbf{v})|_e \in \mathcal{H}(\Omega_e)$ ($e = 1, \ldots, n$).

The space $\mathcal{V}_\Omega$ is a pre-HILBERT space when endowed with the inner product

$$(\,\mathbf{u}\,,\,\mathbf{v}\,)_{\mathcal{V}_\Omega} := \int_\Omega \mathbf{u} \cdot \mathbf{v}\; d\mu + \int_\Omega \mathbf{B}\mathbf{u} : \mathbf{B}\mathbf{v}\; d\mu = (\,\mathbf{u}\,,\,\mathbf{v}\,) + (\!(\,\mathbf{B}\mathbf{u}\,,\,\mathbf{B}\mathbf{v}\,)\!),$$

and the induced norm

$$\|\,\mathbf{u}\,\|_\mathcal{V} = \left[(\!(\,\mathbf{u}\,,\,\mathbf{u}\,)\!) + (\,\mathbf{B}\mathbf{u}\,,\,\mathbf{B}\mathbf{u}\,)\right]^{1/2}.$$

which is equivalent to the sum of the norms, since

$$\|\,\mathbf{u}\,\|_\mathcal{V} \leq \|\,\mathbf{u}\,\|_{H(\Omega)} + \|\,\mathbf{B}\mathbf{u}\,\|_{\mathcal{H}(\Omega)} \leq \sqrt{2}\,\|\,\mathbf{u}\,\|_\mathcal{V}.$$

The kinematic operator $\mathbf{B} \in BL\left\{\mathcal{V}_\Omega, \mathcal{H}(\Omega)\right\}$ is a bounded linear map from $\mathcal{V}_\Omega$ into $\mathcal{H}(\Omega)$ which provides the regular part $\mathbf{B}\mathbf{v} \in \mathcal{H}(\Omega)$ of the distributional strain $\mathbb{B}\mathbf{v} \in \mathbb{D}'(\Omega; \mathrm{W})$ corresponding to the kinematic field $\mathbf{u} \in \mathcal{V}_\Omega$.

The space $\mathcal{F}_{\Omega}$ of force systems is the topological dual of $\mathcal{V}_{\Omega}$. The kinematic and the equilibrium operators $\mathbf{B} \in BL\left\{\mathcal{V}_{\Omega}, \mathcal{H}(\Omega)\right\}$ and $\mathbf{B}' \in BL\left\{\mathcal{H}(\Omega), \mathcal{F}_{\Omega}\right\}$ are dual counterparts associated with the fundamental bilinear form $\mathbf{b}$ which describes the kinematic properties of the model:

$$\mathbf{b}\,(\mathbf{v}, \boldsymbol{\sigma}) := (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{v}\,)\!) = \langle\, \mathbf{B}'\boldsymbol{\sigma}\,,\, \mathbf{v}\,\rangle\,, \qquad \boldsymbol{\sigma} \in \mathcal{H}(\Omega),\, \mathbf{v} \in \mathcal{V}_{\Omega}\,.$$

The formal adjoint of $\mathbf{B} \in BL\left\{\mathcal{V}_{\Omega}, \mathcal{H}(\Omega)\right\}$ is the distributional differential operator $\mathbb{B}'_o : \mathcal{H}(\Omega) \to \mathbb{D}'(\Omega)$ of order m defined by the identity

$$\langle\, \mathbb{B}'_o\boldsymbol{\sigma}\,,\, \mathbf{v}\,\rangle := (\!(\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{v}\,)\!)\,, \qquad \forall\, \mathbf{v} \in \mathbb{D}(\Omega; \mathrm{V})\,, \quad \forall\, \boldsymbol{\sigma} \in \mathcal{H}(\Omega)\,.$$

The space $\mathcal{S}_{\Omega}$ of piecewise GREEN-regular stress fields on Ω is then defined as the linear space of stress fields $\boldsymbol{\sigma} \in \mathcal{H}(\Omega)$ such that the corresponding body force distribution $\mathbb{B}'_o\boldsymbol{\sigma} \in \mathbb{D}'(\Omega; \mathrm{V})$ is representable by a piecewice square integrable field on Ω:

$$\mathcal{S}_{\Omega} := \left\{\boldsymbol{\sigma} \in \mathcal{H}(\Omega) \mid \exists\, \mathcal{T}_{\boldsymbol{\sigma}}(\Omega) \;:\; (\mathbb{B}'_o\boldsymbol{\sigma})|_e \in H(\Omega_e)\right\}.$$

The regular part of the body force $\mathbb{B}'_o\boldsymbol{\sigma} \in \mathbb{D}'(\Omega; \mathrm{V})$, denoted by $\mathbf{B}'_o\boldsymbol{\sigma} \in H(\Omega)$, is defined to be the list of square integrable fields $(\mathbb{B}'_o\boldsymbol{\sigma})|_e \in H(\Omega_e)$ $(e = 1, \ldots, n)$.

The space $\mathcal{S}_{\Omega}$ is a pre-HILBERT space when endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{S}} := \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau}\; d\mu + \int_{\Omega} \mathbf{B}'_o\boldsymbol{\sigma} \cdot \mathbf{B}'_o\boldsymbol{\tau}\; d\mu\,, \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{S}_{\Omega}\,,$$

and the induced norm

$$\|\, \boldsymbol{\sigma}\,\|_{\mathcal{S}} = \Big[\|\, \boldsymbol{\sigma}\,\|_{\mathcal{H}(\Omega)} + \|\, \mathbf{B}'_o\boldsymbol{\sigma}\,\|_{H(\Omega)}\Big]^{1/2},$$

which is equivalent to the sum of the norms:

$$\|\, \boldsymbol{\sigma}\,\|_{\mathcal{S}} \le \|\, \boldsymbol{\sigma}\,\|_{\mathcal{H}(\Omega)} + \|\, \mathbf{B}'_o\boldsymbol{\sigma}\,\|_{H(\Omega)} \le \sqrt{2}\,\|\, \boldsymbol{\sigma}\,\|_{\mathcal{S}}\,.$$

The body equilibrium operator $\mathbf{B}'_o \in BL\left\{\mathcal{S}_{\Omega}, H(\Omega)\right\}$ is a bounded linear map of the stress fields $\boldsymbol{\sigma} \in \mathcal{S}_{\Omega}$ into the regular part $\mathbf{B}'_o\boldsymbol{\sigma} \in H(\Omega)$ of the distributional body force $\mathbb{B}'_o\boldsymbol{\sigma} \in \mathbb{D}'(\Omega; \mathrm{V})$.

2 Green's formula

In mathematical physics, and in particular in continuum mechanics, a fundamental role is played by the classic GREEN's formula which is the fundamental tool for the formulation of Boundary Values Problems.

Let us consider a kinematic field $\mathbf{v} \in \mathcal{V}_\Omega$ and the stress field $\boldsymbol{\sigma} \in \mathcal{S}_\Omega$ and let $\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega) = \mathcal{T}_{\mathbf{v}}(\Omega) \vee \mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$ be a decomposition finer than $\mathcal{T}_{\mathbf{v}}(\Omega)$ and $\mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$.
The GREEN's formula for the operator $\mathbf{B} \in BL\left\{\mathcal{V}_\Omega, \mathcal{H}(\Omega)\right\}$ can be written

$$(\!(\, \boldsymbol{\sigma}\,, \mathbf{B}\mathbf{v}\,)\!) = (\, \mathbf{B}_o^{'}\boldsymbol{\sigma}\,, \mathbf{v}\,) + \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}\,, \boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle\,, \quad \forall\, \mathbf{v} \in \mathcal{V}_\Omega, \quad \forall\, \boldsymbol{\sigma} \in \mathcal{S}_\Omega,$$

where by definition

$$(\!(\, \boldsymbol{\sigma}\,, \mathbf{B}\mathbf{v}\,)\!) \,:= \int_\Omega \boldsymbol{\sigma} : \mathbf{B}\mathbf{v}\ \mathrm{d}\mu\,, \qquad (\, \mathbf{B}_o^{'}\boldsymbol{\sigma}\,, \mathbf{v}\,) \,:= \int_\Omega \mathbf{B}_o^{'}\boldsymbol{\sigma} \cdot \mathbf{v}\ \mathrm{d}\mu\,,$$

and the duality pairing $\langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}\,, \boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle$ is the extension by continuity of the following sum of boundary integrals over $\partial\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega) = \cup\, \partial\Omega_e,\ e = 1, \ldots, n$:

$$\int_{\partial\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega)} \mathbf{N}\boldsymbol{\sigma} \cdot \boldsymbol{\Gamma}\mathbf{v}\ \mathrm{d}\sigma\,.$$

The operators $\boldsymbol{\Gamma}$ and $\mathbf{N}$ are differential operators of order ranging from 0 to $m-1$ defined by the rule of integration by parts.

3 Bilateral constraints

A basic constraint in mechanics is the requirement of piecewise regularity of kinematic fields. Let $\mathcal{V} = \mathcal{V}(\mathcal{T}(\Omega)) \subseteq \mathcal{V}_\Omega$ and $\mathcal{S} = \mathcal{S}(\mathcal{T}(\Omega)) \subseteq \mathcal{S}_\Omega$ be the closed linear spaces of kinematic and stress fields which are GREEN-regular in correspondence to a given subdivision $\mathcal{T}(\Omega)$. The spaces $\mathcal{V}$ and $\mathcal{S}$ are HILBERT spaces when endowed with the topology inherited by $\mathcal{V}_\Omega$ and $\mathcal{S}_\Omega$. We shall further denote by $\mathcal{F} = \mathcal{F}(\mathcal{T}(\Omega))$ the space of force systems in duality with $\mathcal{V}(\mathcal{T}(\Omega))$.

Once a regularity subdivision $\mathcal{T}(\Omega)$ has beeen fixed, the boundary operators appearing in GREEN's formula can be qualified as bounded linear operators between suitable functional spaces. Let us denote by $\partial\mathcal{V} = \partial\mathcal{V}(\mathcal{T}(\Omega))$ the linear space of boundary fields that are traces of fields in $\mathcal{V}$, so that $\partial\mathcal{V} := \boldsymbol{\Gamma}\mathcal{V}$. The space $\partial\mathcal{V}$ is an HILBERT space when endowed with the topology of the isomorphic quotient space $\mathcal{V}/\,\mathrm{Ker}\,\boldsymbol{\Gamma}$ [8]. The flux boundary operator $\mathbf{N} \in BL\left\{\mathcal{S}, \partial\mathcal{F}\right\}$ takes its values in the dual HILBERT space of boundary forces $\partial\mathcal{F} = \partial\mathcal{F}(\mathcal{T}(\Omega))$. The operator $\mathbf{N}$ yields the boundary tractions $\mathbf{N}\boldsymbol{\sigma} \in \partial\mathcal{F}$ due to the stress fields $\boldsymbol{\sigma} \in \mathcal{S}$. The operator $\boldsymbol{\Gamma}$ yields the boundary traces $\boldsymbol{\Gamma}\mathbf{v} \in \partial\mathcal{V}$ of the displacement fields $\mathbf{v} \in \mathcal{V}$.

The operators $\boldsymbol{\Gamma} \in BL\left\{\mathcal{V}, \partial\mathcal{V}\right\}$ and $\mathbf{N} \in BL\left\{\mathcal{S}, \partial\mathcal{F}\right\}$ are surjective: $\mathrm{Im}\,\boldsymbol{\Gamma} = \partial\mathcal{V}$ and $\mathrm{Im}\,\mathbf{N} = \partial\mathcal{F}$. Moreover $\mathrm{Ker}\,\boldsymbol{\Gamma}$ is dense in H and $\mathrm{Ker}\,\mathbf{N}$ is dense in $\mathcal{H}$ [8].

Since $\mathrm{Im}\,\boldsymbol{\Gamma} = \partial\mathcal{V}$, by the closed range theorem the dual operator $\boldsymbol{\Gamma}' \in BL\left\{\partial\mathcal{F}, \mathcal{F}\right\}$ is injective being $\mathrm{Ker}\,\boldsymbol{\Gamma}' = [\,\mathrm{Im}\,\boldsymbol{\Gamma}]^\perp = \{\mathbf{o}\}$.

Affine constraints are usually considered in mechanics so that admissible kinematic fields belong to a closed linear variety $\mathcal{V}_a = \mathcal{V}_a(\mathcal{T}(\Omega)) \subseteq \mathcal{V}(\mathcal{T}(\Omega))$ defined by $\mathcal{V}_a := \mathbf{w} + \mathcal{L}$ where $\mathbf{w} \in \mathcal{V}(\mathcal{T}(\Omega))$ and $\mathcal{L} = \mathcal{L}(\mathcal{T}(\Omega)) \subset \mathcal{V}(\mathcal{T}(\Omega))$ is the closed linear subspace of conforming kinematisms.

The linear space $\mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}}(\mathcal{T}(\Omega))$ of active forces is the topological dual of the HILBERT space $\mathcal{L} \subset \mathcal{V}$ endowed with the topology inherited by $\mathcal{V}$.

It can be proved that there exists an isometric isomorphism between the space $\mathcal{F}_{\mathcal{L}}$ and the quotient space $\mathcal{F}/\mathcal{L}^{\perp}$ [8].

To derive the main result concerning the existence of a stress field, it is convenient to introduce the following pair of dual operators:

- the conforming kinematic operator $\mathbf{B}_{\mathcal{L}} \in BL\left\{\mathcal{L}, \mathcal{H}\right\}$, defined as the restriction of $\mathbf{B} \in BL\left\{\mathcal{V}, \mathcal{H}\right\}$ to $\mathcal{L}$,
- the conforming equilibrium operator $\mathbf{B}^{'}_{\mathcal{L}} \in BL\left\{\mathcal{S}, \mathcal{F}/\mathcal{L}^{\perp}\right\}$, defined by the position $\mathbf{B}^{'}_{\mathcal{L}}\boldsymbol{\sigma} := \mathbf{B}'\boldsymbol{\sigma} + \mathcal{L}^{\perp}$.

The kernels and the images of these operators are given by

$$\operatorname{Ker}\mathbf{B}_{\mathcal{L}} = \operatorname{Ker}\mathbf{B} \cap \mathcal{L}\,, \quad \operatorname{Ker}\mathbf{B}^{'}_{\mathcal{L}} = (\mathbf{B}')^{-1}\mathcal{L}^{\perp} = (\mathbf{B}\mathcal{L})^{\perp}\,,$$

$$\operatorname{Im}\mathbf{B}_{\mathcal{L}} = \mathbf{B}\mathcal{L}\,, \quad \operatorname{Im}\mathbf{B}^{'}_{\mathcal{L}} = (\operatorname{Im}\mathbf{B}' + \mathcal{L}^{\perp})/\mathcal{L}^{\perp}\,.$$

The mechanical property of firm, bilateral and smooth constraints is modeled by requiring that the constraint reactions must be orthogonal to conforming kinematisms:

$$\mathcal{R} = \mathcal{L}^{\perp} = \left\{\mathbf{r} \in \mathcal{F} \mid \langle\, \mathbf{r}\,,\, \mathbf{v}\,\rangle = 0 \quad \forall\, \mathbf{v} \in \mathcal{L}\right\}.$$

The closed linear subspace $\mathcal{V}_{\mathrm{RIG}} := \operatorname{Ker}\mathbf{B}_{\mathcal{L}} \subset \mathcal{V}$ of conforming rigid kinematisms has a special relevance in structural mechanics since its elements appear as test fields in the equilibrium condition of a system of active forces:

$$\ell \in \mathcal{V}_{\mathrm{RIG}}^{\perp} \iff \langle\, \mathbf{f}\,,\, \mathbf{v}\,\rangle = 0 \qquad \forall\, \mathbf{v} \in \mathcal{V}_{\mathrm{RIG}}\,.$$

The elimination of the rigidity constraint is the central issue of continuum mechanics and is performed by a technique of LAGRANGE multipliers originarily envisaged by GABRIO PIOLA in 1833 [1]. The issue will be discussed in the next sections.

4 Korn's inequality

In continuum mechanics the fundamental theorems concerning the variational formulations of equilibrium and compatibility are founded on the property that, for any closed linear subspace of conforming kinematisms, the corresponding conforming kinematic operator has a closed range and a finite dimensional kernel.

It can be proved [7] that this property is fulfilled if and only if the kinematic operator $\mathbf{B} \in BL\{\mathcal{V}, \mathcal{H}\}$ meets an inequality of KORN's type:

$$\| \mathbf{Bv} \|_{\mathcal{H}} + \| \mathbf{v} \|_{H} \geq \alpha \, \| \mathbf{v} \|_{m} \,, \qquad \forall \, \mathbf{v} \in H^m(\mathcal{T}(\Omega); \mathrm{V}) \,,$$

where $H^m(\mathcal{T}(\Omega))$ is a SOBOLEV space of order m subordinated to the subdivision $\mathcal{T}(\Omega)$.

If KORN's inequality holds, the space $\mathcal{V}(\mathcal{T}(\Omega))$ endowed with the norm

$$\| \mathbf{v} \|_{\mathbf{B}} := \left[\| \mathbf{v} \|_{H}^2 + \| \mathbf{Bv} \|_{\mathcal{H}}^2 \right]^{1/2} ,$$

is isomorphic and isometric to $H^m(\mathcal{T}(\Omega); \mathrm{V})$.

KORN's inequality is equivalent to state that for any conforming subspace $\mathcal{L} \subseteq \mathcal{V}$ the reduced kinematic operator $\mathbf{B}_{\mathcal{L}} \in BL\{\mathcal{L}, \mathcal{H}\}$ fulfils the conditions:

$$\begin{cases} \dim \, \mathrm{Ker}\, \mathbf{B}_{\mathcal{L}} < +\infty \,, \\ \| \mathbf{Bv} \|_{\mathcal{H}} \geq c_{\mathbf{B}} \, \| \mathbf{v} \|_{\mathcal{L}/\mathrm{Ker}\,\mathbf{B}_{\mathcal{L}}} \quad \forall \, \mathbf{v} \in \mathcal{L} \end{cases} \iff \quad \mathrm{Im}\, \mathbf{B}_{\mathcal{L}} \ \text{ closed in } \ \mathcal{H} \,,$$

where $c_{\mathbf{B}}$ is a positive constant [7].

The well-posedness of the structural model requires that for any conforming subspace $\mathcal{L} \subseteq \mathcal{V}$ the fundamental form $\mathbf{b}$ be closed on $\mathcal{S} \times \mathcal{V}$. This property is expressed by the inf-sup condition [6]

$$\inf_{\boldsymbol{\sigma} \in \mathcal{H}} \ \sup_{\mathbf{v} \in \mathcal{L}} \frac{\mathbf{b}\,(\mathbf{v}, \boldsymbol{\sigma})}{\| \boldsymbol{\sigma} \|_{\mathcal{H}/\mathrm{Ker}\,\mathbf{B}'} \| \mathbf{v} \|_{\mathcal{L}/\mathrm{Ker}\,\mathbf{B}}} = \inf_{\mathbf{v} \in \mathcal{L}} \ \sup_{\boldsymbol{\sigma} \in \mathcal{H}} \frac{\mathbf{b}\,(\mathbf{v}, \boldsymbol{\sigma})}{\| \boldsymbol{\sigma} \|_{\mathcal{H}/\mathrm{Ker}\,\mathbf{B}'} \| \mathbf{v} \|_{\mathcal{L}/\mathrm{Ker}\,\mathbf{B}}} > 0 \,.$$

The reduced kinematic operator $\mathbf{B}_{\mathcal{L}} \in BL\{\mathcal{L}, \mathcal{H}\}$ and the dual reduced equilibrium operator $\mathbf{B}'_{\mathcal{L}} \in BL\{\mathcal{H}, \mathcal{F}_{\mathcal{L}}\}$ have both closed ranges and meet the equivalent inequalities

$$\| \mathbf{Bv} \|_{\mathcal{H}} \geq \ c_{\mathbf{B}} \, \| \mathbf{v} \|_{\mathcal{L}/\mathrm{Ker}\,\mathbf{B}} \quad \forall \, \mathbf{v} \in \mathcal{L} \iff \| \mathbf{B}'\boldsymbol{\sigma} \|_{\mathcal{F}_{\mathcal{L}}} \geq \ c_{\mathbf{B}} \, \| \boldsymbol{\sigma} \|_{\mathcal{H}/\mathrm{Ker}\,\mathbf{B}'}$$

for all $\boldsymbol{\sigma} \in \mathcal{H}$.

5 Basic theorems

Making appeal to BANACH's closed range theorem [2] we get the proof of the following basic theorem [8] which provides a rigorous basis to the LAGRANGE multipliers method applied by PIOLA in [1].

Proposition 5.1. Theorem of Virtual Powers. *Given a system of active forces* $\ell \in \left[\mathrm{Ker}\, \mathbf{B}_{\mathcal{L}} \right]^{\perp}$ *in equilibrium on the constrained structure* $\mathcal{M}\{\Omega, \mathcal{L}, \mathbf{B}\}$ *there exists at least a stress state* $\boldsymbol{\sigma} \in \mathcal{H}$ *such that the virtual power performed by* $\ell \in \left[\mathrm{Ker}\, \mathbf{B}_{\mathcal{L}} \right]^{\perp}$ *for any conforming kinematic field* $\mathbf{v} \in \mathcal{L}$ *be equal to the virtual power performed by the stress state* $\boldsymbol{\sigma} \in \mathcal{H}$ *for the corresponding tangent strain field* $\mathbf{B}\,\mathbf{v} \in \mathcal{H}$*, i,e.*

$$\ell \in \left[\mathrm{Ker}\, \mathbf{B}_{\mathcal{L}} \right]^{\perp} \subseteq \mathcal{F}_{\mathcal{L}} \iff \exists \, \boldsymbol{\sigma} \in \mathcal{H} \, : \, \langle \, \ell \,, \, \mathbf{v} \, \rangle = (\!(\, \boldsymbol{\sigma} \,, \, \mathbf{Bv} \,)\!) \,, \quad \forall \, \mathbf{v} \in \mathcal{L} \,.$$

Proof. Since the kinematic operator $\mathbf{B} \in BL\left\{\mathcal{V};\mathcal{H}\right\}$ meets KORN's inequality, we infer from BANACH's closed range theorem that $\operatorname{Im}\mathbf{B}^{'}_{\mathcal{L}} = (\operatorname{Ker}\mathbf{B}_{\mathcal{L}})^{\perp}$ where $\mathbf{B}^{'}_{\mathcal{L}} \in BL\left\{\mathcal{H};\mathcal{F}_{\mathcal{L}}\right\}$ is the dual of $\mathbf{B}_{\mathcal{L}} \in BL\left\{\mathcal{L};\mathcal{H}\right\}$. The equilibrium condition reads then $\ell \in \operatorname{Im}\mathbf{B}^{'}_{\mathcal{L}}$ and this ensures the existence of a stress state $\boldsymbol{\sigma} \in \mathcal{H}$ such that $\mathbf{B}^{'}_{\mathcal{L}}\boldsymbol{\sigma} = \ell$, i.e.

$$(\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!) = \langle\, \mathbf{B}^{'}_{\mathcal{L}}\boldsymbol{\sigma} \,,\, \mathbf{v} \,\rangle = \langle\, \ell \,,\, \mathbf{v} \,\rangle\,, \quad \forall\, \mathbf{v} \in \mathcal{L}\,.$$

The statement has been thus proved. $\square$

According to this approach a stress state is introduced as a field of LAGRANGE multipliers suitable to eliminate the rigidity constraint on the conforming virtual kinematisms.

Uniqueness of the stress field in equilibrium holds to within elements of the closed linear subspace of self-stresses, defined by

$$\begin{aligned}
\mathcal{S}_{\text{SELF}} &:= \left\{\boldsymbol{\sigma} \in \mathcal{H} \,:\, (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!) = 0\,, \quad \forall\, \mathbf{v} \in \mathcal{L}\right\} = (\mathbf{B}\mathcal{L})^{\perp} = \\
&= \left\{\boldsymbol{\sigma} \in \operatorname{Ker}\mathbf{B}^{'}_{o} \,:\, \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma} \,,\, \boldsymbol{\Gamma}\mathbf{v} \,\rangle\!\rangle = 0\,, \quad \forall\, \mathbf{v} \in \mathcal{L}\right\} = \\
&= \left\{\boldsymbol{\sigma} \in \mathcal{S} \,:\, \mathbf{B}^{'}_{o}\boldsymbol{\sigma} = \mathbf{o}\,,\ \mathbf{N}\boldsymbol{\sigma} \in [\boldsymbol{\Gamma}\,\mathcal{L}]^{\perp}\right\} = \operatorname{Ker}\mathbf{B}^{'}_{o} \cap \Sigma\,,
\end{aligned}$$

where

$$\begin{aligned}
\Sigma &:= \left\{\boldsymbol{\sigma} \in \mathcal{S} \,:\, (\, \mathbf{B}^{'}_{o}\boldsymbol{\sigma} \,,\, \mathbf{v} \,) = (\!(\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)\!) \quad \forall\, \mathbf{v} \in \mathcal{L}\right\} = \\
&= \left\{\boldsymbol{\sigma} \in \mathcal{S} \,:\, \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma} \,,\, \boldsymbol{\Gamma}\mathbf{v} \,\rangle\!\rangle = 0 \quad \forall\, \mathbf{v} \in \mathcal{L}\right\},
\end{aligned}$$

is the space of conforming stress fields, a closed linear subspace of $\mathcal{S}$.

6 Boundary value problems

Boundary value problems are characterized by the fact that constraints are imposed only on the boundary trace of $\mathcal{T}(\Omega)$-regular kinematic fields $\mathbf{v} \in \mathcal{V}(\mathcal{T}(\Omega))$. Hence in boundary value problems all the $\mathcal{T}(\Omega)$-regular kinematisms with vanishing trace on $\partial\mathcal{T}(\Omega)$ are conforming, a property expressed by the inclusion

$$\operatorname{Ker}\boldsymbol{\Gamma} \subseteq \mathcal{L}\,.$$

As we shall see hereafter in proving an abstract version of CAUCHY theorem, this property is essential in order that variational and differential formulations of equilibrium condition be equivalent one another.

The presence of rigid frictionless bilateral constraints on the boundary $\partial\mathcal{T}(\Omega)$ can be described by considering the pairs of dual HILBERT spaces $\{\Lambda, \Lambda'\}$ and $\{\mathcal{P}, \mathcal{P}'\}$ and the bounded linear operators $\mathbf{L} \in BL\left\{\partial\mathcal{V}, \Lambda'\right\}$ and $\boldsymbol{\Pi} \in BL\left\{\mathcal{P}, \partial\mathcal{V}\right\}$. The operators $\mathbf{L}$ and $\boldsymbol{\Pi}$ provide respectively implicit and explicit descriptions of the boundary constraints. We assume that $\mathbf{L}$ and $\boldsymbol{\Pi}$ have closed ranges so that, denoting by $\mathbf{L}' \in BL\left\{\Lambda, \partial\mathcal{F}\right\}$ and

$\mathbf{\Pi}' \in BL\left\{\partial\mathcal{F}, \mathcal{P}'\right\}$ the dual operators, BANACH's theorem tell us that $\operatorname{Im}\mathbf{L}' = (\operatorname{Ker}\mathbf{L})^{\perp}$ and $\operatorname{Im}\mathbf{\Pi} = (\operatorname{Ker}\mathbf{\Pi}')^{\perp}$ [8].

The closed linear subspace of conforming displacement fields is then characterized by

$$\mathcal{L} = \left\{\mathbf{v} \in \mathcal{V} \mid \mathbf{\Gamma}\mathbf{v} \in \operatorname{Im}\mathbf{\Pi} = \operatorname{Ker}\mathbf{L}\right\},$$

In boundary value problems the orthogonality property $\mathcal{R} = \mathcal{L}^{\perp}$ yields the condition $\mathcal{R} \subseteq (\operatorname{Ker}\mathbf{\Gamma})^{\perp} = \operatorname{Im}\mathbf{\Gamma}'$ where $\mathbf{\Gamma}' \in BL\left\{\partial\mathcal{F}, \mathcal{F}\right\}$ is the dual of $\mathbf{\Gamma} \in BL\left\{\mathcal{V}, \partial\mathcal{V}\right\}$. Hence there exists a boundary reaction $\boldsymbol{\rho} \in \partial\mathcal{F}$ such that $\mathbf{\Gamma}'\boldsymbol{\rho} = \mathbf{r}$ that is

$$\langle\, \mathbf{r}\,, \mathbf{v}\,\rangle = \langle\, \boldsymbol{\rho}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\,, \quad \forall\, \mathbf{v} \in \mathcal{V}\,.$$

This means that constraint reactions consists only of boundary reactions which are elements of the subspace

$$\partial\mathcal{R} = \left\{\boldsymbol{\rho} \in \partial\mathcal{F} \mid \langle\!\langle\, \boldsymbol{\rho}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle = 0 \quad \forall\, \mathbf{v} \in \mathcal{L}\right\} = (\mathbf{\Gamma}\mathcal{L})^{\perp} = \operatorname{Im}\mathbf{L}' = \operatorname{Ker}\mathbf{\Pi}'\,.$$

Uniqueness of the parametric representations of $\mathcal{L}$ and $\partial\mathcal{R}$ requires that $\operatorname{Ker}\mathbf{\Pi} = \{\mathbf{o}\}$ and $\operatorname{Ker}\mathbf{L}' = \{\mathbf{o}\}$ respectively.

It is now possible to provide a simple proof of an abstract version of CAUCHY's fundamental theorem for boundary value problems in the statics of continua.

Proposition 6.1. **CAUCHY's Theorem.** *Let us consider a constrained model* $\mathcal{M}\{\Omega, \mathcal{L}, \mathbf{B}\}$ *with kinematic constraint conditions imposed on the boundary* $\partial\mathcal{T}(\Omega)$ *of a subdivision* $\mathcal{T}(\Omega)$. *Then a system of body and contact forces* $\{\mathbf{b}, \mathbf{t}\} \in H \times \partial\mathcal{F}$ *and a stress state* $\boldsymbol{\sigma} \in \mathcal{H}$ *meet the variational condition of equilibrium*

$$(\,\mathbf{b}\,, \mathbf{v}\,) + \langle\!\langle\, \mathbf{t}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle = (\!(\, \boldsymbol{\sigma}\,, \mathbf{B}\mathbf{v}\,)\!)\,, \quad \boldsymbol{\sigma} \in \mathcal{H}\,, \quad \forall\, \mathbf{v} \in \mathcal{L}$$

if and only if they satisfy the CAUCHY *equilibrium equations*

$$\mathbf{B}_o^{'}\boldsymbol{\sigma} = \mathbf{b}\,, \qquad \textit{body equilibrium}\,,$$

$$\mathbf{N}\boldsymbol{\sigma} = \mathbf{t} + \boldsymbol{\rho} \quad \textit{boundary equilibrium}\,,$$

where $\boldsymbol{\sigma} \in \mathcal{S}$ *and* $\boldsymbol{\rho} \in [\mathbf{\Gamma}\,\mathcal{L}]^{\perp}$ *is a reactive system acting on* $\partial\mathcal{T}(\Omega)$.

Proof. Let the variational condition of equilibrium be met and assume as test fields the kinematisms $\boldsymbol{\varphi} \in \mathbb{D}(\mathcal{T}(\Omega); \mathrm{V}) \subset \operatorname{Ker}\mathbf{\Gamma} \subseteq \mathcal{L} \subseteq \mathcal{V}$. From the distributional definition of the operator $\mathbb{B}_o' : \mathcal{H} \mapsto \mathbb{D}'(\mathcal{T}(\Omega); \mathrm{V})$ we get the relation

$$(\,\mathbf{b}\,, \boldsymbol{\varphi}\,) = (\!(\, \boldsymbol{\sigma}\,, \mathbf{B}\boldsymbol{\varphi}\,)\!) = (\,\mathbb{B}_o'\boldsymbol{\sigma}\,, \boldsymbol{\varphi}\,)\,, \quad \forall\, \boldsymbol{\varphi} \in \mathbb{D}(\mathcal{T}(\Omega); \mathrm{V})\,.$$

It follows that $\boldsymbol{\sigma} \in \mathcal{S}$ and $\mathbf{B}_o^{'}\boldsymbol{\sigma} = \mathbf{b}$ and GREEN's formula can be applied to prove that

$$(\!(\, \boldsymbol{\sigma}\,, \mathbf{B}\mathbf{v}\,)\!) = (\,\mathbf{B}_o^{'}\boldsymbol{\sigma}\,, \mathbf{v}\,) + \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle\,, \quad \forall\, \mathbf{v} \in \mathcal{L}\,, \quad \boldsymbol{\sigma} \in \mathcal{S}\,.$$

From the variational condition of equilibrium we finally get

$$\langle\!\langle\, \mathbf{t}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle = \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle\,, \quad \boldsymbol{\sigma} \in \mathcal{S} \quad \forall\, \mathbf{v} \in \mathcal{L}\,,$$

or equivalently

$$\mathbf{N}\boldsymbol{\sigma} - \mathbf{t} \in [\mathbf{\Gamma}\,\mathcal{L}]^{\perp} = \partial\mathcal{R}\,.$$

On the other hand, if CAUCHY's equilibrium conditions are met, observing that

$$\langle\!\langle\, \boldsymbol{\rho}\,, \mathbf{\Gamma}\mathbf{v}\,\rangle\!\rangle = 0\,, \quad \forall\, \boldsymbol{\rho} \in [\mathbf{\Gamma}\,\mathcal{L}]^{\perp}\,, \quad \forall\, \mathbf{v} \in \mathcal{L}\,,$$

the variational condition of equilibrium is readily inferred from GREEN's formula. □

The closedness of $\operatorname{Im}\mathbf{B}_{\mathcal{L}} = \mathbf{B}\mathcal{L}$ and the definition $\mathcal{S}_{\mathrm{SELF}} := (\mathbf{B}\mathcal{L})^{\perp}$ yield the equality $\mathbf{B}\mathcal{L} = (\mathbf{B}\mathcal{L})^{\perp\perp} = \mathcal{S}^{\perp}_{\mathrm{SELF}}$ which provides another basic existence result in structural mechanics and leads to the variational method for kinematic compatibility stated below.

Proposition 6.2. *Let $\mathcal{M}\{\Omega, \mathcal{L}, \mathbf{B}\}$ be a constrained structure and let $\{\boldsymbol{\varepsilon}, \mathbf{w}\} \in \mathcal{H}\times\mathcal{V}$ be a kinematic system formed by an imposed distorsion $\boldsymbol{\varepsilon} \in \mathcal{H}$ and an impressed kinematism $\mathbf{w} \in \mathcal{V}$. Then we have the equivalence*

$$(\!(\, \boldsymbol{\sigma} \,,\, \boldsymbol{\varepsilon} - \mathbf{B}\mathbf{w} \,)\!) = 0 \quad \forall\, \boldsymbol{\sigma} \in \mathcal{S}_{\mathrm{SELF}} \iff \exists\, \mathbf{u} \in \mathbf{w} + \mathcal{L} \,:\, \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}\,.$$

Proof. By BANACH's closed range theorem we have that $\operatorname{Im}\mathbf{B}_{\mathcal{L}} = \left(\operatorname{Ker}\mathbf{B}^{'}_{\mathcal{L}}\right)^{\perp}$. Hence $\boldsymbol{\varepsilon} - \mathbf{B}\mathbf{w} \in \mathcal{S}^{\perp}_{\mathrm{SELF}}$ is equivalent to $\boldsymbol{\varepsilon} - \mathbf{B}\mathbf{w} \in \operatorname{Im}\mathbf{B}_{\mathcal{L}}$. □

The result in proposition 6.2 leads also to the following decomposition property.

Decomposition of the space $\mathcal{H}$.

■ The linear subspace $\mathbf{B}\mathcal{L}$ of tangent strains which are compatible with conforming kinematisms and the linear subspace $\mathcal{S}_{\mathrm{SELF}}$ of self stresses provide a decomposition of the HILBERT space $\mathcal{H}$ of square integrable tangent strain fields into the direct sum of two orthogonal complements

$$\mathcal{H} = \mathcal{S}_{\mathrm{SELF}} \dotplus \mathbf{B}\mathcal{L} \quad \text{with} \quad \begin{cases} \mathcal{S}_{\mathrm{SELF}} = \left[\mathbf{B}\mathcal{L}\right]^{\perp}, \\ \mathbf{B}\mathcal{L} = \left[\mathcal{S}_{\mathrm{SELF}}\right]^{\perp}, \end{cases}$$

where the symbol $\dotplus$ denotes the direct sum and orthogonality has to be taken in the mean square sense in Ω, that is according to the hilbertian topology of the space $\mathcal{H}$.

The theory developed above allows us to establish a number of useful results which could not be deduced if a more naïve analysis were performed.

Among these we quote several new representation formulas which are relevant in the complementary formulations of equilibrium and compatibility and in the statement of primal and complementary mixed and hybrid variational principles in elastostatics [3], [4], [7].

From the basic orthogonal decomposition of the space $\mathcal{H}$ another decomposition formula which plays a basic role in homogenization theory (see e.g. [5] and reference therein) can be directly inferred.

To this end let $\mathbf{M}_{\Omega} \in BL\left\{\mathcal{H}; \mathrm{W}\right\}$ be the averaging operator which provides the mean value in Ω of fields $\boldsymbol{\varepsilon} \in \mathcal{H}$. It is easy to see that $\operatorname{Im}\mathbf{M}_{\Omega} = \mathrm{W}$ and that the adjoint operator $\mathbf{M}'_{\Omega} \in BL\left\{\mathrm{W}; \mathcal{H}\right\}$ maps $\mathbf{D} \in \mathrm{W}$ into the constant field $\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{D} \quad \forall\, \mathbf{x} \in \Omega$. By the closed range theorem we have that

$$\operatorname{Im}\mathbf{M}_{\Omega} = (\,\operatorname{Ker}\mathbf{M}'_{\Omega})^{\perp}\,, \qquad \operatorname{Im}\mathbf{M}'_{\Omega} = (\,\operatorname{Ker}\mathbf{M}_{\Omega})^{\perp}\,.$$

We have the following result.

Proposition 6.3. *Let* $\mathcal{M}\{\Omega, \mathcal{L}, \mathbf{B}\}$ *be a structural model such that the space* $\mathbf{B}\mathcal{L}$ *of conforming strains includes the constant fields:*

$$\operatorname{Im} \mathbf{M}'_{\Omega} \subset \mathbf{B}\mathcal{L} \,.$$

Then the following decomposition into the direct sum of orthogonal complements holds:

$$\mathcal{H} = \operatorname{Im} \mathbf{M}'_{\Omega} \dotplus \mathbf{B}\mathcal{L} \cap \operatorname{Ker} \mathbf{M}_{\Omega} \dotplus (\mathbf{B}\mathcal{L})^{\perp} \,.$$

where

$$\begin{cases} \operatorname{Im} \mathbf{M}'_{\Omega} & \textit{constant fields,} \\ \mathbf{B}\mathcal{L} \cap \operatorname{Ker} \mathbf{M}_{\Omega} & \textit{zero mean conforming strain fields,} \\ (\mathbf{B}\mathcal{L})^{\perp} & \textit{zero mean selfequilibrated stress fields.} \end{cases}$$

Proof. The result follows from the formula

$$\mathbf{B}\mathcal{L} = \operatorname{Im} \mathbf{M}'_{\Omega} + \mathbf{B}\mathcal{L} \cap (\operatorname{Im} \mathbf{M}'_{\Omega})^{\perp} = \operatorname{Im} \mathbf{M}'_{\Omega} + \mathbf{B}\mathcal{L} \cap \operatorname{Ker} \mathbf{M}_{\Omega} \,,$$

and from the equivalence $\operatorname{Im} \mathbf{M}'_{\Omega} \subset \mathbf{B}\mathcal{L} \iff (\mathbf{B}\mathcal{L})^{\perp} \subset \operatorname{Ker} \mathbf{M}_{\Omega}$. □

In periodic homogenization theory the closed linear subspace of conforming kinematisms is defined to be $\mathcal{L}(C) := \operatorname{Im} \mathbf{M}'_{\Omega} \dotplus \mathcal{L}_{\mathrm{PER}}(C)$. Here C is the periodicity cell and $\mathcal{L}_{\mathrm{PER}}(C)$ is the closed linear subspace of GREEN-regular periodic kinematisms defined by $\mathcal{L}_{\mathrm{PER}}(C) := \{\mathbf{v} \in \mathcal{V}(C) \mid \mathbf{B}\mathbf{v}_{\sharp} \in L^2(K; \mathrm{V})\}$ being K any compact neighborhood of the periodicity cell C and $\mathbf{v}_{\sharp}$ the extension by periodicity of the kinematism $\mathbf{v} \in \mathcal{V}(C)$. It is easy to see that $\mathcal{L}_{\mathrm{PER}}(C) \subset \operatorname{Ker} \mathbf{M}_{\Omega}$. Hence $\mathcal{L}(C)$ is closed being the sum of two orthogonal closed linear subspaces.

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