Vol. 63 (2009) *REPORTS ON MATHEMATICAL PHYSICS* No. 3

ON MAUPERTUIS PRINCIPLE IN DYNAMICS

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(Received March 14, 2008 – Revised March 21, 2009)

A new statement of Maupertuis principle of extremal action is contributed on the basis of a constrained action principle in the velocity phase-space in which the condition of energy conservation is imposed on virtual velocities. Dynamical systems governed by time-dependent Lagrangians on nonlinear configuration manifolds and subject to the action of time-dependent forces are considered. In time-independent systems, and in particular in conservative systems, the constrained action principle specializes to a formulation of the original Maupertuis least action principle in which however conservation of energy along the trajectory is a natural consequence of the variational principle and not an a priori assumption as in classical statements.

Keywords: dynamics, action principle, Maupertuis principle, Lagrange's multipliers, Banach's theorem, functional analysis.

1. Introduction

The long lasting controversy about the principle of least action in dynamics was initiated in 1751 by the ugly dispute between Pierre Louis Moreau de Maupertuis and Samuel König who claimed that Maupertuis had plagiarized a result due to Leibniz who communicated it to Jacob Hermann in a letter dated 1707. Voltaire, in support of König, on one hand, and d'Alembert, Euler and the king of Prussia Frederick the Great, in support of Maupertuis, where involved at the center of the dispute, but the original of the incriminating letter was never found. The least action principle was enunciated by Maupertuis on 15 April 1744 [1] and in the same year by Euler in [2]. The principle appears to have been persecuted by a curse and its very formulation has longly been problematic and up to now appears to be still unsatisfactorily grown up.

In [3], footnote on page 243, Arnold quotes: "In almost all textbooks, even the best, this principle is presented so that it is impossible to understand" (C. Jacobi, *Lectures on Dynamics*). I do not choose to break with tradiction. A very interesting "proof" of Maupertuis principle is in Section 44 of the mechanics textbook of Landau and Lifšhits (*Mechanics*, Oxford, Pergamon 1960). In [4], footnote on p. 249, Abraham and Marsden write: We thank M. Spivak for helping us to formulate this theorem correctly. The authors, like many others (we were happy to learn), were confused by the standard textbook statements. For instance the misterious variation

"Δ" in Goldstein [1950, p. 228] corresponds to our enlargement of the variables by $c \rightarrow (\tau, c)$.

The difficulties faced in providing a proper formulation of Maupertuis principle are imputable to two drawbacks which may affect treatments of dynamics. A first issue concerns the formulation of action principles which should most suitably be developed in the phase-space (the tangent bundle to the configuration manifold) and not in the configuration manifold itself. Although less direct, the formulation in the phase-space is natural in dynamics and is in fact compelling if constraints involving velocities are considered. The Maupertuis principle is in this respect rather paradigmatic since the constraint of energy conservation is such. A second issue deals with the fact that it should be recognized that, in the extremality conditions of the geometric action principle, only virtual velocities in the phase-space do play a role. As a consequence, linear constraints on the variations have only be imposed on virtual velocities and must not necessarily be defined in the large, that is, on finitely varied trajectories [5].

The formulation of the Maupertuis principle contributed in the present paper is based on the action principle of dynamics in the velocity phase-space. No *a priori* assumption of energy conservation along the trajectory is needed and hence the new statement of the principle is not confined to conservative dynamical systems as its standard version [3, 4, 6–8]. Impulsive forces are not explicitly considered for brevity but could be easily accounted for. This formulation should end the long track followed by this extremality principle, still often referred to as Maupertuis least action principle for historical reasons.

The plan of the paper is the following. Preliminarily some tools and definitions of calculus on manifolds are recalled to clarify the notation adopted in the main body of the paper. On the basis of the functional analytic version of Lagrange's multipliers method based on Banach's closed range theorem, we provide a direct and clear mathematical proof of the equivalence between the geometric action principle in the velocity-time manifold and a constrained action principle in which the condition of energy conservation is imposed on virtual velocities in this manifold. When energy and force systems are time-independent, i.e. they do not depend directly on time, this principle specializes to an action principle in the velocity phase-space, with the time playing no role. This result provides an extended formulation of Maupertuis principle. In conservative systems the constraint of energy conservation becomes condition of vanishing energy variation and Maupertuis principle takes the classical form but still retaining a more general formulation since constancy of the energy along the trajectory is deduced as a natural consequence of the principle, rather than being assumed *a priori* as in previous statements and no fixed-ends condition is imposed on the variations. This result is a direct consequence of the new approach developed in this paper. Classical as it stands, Maupertuis principle is playing an active role in formulations of dynamics and related fields, still in recent times (see e.g. [9, 10]). Among general variational principles in science, Maupertuis least action principle shares the spotlights with Fermat principle in optics.

2. Calculus on manifolds and action principle

In the sequel a dot $\cdot$ denotes linear dependence. A basic tool of calculus on manifolds is Stokes' formula stating that the integral of a $(k-1)$-form $\boldsymbol{\omega}^{k-1}$ on the boundary $\partial\Sigma$ of a k-chain Σ in a manifold $\mathbb{M}$, with $\dim \mathbb{M} > k$, is equal to the integral of its exterior derivative $d\boldsymbol{\omega}^{k-1}$, a k-form, on Σ, i.e.

$$\int_{\Sigma} d\boldsymbol{\omega}^{k-1} = \oint_{\partial\Sigma} \boldsymbol{\omega}^{k-1}.$$

This equality can be assumed to be the very definition of exterior derivative of a k-form. Denoting by $\downarrow$ the pull back ($\uparrow$ the push forward), the Lie's derivative of a vector field $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}; \mathbb{TM})$ along a flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\Sigma; \mathbb{M})$, with velocity $\mathbf{v} = \partial_{\lambda=0}\, \boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\Sigma\,; \mathbb{TM})$, is given by

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = \partial_{\lambda=0}\ (\boldsymbol{\varphi}_\lambda\!\downarrow\!\mathbf{w}) = \partial_{\lambda=0}\ (T\boldsymbol{\varphi}_\lambda \circ \mathbf{w} \circ \boldsymbol{\varphi}_\lambda),$$

and is equal to the antisymmetric Lie-bracket $\mathcal{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$. The Lie derivative, of a field $\boldsymbol{\omega}^k(\mathbf{x}) \in \Lambda^k(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ of k-forms on Σ, is given by $\mathcal{L}_{\mathbf{v}}\boldsymbol{\omega}^k = \partial_{\lambda=0}\ (\boldsymbol{\varphi}_\lambda\!\downarrow\!\boldsymbol{\omega}^k)$ with the pull back defined by invariance. Reynolds' transport formula, Stokes' formula and Fubini's theorem lead to the integral *extrusion formula* [5]:

$$\partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_{\Sigma} (d\boldsymbol{\omega}^k) \cdot \mathbf{v} + \int_{\partial\Sigma} \boldsymbol{\omega}^k \cdot \mathbf{v},$$

and to the related differential Henri Cartan's magic formula [5, 11, 12] (also called *homotopy formula* [3]),

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^k = (d\boldsymbol{\omega}^k) \cdot \mathbf{v} + d(\boldsymbol{\omega}^k \cdot \mathbf{v}),$$

where $\boldsymbol{\omega}^k \cdot \mathbf{v}$ denotes the $(k-1)$-form which is the contraction performed by taking $\mathbf{v}$ as the first argument of $\boldsymbol{\omega}^k$. The homotopy formula is readily inverted to get Palais formula for the exterior derivative of a 1-form. By Leibniz rule for the Lie derivative, for any two vector fields $\mathbf{v}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}; \mathbb{TM})$,

$$\begin{aligned} d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{w} &= (\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^1) \cdot \mathbf{w} - d(\boldsymbol{\omega}^1 \cdot \mathbf{v}) \cdot \mathbf{w} \\ &= d_{\mathbf{v}}\, (\boldsymbol{\omega}^1 \cdot \mathbf{w}) - \boldsymbol{\omega}^1 \cdot [\mathbf{v}, \mathbf{w}] - d_{\mathbf{w}}\, (\boldsymbol{\omega}^1 \cdot \mathbf{v}). \end{aligned}$$

The expression at the r.h.s. of Palais formula fulfills the tensoriality criterion, see e.g. [5, 12]. The exterior derivative of a differential one-form is thus well defined as a differential two-form, since its value at a point depends only on the values of the argument vector fields at that point. The same algebra may be repeatedly applied to deduce the formula for the exterior derivative of a k-form [13].

The geometric point of view is underlying all the classical variational principles of mathematical physics which were inspired by the early discoveries by Heron of Alexandria, at the beginning of the *Christian era*, about the shortest path followed by reflected light-rays. The extension to refraction of light is due, about one thousand years later, to the muslim scientist Ibn al-Haytham, the *father of optics*, in his book

Kitab al Manazir (Book of Optics). The principle of minimum optical length was enunciated by Fermat on January 1, 1662 in a letter to Cureau de la Chambre.

We will consider piecewise regular trajectories. Discontinuities may occur due to abrupt changes in time of the system properties (e.g. mass loss or impulsive actions) or to abrupt changes in the configuration space properties (e.g. collisions between bodies or change of refraction properties in optical media). In abstract terms, the extremality condition for an action integral along a path (a 1-chain) in a manifold $\mathbb{M}$ is enunciated as follows [5].

DEFINITION 1. (Action principle) A *trajectory* of a system governed by a piecewise regular action one-form $\boldsymbol{\omega}^1$ with a 1D kernel on $\mathbb{M}$, is a piecewise regular path $\Gamma \in C^1(I\,;\mathbb{M})$ with $\boldsymbol{\Gamma} := \Gamma(I)$ fulfilling the variational condition

$$\partial_{\lambda=0} \int_{\mathbf{Fl}^{\mathbf{v}}_{\lambda}(\boldsymbol{\Gamma})} \boldsymbol{\omega}^1 = \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega}^1 \cdot \mathbf{v},$$

for all virtual flows $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in C^1(\boldsymbol{\Gamma}\,;\mathbb{M})$, with $\mathbf{Fl}^{\mathbf{v}}_{0}$ the identity map and virtual velocity field $\mathbf{v} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in C^1(\boldsymbol{\Gamma}\,;\mathbb{TM})$ fulfilling, at singular interfaces, conditions apt to ensure that duality with covectors is well defined.

Since the action one-form $\boldsymbol{\omega}^1$ has a 1D kernel, the trajectory is detected by the action principle to within an arbitrary reparametrization and thus the variational principle is purely geometrical. The local characterization of a trajectory is performed by the following result (see e.g. [5, 14]) which translates in geometric differential terms the original idea due to the genius of Leonhard Euler [2] and is a direct consequence of the extrusion formula and of the fundamental lemma of calculus of variations.

THEOREM 1. (Euler's conditions) *A path* $\Gamma \in C^1(I\,;\mathbb{M})$ *fulfills the action principle if and only if the tangent vector field* $\mathbf{v}_{\Gamma} \in C^1(\boldsymbol{\Gamma}\,;\mathbb{T}\boldsymbol{\Gamma})$ *fulfils the homogeneous differential condition*

$$d\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\Gamma} \cdot \mathbf{v} = 0,$$

and, at the singularity interfaces, the jump conditions $[\![\boldsymbol{\omega}^1]\!] \cdot \mathbf{v} = 0$ *for all virtual tangent vector fields* $\mathbf{v} \in C^1(\boldsymbol{\Gamma}\,;\mathbb{TM})$.

3. Kinematics of continua

The kinematics of continua deals with the placements of a *continuous body*, a compact connected manifold $\mathbb{B}$ with boundary, in an *ambient space* assumed to be a Riemannian manifold $\{\mathcal{S}\,,\mathbf{g}\}$ (classically the Euclidean space) with metric tensor field $\mathbf{g}$. Configurations $\boldsymbol{\chi} \in C^1(\mathbb{B};\mathcal{S})$ are injective maps which are diffeomorphic transformations onto their ranges (embeddings). Given a set X and a Banach space Y, the Banach space of bounded linear maps from X to Y is denoted by $BL(X\,;Y)$. The Whitney product of two vector bundles $(\mathbb{E}\,,\mathbf{p}\,,\mathbb{M})$ and $(\mathbb{H}\,,\boldsymbol{\pi}\,,\mathbb{M})$, over the same base $\mathbb{M}$, is the vector bundle defined by [15]: $\mathbb{E} \times_{\mathbb{M}} \mathbb{H} := \{(\mathbf{e}\,,\mathbf{h}) \in \mathbb{E} \times \mathbb{H} \mid \mathbf{p}(\mathbf{e}) = \boldsymbol{\pi}(\mathbf{h})\}$. The *configuration-space* $\mathbb{C} := C^1(\mathbb{B};\mathcal{S})$ is a differentiable manifold of maps modelled on a Banach space. Tangent and cotangent bundles are denoted by $\mathbb{TC}$ and $\mathbb{T}^*\mathbb{C}$ and respectively called *velocity* and *covelocity phase-space*.

The *velocity-time* and *covelocity-time spaces* are the cartesian products $\mathbb{TC} \times I$ and $\mathbb{T}^*\mathbb{C} \times I$ with a compact time interval I. These state-spaces are respectively adopted in Lagrangian and Hamiltonian descriptions of dynamics. Vectors tangent to the state-space $\mathbb{TC} \times I$ are in the bundle $\mathbb{TTC} \times \mathbb{T}I$ whose elements are pairs $\{\mathbf{X_v}, \Theta_t\} \in \mathbb{T_v TC} \times \mathbb{T}_t I$.

The *tangent map* $T\boldsymbol{\varphi} \in \mathrm{C}^0(\mathbb{TM}; \mathbb{TN})$ of a morphism $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}; \mathbb{N})$ between manifolds is the linear vector bundle homomorphism, i.e. the fiber preserving and fiber-linear map which transforms tangent vectors $\mathbf{v_x} \in \mathbb{T_x M}$ into the differentials $T_\mathbf{x}\boldsymbol{\varphi} \cdot \mathbf{v_x} \in \mathbb{T}_{\boldsymbol{\varphi}(\mathbf{x})}\mathbb{N}$. The tangent bundle is fibred by the surjective submersion $\boldsymbol{\tau}_\mathbb{C} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{C})$ and the tangent of the fibration $T\boldsymbol{\tau}_\mathbb{C} \in \mathrm{C}^0(\mathbb{TTC}; \mathbb{TC})$ maps a vector $\mathbf{X_v} \in \mathbb{T}_\mathbf{v}\boldsymbol{\Gamma}$, tangent to a line $\boldsymbol{\Gamma} \subset \mathbb{TC}$, into the velocity of the projected line $\boldsymbol{\gamma} = \boldsymbol{\tau}_\mathbb{C}(\boldsymbol{\Gamma}) \subset \mathbb{C}$ at $\boldsymbol{\tau}_\mathbb{C}(\mathbf{v}) \in \boldsymbol{\gamma}$.

A vector $\mathbf{X_v} \in \mathbb{T_v TC}$ such that $\boldsymbol{\tau}_\mathbb{TC}(\mathbf{X_v}) = T_\mathbf{v}\boldsymbol{\tau}_\mathbb{C} \cdot \mathbf{X_v}$ is *second order*. Vectors $\mathbf{X_v} \in \mathbb{T_v TC}$ such that $T_\mathbf{v}\boldsymbol{\tau}_\mathbb{C} \cdot \mathbf{X_v} = 0$ are *vertical*. The *vertical lift* at $\mathbf{v} \in \mathbb{TC}$ is the linear map $\mathbf{vl}_\mathbb{TC}(\mathbf{v}) \in \mathrm{C}^1(\mathbb{T}_{\boldsymbol{\tau}_\mathbb{C}(\mathbf{v})}\mathbb{C}; \mathbb{T_v TC})$ defined by $\mathbf{vl}_\mathbb{TC}(\mathbf{v}) \cdot \mathbf{w} := \partial_{\lambda=0}(\mathbf{v} + \lambda\mathbf{w})$ with $(\mathbf{v}, \mathbf{w}) \in \mathbb{TC} \times_\mathbb{C} \mathbb{TC}$.

The fiber derivative of a Lagrangian functional $L \in \mathrm{C}^2(\mathbb{TC}; \Re)$ is the morphism $d_\mathrm{F}L \in \mathrm{C}^1(\mathbb{TC}; \mathbb{T}^*\mathbb{C})$ defined by

$$d_\mathrm{F}L(\mathbf{v}) \cdot \mathbf{w} := \partial_{\lambda=0} L(\mathbf{v} + \lambda\mathbf{w}) = \langle TL(\mathbf{v}), \mathbf{vl}_\mathbb{TC}(\mathbf{v}) \cdot \mathbf{w} \rangle,$$

for all $(\mathbf{v}, \mathbf{w}) \in \mathbb{TC} \times_\mathbb{C} \mathbb{TC}$.

The Fenchel–Legendre transform relates fiberwise convex Hamiltonians $H : \mathbb{T}^*\mathbb{C} \to \Re \cup +\infty$ and fiberwise convex Lagrangians $L : \mathbb{TC} \to \Re \cup +\infty$ according to the conjugacy relations [16], [17]:

$$H(\mathbf{v}^*) = L^*(\mathbf{v}^*) := \sup_{\mathbf{v} \in \mathbb{T}_{\boldsymbol{\pi}^*_\mathbb{C}(\mathbf{v}^*)}\mathbb{C}} \{\langle \mathbf{v}^*, \mathbf{v} \rangle - L(\mathbf{v})\},$$

$$L(\mathbf{v}) = H^*(\mathbf{v}) := \sup_{\mathbf{v}^* \in \mathbb{T}^*_{\boldsymbol{\tau}_\mathbb{C}(\mathbf{v})}\mathbb{C}} \{\langle \mathbf{v}^*, \mathbf{v} \rangle - H(\mathbf{v}^*)\}.$$

The involutive property $L^{**} = L$ and $H^{**} = H$ holds under the assumption that the functionals are fiber-subdifferentiable. Then the Fenchel–Legendre transform is the maximal monotone and conservative [18] (i.e. cyclically monotone) graph associated with the multivalued subdifferential maps

$$\mathbf{v}^* \in \partial_\mathrm{F}L(\mathbf{v}) \iff \mathbf{v} \in \partial_\mathrm{F}H(\mathbf{v}^*),$$

defined by the convex sets:

$$\partial_\mathrm{F}L(\mathbf{v}) := \{\mathbf{v}^* \in \mathbb{T}^*_{\boldsymbol{\tau}_\mathbb{C}(\mathbf{v})}\mathbb{C} \, : \, \langle \mathbf{v}^*, \mathbf{u} - \mathbf{v} \rangle \le L(\mathbf{u}) - L(\mathbf{v}), \qquad \forall\, \mathbf{u} \in \mathbb{T}_{\boldsymbol{\tau}_\mathbb{C}(\mathbf{v})}\mathbb{C}\},$$

$$\partial_\mathrm{F}H(\mathbf{v}^*) := \{\mathbf{v} \in \mathbb{T}_{\boldsymbol{\tau}^*_\mathbb{C}(\mathbf{v}^*)}\mathbb{C} \, : \, \langle \mathbf{v}, \mathbf{u}^* - \mathbf{v}^* \rangle \le H(\mathbf{u}^*) - H(\mathbf{v}^*), \qquad \forall\, \mathbf{u}^* \in \mathbb{T}^*_{\boldsymbol{\tau}^*_\mathbb{C}(\mathbf{v}^*)}\mathbb{C}\}.$$

For fiber-differentiable Lagrangians, the subdifferetial $\partial_\mathrm{F}L(\mathbf{v})$ is the singleton $d_\mathrm{F}L(\mathbf{v})$

and Legendre transform yields the relation: $E(\mathbf{v}) := H(d_{\mathrm{F}}L(\mathbf{v})) = \langle d_{\mathrm{F}}L(\mathbf{v}), \mathbf{v}\rangle - L(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{TC}$.

An interesting revisitation of Legendre transformation has been recently provided in [19]. A form on $\mathbb{TC}$ ($\mathbb{T}^*\mathbb{C}$) is *horizontal* if it vanishes when any of its arguments is a vertical tangent vector to $\mathbb{TC}$ ($\mathbb{T}^*\mathbb{C}$) [12]. This concept is independent of the choice of a connection. The Liouville one-form on the cotangent bundle is the horizontal form intrinsically defined in variational terms,

$$\langle \boldsymbol{\theta}(\mathbf{v}^*), \mathbf{Y}_{\mathbf{v}^*}\rangle = \langle \mathbf{v}^*, T_{\mathbf{v}^*}\boldsymbol{\tau}^*_{\mathbb{C}} \cdot \mathbf{Y}_{\mathbf{v}^*}\rangle = \langle T^*_{\mathbf{v}^*}\boldsymbol{\tau}^*_{\mathbb{C}} \cdot \mathbf{v}^*, \mathbf{Y}_{\mathbf{v}^*}\rangle, \qquad \forall\, \mathbf{Y}_{\mathbf{v}^*} \in \mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C}.$$

A basic property is that the exterior derivative $d\boldsymbol{\theta}(\mathbf{v}^*)$ is a two-form with a trivial kernel [11]. The counterpart in the tangent bundle is the horizontal Poincaré–Cartan one-form $\boldsymbol{\theta}_L \in \mathrm{C}^1(\mathbb{TC}\,; \mathbb{T}^*\mathbb{TC})$, defined in variational terms by

$$\langle \boldsymbol{\theta}_L(\mathbf{v}), \mathbf{Y}_{\mathbf{v}}\rangle = \langle d_{\mathrm{F}}L(\mathbf{v}), T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}_{\mathbf{v}}\rangle = \langle T^*_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{C}} \cdot d_{\mathrm{F}}L(\mathbf{v}), \mathbf{Y}_{\mathbf{v}}\rangle, \qquad \forall\, \mathbf{Y}_{\mathbf{v}} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC}.$$

The *Lagrange's action one-form* $\boldsymbol{\omega}^1_L \in \mathrm{C}^1(\boldsymbol{\Gamma}_I\,; \mathbb{T}^*(\mathbb{TC} \times I))$ in the velocity-time state-space is defined by $\boldsymbol{\omega}^1_L(\mathbf{v}, t) := \boldsymbol{\theta}_{L_t}(\mathbf{v}) - E_t(\mathbf{v})\,dt$ where $E_t(\mathbf{v}) = H_t(d_{\mathrm{F}}L_t(\mathbf{v}))$, with $\mathbf{v} \in \mathbb{TC}$, is the energy functional. The subscript $_t$ denotes a direct dependence on time.

A trajectory in the velocity-time state-space is denoted by $\Gamma_I \in \mathrm{C}^1(I\,; \mathbb{TC} \times I)$ with cartesian projection on the velocity phase-space given by $\Gamma \in \mathrm{C}^1(I\,; \mathbb{TC})$, so that $\Gamma_I(t) = (\Gamma(t)\,, t)$. The trajectory in the configuration space is then $\gamma = \boldsymbol{\tau}_{\mathbb{C}} \circ \Gamma$ and the trajectory images are denoted by $\boldsymbol{\gamma} := \gamma(I)$, $\boldsymbol{\Gamma} := \Gamma(I)$ and $\boldsymbol{\Gamma}_I := \boldsymbol{\Gamma} \times I$. We set $\Gamma = \mathbf{v} \circ \gamma = T\gamma \cdot 1$ where $1 \in \mathrm{C}^1(I\,; \mathbb{T}I)$ with $\boldsymbol{\tau}_I \circ 1 = \mathbf{id}_I$ the unit section. For brevity we set $\mathbf{v}_t = \mathbf{v}(\gamma_t)$.

A virtual flow $\mathbf{Fl}^{\mathbf{v}}_\lambda \in \mathrm{C}^1(\boldsymbol{\gamma}\,; \mathbb{C})$ in the configuration manifold and a virtual flow $\mathbf{Fl}^{\Theta}_\lambda \in \mathrm{C}^1(I\,; \Re)$ along the time axis define altogether an asynchronous flow $\mathbf{Fl}^{\mathbf{v}}_\lambda \times \mathbf{Fl}^{\Theta}_\lambda \in \mathrm{C}^1(\boldsymbol{\gamma} \times I\,; \mathbb{C} \times \Re)$ in the configuration-time manifold. A vanishing time-velocity Θ at every time $t \in I$ defines a synchronous flow $\mathbf{Fl}^{\mathbf{v}}_\lambda \times \mathbf{id}_I \in \mathrm{C}^1(\boldsymbol{\gamma} \times I\,; \mathbb{C} \times \Re)$ in the configuration-time manifold.

Virtual velocity fields along the lifted trajectory $\boldsymbol{\Gamma} \subset \mathbb{TC}$ in the velocity phase-space are vector fields $\mathbf{Y} \in \mathrm{C}^1(\boldsymbol{\Gamma}\,; \mathbb{TTC})$ which project onto virtual velocity fields $\mathbf{v}_\varphi \in \mathrm{C}^2(\boldsymbol{\gamma}\,; \mathbb{TC})$ in the configuration manifold. We will refer to this vector fields as bi-velocities. A suitable extension of $\mathbf{Y} \in \mathrm{C}^1(\boldsymbol{\Gamma}\,; \mathbb{TTC})$ outside the trajectory $\boldsymbol{\Gamma} \subset \mathbb{TC}$ defines a virtual flow $\mathbf{Fl}^{\mathbf{Y}}_\lambda \in \mathrm{C}^1(\boldsymbol{\Gamma}\,; \mathbb{TC})$ which projects to a virtual flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^2(\boldsymbol{\gamma}\,; \mathbb{C})$ with initial velocity $\mathbf{v}_\varphi \in \mathrm{C}^2(\boldsymbol{\gamma}\,; \mathbb{TC})$, so that $\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{Fl}^{\mathbf{Y}}_\lambda = \boldsymbol{\varphi}_\lambda \circ \boldsymbol{\tau}_{\mathbb{C}}$ and $T\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{Y} = \mathbf{v}_\varphi \circ \boldsymbol{\tau}_{\mathbb{C}}$.

4. Force systems

Nonpotential forces acting on the mechanical system along a trajectory in the configuration manifold are represented by a time-dependent field of one-forms $\mathbf{f}_t \in \mathrm{C}^1(\boldsymbol{\gamma}\,; \mathbb{T}^*\mathbb{C})$, so that $\mathbf{f}_t(\gamma_t) \in \mathbb{T}^*_{\gamma_t}\mathbb{C}$. To formulate the law of dynamics on the tangent bundle, forces must be expressed as one-forms on this bundle. Physical consistency requires that force forms be represented by horizontal forms on the tangent bundle since their virtual work must vanish for vanishing velocities of the base point in the configuration manifold. The correspondence between force one-forms $\mathbf{f}_t \in C^1(\boldsymbol{\gamma}\,;\mathbb{T}^*\mathbb{C})$ acting along the trajectory in the configuration manifold and horizontal one-forms $\mathbf{F}_t \in C^1(\boldsymbol{\Gamma}\,;\mathbb{T}^*\mathbb{T}\mathbb{C})$ acting along the lifted trajectory in the tangent bundle is the linear isomorphism defined by

$$\begin{aligned}\langle \mathbf{F}_t(\mathbf{v}_t), \mathbf{Y}(\mathbf{v}_t)\rangle &= \langle \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)), T_{\mathbf{v}_t}\boldsymbol{\tau}_{\mathbb{C}}\cdot \mathbf{Y}(\mathbf{v}_t)\rangle \\ &= \langle \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)), \mathbf{v}_{\varphi}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t))\rangle, \qquad \forall\, \mathbf{Y}_{\mathbf{v}_t} \in \mathbb{T}_{\mathbf{v}_t}\mathbb{T}\mathbb{C}.\end{aligned}$$

In the velocity-time state-space, forces are represented by *force two-forms* $(\mathbf{F}\wedge dt)(\mathbf{v}_t, t)$ defined by

$$(\mathbf{F}\wedge dt)_{(\mathbf{v}_t,t)}\cdot(\mathbf{Y}\,,\Theta)\cdot(\mathbf{X}\,,1) = \mathbf{F}_t(\mathbf{v}_t)\cdot\mathbf{Y}(\mathbf{v}_t) - (\mathbf{F}_t(\mathbf{v}_t)\cdot\mathbf{X}(\mathbf{v}_t))\,\Theta_t,$$

which, for synchronous virtual velocities, gives

$$(\mathbf{F}\wedge dt)_{(\mathbf{v}_t,t)}\cdot(\mathbf{Y}\,,0)\cdot(\mathbf{X}\,,1) = \mathbf{F}_t(\mathbf{v}_t)\cdot\mathbf{Y}(\mathbf{v}_t).$$

5. Dynamics

We may now state the geometric *action principle* in the velocity-time space. In this respect we notice that the kinetic energy of a dynamical system is defined along the trajectory in the velocity phase-space, where the instantaneous mass-form pertaining to the body is well defined. Accordingly also the Lagrangian functional is defined only along the trajectory. To state the law of dynamics as a extremality principle for the action integral associated with the Lagrange one-form, the Lagrangian functional must be suitably extended outside the trajectory, more precisely it is to be defined in any 2D sheet spanned by a virtual flow dragging the trajectory in the velocity phase-space. This issue is a distinctive feature of action principles in dynamics, compared with Fermat's principle in optics which deals with rays in the physical ambient space and their variations.

The extension is performed by pushing along the flow the velocity of the trajectory and the mass form, which means that conservation of mass is assumed in varying the trajectory [5, 20].

PROPOSITION 1. (Geometrical action principle) *A trajectory in velocity-time state-space is a piecewise regular path $\Gamma_I : I \to \mathbb{T}\mathbb{C}\times I$ such that the Lagrangian one-form $\boldsymbol{\omega}_L^1 \in C^1(\boldsymbol{\Gamma}_I\,;\mathbb{T}^*(\mathbb{T}\mathbb{C}\times I))$ fulfils the asynchronous action principle,*

$$\partial_{\lambda=0}\int_{\mathbf{Fl}_\lambda^{(\mathbf{Y}\,,\Theta)}(\boldsymbol{\Gamma}_I)}\boldsymbol{\omega}_L^1 - \oint_{\partial\boldsymbol{\Gamma}_I}\boldsymbol{\omega}_L^1\cdot(\mathbf{Y}\,,\Theta) = -\int_{\boldsymbol{\Gamma}_I}(\mathbf{F}\wedge dt)\cdot(\mathbf{Y}\,,\Theta),$$

for any time-flow $\mathbf{Fl}_\lambda^{\Theta} \in C^1(I\,;\Re)$, with velocity field $\Theta \in C^1(I\,;\mathbb{T}I)$ and for any fiber preserving flow $\mathbf{Fl}_\lambda^{\mathbf{Y}} \in C^1(\boldsymbol{\Gamma}\,;\mathbb{T}\mathbb{C})$ projecting on a flow $\boldsymbol{\varphi}_\lambda \in C^2(\boldsymbol{\gamma}\,;\mathbb{C})$. Here $\partial\boldsymbol{\Gamma}_I$ is the boundary 0-chain.

The extrusion formula

$$\partial_{\lambda=0} \int_{\mathbf{Fl}_\lambda^{(\mathbf{Y},\Theta)}(\boldsymbol{\Gamma}_I)} \boldsymbol{\omega}_L^1 - \oint_{\partial\boldsymbol{\Gamma}_I} \boldsymbol{\omega}_L^1 \cdot (\mathbf{Y}, \Theta) = \int_{\boldsymbol{\Gamma}_I} d\boldsymbol{\omega}_L^1 \cdot (\mathbf{Y}, \Theta),$$

and a direct computation of the exterior derivative show that the variational statement of Proposition 1 is equivalent to the one in which only synchronous variations are considered, by setting $\Theta = 0$ identically, and to the following Euler's condition of extremality [5],

$$d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t) = (\mathbf{F}_t - dE_t)(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t),$$

with the jump conditions at singular points on the trajectory given by

$$\begin{cases} [\![d_{\mathrm{F}} L_t(\mathbf{v}_t)]\!] = 0, \\ [\![E_t(\mathbf{v}_t)]\!] = 0. \end{cases}$$

By the skew-symmetry of the exterior derivative $d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t)$ we infer that

$$d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) = (\mathbf{F}_t - dE_t)(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) = 0,$$

which expresses conservation of energy: the rate of change along the trajectory of the energy functional, considered as not explicitly dependent on time, must be equal to the power expended by the external forces, i.e.

$$\langle dE_t(\mathbf{v}_t), \mathbf{X}(\mathbf{v}_t) \rangle = \langle \mathbf{F}_t(\mathbf{v}_t), \mathbf{X}(\mathbf{v}_t) \rangle,$$

which, being $dE_t(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) = \partial_{\tau=t} E_t(\mathbf{v}_\tau)$, may be written in terms of total and partial time derivatives as

$$\partial_{\tau=t} E_\tau(\mathbf{v}_\tau) = \langle \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)), \mathbf{v}_t \rangle + \partial_{\tau=t} E_\tau(\mathbf{v}_t).$$

The virtual bi-velocity $\mathbf{Y} \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{TTC})$ and the bi-velocity $\mathbf{v}_{T\boldsymbol{\varphi}} \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{TTC})$ of the lifted flow $T\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{TC})$ project both to the same virtual velocity field $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^2(\boldsymbol{\gamma}; \mathbb{TC})$ according to the relation $T\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{Y} = T\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{v}_{T\boldsymbol{\varphi}} = \mathbf{v}_{\boldsymbol{\varphi}} \circ \boldsymbol{\tau}_{\mathbb{C}}$. It follows that $\mathbf{Y} = \mathbf{v}_{T\boldsymbol{\varphi}} + \mathbf{V}$ with $\mathbf{V} \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{TTC})$ vertical vector field. The next lemma shows the role of the vertical virtual bi-velocity field in Euler's condition of extremality [5].

LEMMA 1. *If the linear map $d_{\mathrm{F}}^2 L_t(\mathbf{v}) \in BL(\mathbb{T}_{\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v})}\mathbb{C}; \mathbb{T}^*_{\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v})}\mathbb{C})$ is invertible, the fulfillment of the variational condition*

$$d\boldsymbol{\theta}_{L_t}(\mathbf{v}) \cdot \mathbf{X}(\mathbf{v}) \cdot \mathbf{V}(\mathbf{v}) = -\langle dE_t(\mathbf{v}), \mathbf{V}(\mathbf{v}) \rangle,$$

for any vertical vector $\mathbf{V}(\mathbf{v}) \in \mathbb{T}_{\mathbf{v}}\mathbb{T}_{\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v})}\mathbb{C} \simeq \mathbb{T}_{\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)}\mathbb{C}$, is equivalent to require that $\mathbf{X} \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{TTC})$ is second order: $T\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{X} = \boldsymbol{id}_{\mathbb{TC}}$.

Lemma 1 ensures that the integral curve of the vector field $\mathbf{X} \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{TTC})$, solution of Euler's differential equation in the velocity phase-space, is indeed the time derivative of the trajectory speed in the configuration manifold, that is $\mathbf{X}(\mathbf{v}_t) = \dot{\mathbf{v}}_t = \partial_{\tau=t} \mathbf{v}_\tau$. Hence $\partial_{\tau=t} \gamma_\tau = \partial_{\tau=t} (\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{v}_\tau) = T_{\mathbf{v}_t}\boldsymbol{\tau}_{\mathbb{C}} \cdot \partial_{\tau=t} \mathbf{v}_\tau = T_{\mathbf{v}_t}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{X}(\mathbf{v}_t) = \mathbf{v}_t$. Then Euler's condition of extremality writes [5]

$$\begin{aligned} d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \dot{\mathbf{v}}_t \cdot \mathbf{v}_{T\varphi}(\mathbf{v}_t) &= \langle \mathbf{F}_t(\mathbf{v}_t) - dE_t(\mathbf{v}_t), \mathbf{v}_{T\varphi}(\mathbf{v}_t) \rangle \\ &= \langle \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)), \mathbf{v}_\varphi(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)) \rangle - \langle dE_t(\mathbf{v}_t), \mathbf{v}_{T\varphi}(\mathbf{v}_t) \rangle. \end{aligned}$$

From this formula it is apparent that the condition to be fulfilled by the virtual velocity field $\mathbf{v}_\varphi \in \mathrm{C}^2(\boldsymbol{\gamma}\,; \mathbb{TC})$, in order that the r.h.s. of Euler's condition vanish, involves also the map $\mathbf{v}_{T\varphi} \in \mathrm{C}^1(\boldsymbol{\Gamma}\,; \mathbb{TTC})$ which is related to the tangent map $T\mathbf{v}_\varphi \in \mathrm{C}^1(\boldsymbol{\Gamma}\,; \mathbb{TTC})$ by the relation $\mathbf{v}_{T\varphi} = \mathbf{k} \circ T\mathbf{v}_\varphi$ with $\mathbf{k} \in \mathrm{C}^1(\mathbb{TTC}\,; \mathbb{TTC})$ the canonical flip [12]. This fact may explain the difficulties faced by treatments of Maupertuis principle developed in the configuration manifold.

6. The constrained action principle

To the authors knowledge, a first satisfactory proof of the classical Maupertuis principle was given in [4], Theorem 3.8.5 at page 249, already cited in the introduction. The variational principle in [4] considers a trajectory in the configuration manifold and its asynchronous variations by means of transformations in which the end-points and the value of the energy function are held fixed while varying the trajectory and the start and end-time instants. Asynchronous variations are needed since there could be no path joining the end-points with the same constant energy and the same start and end-time, other than the given trajectory. The treatment is developed in terms of coordinates and the variational principle imposes a extremality condition on the action functional defined as the integral of the action function along the varied paths in the configuration-time manifold.

The treatment in [4] shares with other classical ones (see e.g. [3, 6–8]) the assumption that the dynamical system is time independent and conservative and that variations of the action functional are made between trajectories lying in a constant energy leaf. This assumption is not in agreement with the fact that extremality of the geometric action functional requires only bi-velocities as test functions and not finite displacements of the trajectory. The classical assumption of variations in a constant energy leaf, is more stringent than needed. As we will see, it includes among the geometric conditions also a natural one.

The approach developed in this paper is based on the formulation of the geometric action principle in the velocity-time manifold and on the corresponding Euler's extremality condition, illustrated in the previous section. The formulation in terms of the lifted trajectory in the velocity-time manifold is a key tool since variations of the trajectory in the velocity phase-space are virtual bi-velocities and there is no need for asynchronous variations. Moreover, the condition to be imposed on the virtual bi-velocities is apparent from the relevant Euler's extremality condition expressed in variational terms.

The main step towards the formulation of a generalized Maupertuis principle is the result concerning the equivalence between the Geometric Action Principle of Proposition 1 (GAP) and a Constrained Action Principle (CAP) in the velocity-time state-space. No asynchronous variations are needed since the CAP is formulated as a geometric action principle in the velocity-time state-space. Moreover, the condition of energy conservation is imposed pointwise on the virtual velocities, but not along the varied trajectories.

The idea underlying the proof is the following. It is straightforward to see that a trajectory fulfilling the GAP is also solution of the CAP. Not trivial is the converse implication, that the geometric trajectory provided by the CAP is also solution of the GAP. The basic tool is Lagrange's multipliers method which in turn relies upon Banach's closed range theorem in functional analysis. We formulate a variational statement of the CAP valid for any dynamical system, including time-dependent lagrangians and nonpotential or time-dependent forces. The classical Maupertuis least action principle will be later directly recovered under the special assumption of conservativity. A more general principle which we still call Maupertuis principle is got under the assumption that the energy functional and the force system do not depend directly on time.

The Poincaré–Cartan one-form $\boldsymbol{\theta}_L \in \mathrm{C}^1(\mathbf{\Gamma}_I\,;\mathbb{T}^*(\mathbb{T}\mathbb{C}\times I))$, along the trajectory in the velocity-time state-space, is defined by: $\langle \boldsymbol{\theta}_L, (\mathbf{Y}, \Theta)\rangle(\mathbf{v}_t, t) := \langle \boldsymbol{\theta}_{L_t}, \mathbf{Y}\rangle(\mathbf{v}_t)$ and the energy functional $E \in \mathrm{C}^1(\mathbf{\Gamma}_I\,;\Re)$ is given by $E(\mathbf{v}_t, t) := E_t(\mathbf{v}_t)$ with $(\mathbf{v}_t\,, t) = \Gamma_I(t)$. The next lemma provides a preliminary result.

LEMMA 2. (Energy form) *Let the energy one-form* $\boldsymbol{\eta}_E \in \mathrm{C}^1(\mathbf{\Gamma}_I\,;\mathbb{T}^*(\mathbb{T}\mathbb{C}\times I))$ *be defined by* $\boldsymbol{\eta}_E(\mathbf{v}_t, t) := E(\mathbf{v}_t, t)\,dt$. *Then we have that*

$$[d\boldsymbol{\eta}_E\cdot(\mathbf{Y}\,,0)\cdot(\mathbf{X}\,,1)](\mathbf{v}_t, t) = dE_t(\mathbf{v}_t)\cdot\mathbf{Y}(\mathbf{v}_t).$$

Proof: The computation may be performed by Palais formula by extending the vector $(\mathbf{X}(\mathbf{v}_t)\,, 1_t) \in \mathbb{T}_{(\mathbf{v}_t\,,t)}\mathbf{\Gamma}_I$ to a field $\mathcal{F} \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\times I\,;\mathbb{T}(\mathbb{T}\mathbb{C}\times I))$ by pushing it along the flow $\mathbf{Fl}_\lambda^{(\mathbf{Y},0)} \in \mathrm{C}^1(\mathbf{\Gamma}_I\,;\mathbb{T}\mathbb{C}\times I)$, according to the relation:

$$\mathcal{F}(\mathbf{Fl}_\lambda^{(\mathbf{Y},0)}(\mathbf{v}_t, t)) := (\mathbf{Fl}_\lambda^{\mathbf{Y}}\uparrow\mathbf{X}(\mathbf{v}_t)\,, 1_t)_{(\mathbf{Fl}_\lambda^{\mathbf{Y}}(\mathbf{v}_t)\,,t)}.$$

Then Palais formula tells us that

$$\begin{aligned} d\boldsymbol{\eta}_E(\mathbf{v}_t, t)\cdot(\mathbf{Y}(\mathbf{v}_t)\,, 0_t)\cdot(\mathbf{X}(\mathbf{v}_t)\,, 1_t) &= d_{(\mathbf{Y}(\mathbf{v}_t)\,,0_t)}\langle\boldsymbol{\eta}_E, \mathcal{F}\rangle \\ &\quad - d_{(\mathbf{X}(\mathbf{v}_t)\,,1_t)}\langle\boldsymbol{\eta}_E, (\mathbf{Y}\,,0)\rangle - \langle\boldsymbol{\eta}_E, \mathcal{L}_{(\mathbf{Y}\,,0)}\mathcal{F}\rangle(\mathbf{v}_t, t). \end{aligned}$$

Since, by the chosen extension, the Lie derivative $\mathcal{L}_{(\mathbf{Y}\,,0)}\mathcal{F}$ vanishes, we may evaluate as follows:

$$d_{(\mathbf{X}(\mathbf{v}_t)\,,1_t)}\langle\boldsymbol{\eta}_E, (\mathbf{Y}\,,0)\rangle = \partial_{\tau=t}\,\langle\boldsymbol{\eta}_E(\mathbf{v}_\tau, \tau), (\mathbf{Y}(\mathbf{v}_\tau)\,, 0)\rangle = \partial_{\tau=t}\,E_\tau(\mathbf{v}_\tau)\langle\,d\tau, 0\rangle = 0,$$

$$d_{(\mathbf{Y}(\mathbf{v}_t)\,,0_t)}\langle\boldsymbol{\eta}_E, \mathcal{F}\rangle = \partial_{\lambda=0}\,E_t(\mathbf{Fl}_\lambda^{\mathbf{Y}}(\mathbf{v}_t))\,\langle dt, 1_t\rangle = \partial_{\lambda=0}\,E_t(\mathbf{Fl}_\lambda^{\mathbf{Y}}(\mathbf{v}_t)).$$

Summing up and recalling that $\partial_{\lambda=0}\,E_t(\mathbf{Fl}_\lambda^{\mathbf{Y}}(\mathbf{v}_t)) = dE_t(\mathbf{v}_t)\cdot\mathbf{Y}(\mathbf{v}_t)$, we get the result. □

THEOREM 2. (Constrained action principle) *A trajectory* $\mathbf{\Gamma}_I$ *in the velocity-time state-space* $\mathbb{T}\mathbb{C}\times I$ *of a dynamical system, governed by a time-dependent* *energy* $E_t \in C^1(\boldsymbol{\Gamma}\,;\Re)$ *and subject to time-dependent forces* $\mathbf{f}_t \in C^1(\boldsymbol{\gamma}\,;\mathbb{T}^*\mathbb{C})$, *with* $\boldsymbol{\gamma} = \boldsymbol{\tau}_{\mathbb{C}}(\boldsymbol{\Gamma})$, *is a* 1*-chain fulfilling the geometric action principle*

$$\partial_{\lambda=0} \int_{\mathbf{Fl}_{\lambda}^{(\mathbf{Y},0)}(\boldsymbol{\Gamma}_I)} \boldsymbol{\theta}_L = \oint_{\partial \boldsymbol{\Gamma}_I} \boldsymbol{\theta}_L \cdot (\mathbf{Y}\,,0),$$

for any virtual bi-velocity field fulfilling the condition of energy conservation

$$\mathbf{Y}(\mathbf{v}_t) \in \ker((\mathbf{F}_t - dE_t)(\mathbf{v}_t)) \subset \mathbb{T}_{\mathbf{v}_t}\mathbb{T}\mathbb{C}.$$

Proof: Let us prove that the above statement, denoted CAP, is equivalent to the action principle in Proposition 1, denoted GAP. Indeed, in the synchronous case, being $\boldsymbol{\omega}_L^1 = \boldsymbol{\theta}_L - \boldsymbol{\eta}_E$, applying the extrusion formula to the energy form, the GAP may be written as

$$\partial_{\lambda=0} \int_{\mathbf{Fl}_{\lambda}^{(\mathbf{Y},0)}(\boldsymbol{\Gamma}_I)} \boldsymbol{\theta}_L - \oint_{\partial \boldsymbol{\Gamma}_I} \boldsymbol{\theta}_L \cdot (\mathbf{Y}\,,0) = \int_{\boldsymbol{\Gamma}_I} (d\boldsymbol{\eta}_E - \mathbf{F} \wedge dt) \cdot (\mathbf{Y}\,,0),$$

for any field $\mathbf{Y} \in C^0(\boldsymbol{\Gamma}\,;\mathbb{T}\mathbb{T}\mathbb{C})$.

Along a time-parametrized trajectory, by Lemma 2 we have that

$$(d\boldsymbol{\eta}_E - \mathbf{F} \wedge dt)_{(\mathbf{v}_t\,,t)} \cdot (\mathbf{Y}\,,0) \cdot (\mathbf{X}\,,1) = (dE_t - \mathbf{F}_t)(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t).$$

Hence the GAP implies the CAP. The converse implication is proved by comparing Euler's conditions for both action principles. By the extrusion formula, the expression of the GAP becomes

$$\int_{\boldsymbol{\Gamma}_I} (d\boldsymbol{\theta}_L - d\boldsymbol{\eta}_E + \mathbf{F} \wedge dt) \cdot (\mathbf{Y}\,,0) = 0, \qquad \forall\, \mathbf{Y} \in C^0(\boldsymbol{\Gamma}\,;\mathbb{T}\mathbb{T}\mathbb{C}),$$

and the expression of the CAP may be written as

$$\int_{\boldsymbol{\Gamma}_I} d\boldsymbol{\theta}_L \cdot (\mathbf{Y}\,,0) = 0, \qquad \forall\,(\mathbf{Y}\,,0) \in \ker((d\boldsymbol{\eta}_E - \mathbf{F} \wedge dt) \cdot (\mathbf{X}\,,1)).$$

Being

$$(d\boldsymbol{\eta}_E - \mathbf{F} \wedge dt)_{(\mathbf{v}_t\,,t)} \cdot (\mathbf{X}\,,1) = (\mathbf{F}_t - dE_t)(\mathbf{v}_t)$$

and

$$[(d\boldsymbol{\theta}_L)_{(\mathbf{v}_t\,,t)} \cdot (\mathbf{Y}\,,0) \cdot (\mathbf{X}\,,1)] = d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t),$$

the GAP and the CAP are respectively equivalent to the Euler's conditions:

$$d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t) = (\mathbf{F}_t - dE_t)(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t), \qquad \forall\, \mathbf{Y} \in C^0(\boldsymbol{\Gamma}\,;\mathbb{T}\mathbb{T}\mathbb{C}),$$

$$d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t) = 0, \qquad \forall\, \mathbf{Y}(\mathbf{v}_t) \in \ker((\mathbf{F}_t - dE_t)(\mathbf{v}_t)).$$

By the nondegeneracy of the two-form $d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t)$ the former equation admits a unique solution $\mathbf{X}(\mathbf{v}_t) \in \mathbb{T}_{\mathbf{v}_t}\mathbb{T}\mathbb{C}$.

The solution of the latter homogeneous equation is instead definite to within a scalar factor. The former condition clearly implies the latter one, in the sense that the solution of the former is also a solution of the latter. The converse implication, that there is a solution of the latter which is also solution of the former, is proved by Lagrange's multiplier method. The argument is as follows. Setting $\mathbf{F}_{E_t}(\mathbf{v}_t) := (\mathbf{F}_t - dE_t)(\mathbf{v}_t) \in \mathbb{T}^*_{\mathbf{v}_t}\mathbb{TC} = BL(\mathbb{T}_{\mathbf{v}_t}\mathbb{TC}\,; \Re)$, the subspace $\mathbf{im}(\mathbf{F}_{E_t}(\mathbf{v}_t)) = \Re$ is trivially closed and hence $\ker(\mathbf{F}_{E_t}(\mathbf{v}_t))^0 = \mathbf{im}(\mathbf{F}_{E_t}(\mathbf{v}_t)^*)$ by Banach closed range theorem [21]. Here $\mathbf{F}_{E_t}(\mathbf{v}_t)^* \in BL(\Re\,; \mathbb{T}^*_{\mathbf{v}_t}\mathbb{TC})$ is the dual operator. The latter condition writes $d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) \in \ker(\mathbf{F}_{E_t}(\mathbf{v}_t))^0$ and hence the equality above assures the existence of a $\mu(\mathbf{v}_t) \in \Re$ such that

$$d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t) \cdot \mathbf{X}(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t) = \langle \mathbf{F}_{E_t}(\mathbf{v}_t)^* \cdot \mu(\mathbf{v}_t), \mathbf{Y}(\mathbf{v}_t)\rangle, \qquad \forall\, \mathbf{Y}(\mathbf{v}_t) \in \mathbb{T}_{\mathbf{v}_t}\mathbb{TC},$$

equivalent to $d\boldsymbol{\theta}_{L_t}(\mathbf{v}_t)\cdot\mathbf{X}(\mathbf{v}_t) = \mu(\mathbf{v}_t)\,\mathbf{F}_{E_t}(\mathbf{v}_t)$. Then the field $\mathbf{X}(\mathbf{v}_t)/\mu(\mathbf{v}_t)$ is solution of both Euler's conditions. The Lagrange's multipliers provide a field of scaling factors to get the right time schedule along the trajectory. □

According to Lemma 1, Euler's differential condition ensures that the trajectory $\Gamma \in C^1(I\,; \mathbb{TC})$ is the lifting to the tangent bundle of the trajectory $\gamma \in C^2(I\,; \mathbb{C})$, so that $\mathbf{v}_t = \Gamma(t) = \partial_{\tau=t}\,\gamma(\tau)$.

The virtual flow may then be defined as $\mathbf{Fl}^{\mathbf{Y}}_\lambda = T\boldsymbol{\varphi}_\lambda \in C^1(\boldsymbol{\Gamma}\,; \mathbb{TC})$ with $\boldsymbol{\varphi}_\lambda \in C^2(\boldsymbol{\gamma}\,; \mathbb{C})$ and the virtual bi-velocity is given by $\mathbf{Y} = \mathbf{v}_{T\varphi} = \partial_{\lambda=0}\,T\boldsymbol{\varphi}_\lambda \in C^0(\boldsymbol{\Gamma}\,; \mathbb{TTC})$. Accordingly, the variational condition of the constrained action principle of Theorem 2 can be written explicitly, in terms of the *action functional* $\mathrm{A}_t(\mathbf{v}_t) := \langle d_{\mathrm{F}}L_t(\mathbf{v}_t), \mathbf{v}_t\rangle$ associated with the Lagrangian and of the virtual flow in the configuration manifold as

$$\partial_{\lambda=0} \int_I \mathrm{A}_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t))\,dt = \oint_{\partial I} \langle d_{\mathrm{F}}L_t(\mathbf{v}_t), \mathbf{v}_{\varphi}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t))\rangle\,dt,$$

with virtual velocities fulfilling conservation of energy, i.e. such that

$$dE_t(\mathbf{v}_t) \cdot \mathbf{v}_{T\varphi}(\mathbf{v}_t) = \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)) \cdot \mathbf{v}_{\varphi}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t)).$$

The action functional is also referred to in the literature as the *reduced action* functional, to stress that the energy term is missing in comparison with Hamilton's extremality principle for the Lagrangian $L_t := \mathrm{A}_t - E_t$.

We underline that, in spite of the explicit appearance of virtual flows and tangent flows in the expression of the principle, only the virtual velocity $\mathbf{v}_{\varphi}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t))$ along the trajectory $\boldsymbol{\gamma} = \boldsymbol{\tau}_{\mathbb{C}}(\boldsymbol{\Gamma})$ is influential in the formulation of the law of dynamics. In fact tha virtual flows with coincident initial velocities provide the same test condition. This basic property is here hidden by the imposition of the constraint of energy conservation, but may be proven by considering the equivalent geometric action principle. The proof requires the introduction of a connection in the configuration manifold to get a generalized formulation of Lagrange's law of dynamics [5, 14, 20].

The next theorem shows that the CAP may be stated with an equivalent formulation in which the constraint of energy conservation on the virtual bi-velocities is imposed in integral form along the trajectory.

THEOREM 3. (Constrained Action Principle: equivalent form) *A trajectory* $\boldsymbol{\Gamma}_I$ *of a dynamical system in the velocity-time state-space* $\mathbb{TC} \times I$ *is a path fulfilling* *the geometric action principle*

$$\partial_{\lambda=0} \int_{\mathbf{Fl}_\lambda^{(\mathbf{Y},0)}(\mathbf{\Gamma}_I)} \boldsymbol{\theta}_L = \oint_{\partial \mathbf{\Gamma}_I} \boldsymbol{\theta}_L \cdot \mathbf{Y},$$

for any tangent field $\mathbf{Y} \in C^0(\mathbf{\Gamma}\,; \mathbb{TTC})$ *such that*

$$\int_I dE_t(\mathbf{v}_t) \cdot \mathbf{Y}(\mathbf{v}_t)\, dt = \int_I \langle \mathbf{F}_t(\mathbf{v}_t), \mathbf{Y}(\mathbf{v}_t) \rangle\, dt.$$

Proof: A trajectory fulfils the action principle of Proposition 1 and hence a fortiori the constrained principle of Theorem 3 and then again a fortiori the weaker condition of the principle in Theorem 2. Since this latter is equivalent to the action principle of Proposition 1, the circle of implications is closed and the assertion is proven. □

7. Time independent and conservative systems

When the Lagrangian $L \in C^2(\mathbb{TC}\,; \Re)$ is time-independent and the system is subject to a time-independent force system $\mathbf{f} \in C^1(\boldsymbol{\gamma}\,; \mathbb{T}^*\mathbb{C})$, the constraint of energy conservation on the virtual velocity field is independent of time. Then the projected trajectory in the velocity phase-space can be arbitrarily parametrized and the CAP directly yields an extended version of Maupertuis principle in which the dynamical system is not necessarily conservative.

THEOREM 4. (Maupertuis principle) *In a dynamical system governed by a time-independent Lagrangian functional* $L \in C^2(\mathbb{TC}\,; \Re)$ *and subject to time-independent forces* $\mathbf{f} \in C^1(\boldsymbol{\gamma}\,; \mathbb{T}^*\mathbb{C})$, *the trajectories are 1-chains* $\mathbf{\Gamma} \subset \mathbb{TC}$ *in the velocity phase-space with tangent vectors* $\mathbf{X}(\mathbf{v}_\lambda) := \partial_{\mu=\lambda}\, \Gamma(\mu) \in \mathbb{T}_{\mathbf{v}_\lambda}\mathbf{\Gamma}$, *with* $\mathbf{v}_\lambda := \Gamma(\lambda)$, *fulfilling the homogeneous Euler's condition*

$$d\boldsymbol{\theta}_L(\mathbf{v}) \cdot \mathbf{X}(\mathbf{v}) \cdot \mathbf{Y}(\mathbf{v}) = 0, \quad \mathbf{X}(\mathbf{v}) \in \mathbb{T}_{\mathbf{v}}\mathbb{TC},$$

for any virtual bi-velocity field fulfilling the energy conservation condition $\mathbf{Y}(\mathbf{v}) \in \ker(\mathbf{F}(\mathbf{v}) - dE(\mathbf{v})) \subset \mathbb{T}_{\mathbf{v}}\mathbb{TC}$. *The associated geometric action principle in the velocity phase-space* $\mathbb{TC}$ *is expressed by the variational condition:*

$$\partial_{\lambda=0} \int_{\mathbf{Fl}_\lambda^{\mathbf{Y}}(\mathbf{\Gamma})} \boldsymbol{\theta}_L = \oint_{\partial \mathbf{\Gamma}} \boldsymbol{\theta}_L \cdot \mathbf{Y},$$

stating the extremality of the action integral of the Poincaré–Cartan one-form $\boldsymbol{\theta}_L$ *for all virtual flows* $\mathbf{Fl}_\lambda^{\mathbf{Y}} \in C^1(\mathbf{\Gamma}\,; \mathbb{TC})$ *with an energy conserving virtual bi-velocity* $\mathbf{Y}(\mathbf{v}) \in \ker(\mathbf{F}(\mathbf{v}) - dE(\mathbf{v}))$.

An alternative statement can be deduced from the result in Theorem 3. The Maupertuis principle of Theorem 4 is a geometric action principle whose solutions are determined to within an arbitrary reparametrization. The relevant Euler condition is homogeneous in the trajectory speed and hence provides the geometry of the trajectory but not the time law according to which it is travelled by the dynamical system. Anyway, if the dynamical trajectory in the velocity-time state space is projected on the velocity phase-space, both Maupertuis principle and the energy conservation are fulfilled. Therefore, the time schedule is recoverable from the initial condition on the velocity by imposing conservation of energy along the geometrical trajectory evaluated by the Maupertuis principle. For conservative systems the statement in Theorem 4 specializes into the classical formulation of the least action principle due to Maupertuis [1, 22, 23], Euler [2], Lagrange [24], Jacobi [25, 26] which has been reproduced without exceptions in the literature, see e.g. [3, 4, 8, 11]. The principle deduced from Theorem 4 is however more general than the classical one because it is formulated without making the standard assumption of fixed end-points of the base trajectory in the configuration manifold and also without assuming that the trajectory developes in a constant energy leaf. Indeed the new statement underlines that a condition of energy conservation is imposed on virtual bi-velocities in the velocity phase-space but no energy conservation along the trajectory is assumed.

REMARK 1. In the papers [27] and [28] the authors claim that the classical Maupertuis principle for conservative systems can be given an equivalent formulation by assuming that the trajectory is varied under the assumption of an invariant mean value of the energy (they call this statement the general Maupertuis principle GMP). The sketched proof provided in these papers is however inficiated by the misstatement that the fulfilment of the original Maupertuis principle (MP), in which the energy is constant under the variations, implies the fulfilment of the GMP. But this last variational condition has more variational test fields and hence the converse is true. The implication proved in [27] and [28], that GMP implies MP, is then trivial and the nontrivial converse implication is missing. Theorem 3 shows that the pointwise condition $(dE \cdot \mathbf{Y})(\mathbf{v}_t) = 0$, and the integral condition

$$\partial_{\lambda=0} \int_I E(\mathbf{Fl}^{\mathbf{Y}}_{\lambda}(\mathbf{v}_t))\, dt = \int_I (dE \cdot \mathbf{Y})(\mathbf{v}_t)\, dt = 0,$$

on the virtual bi-velocity field lead to equivalent formulations of the classical Maupertuis principle.

REMARK 2. When dealing with finite dimensional (say nD) dynamical systems, it is of common (if not universal) usage to denote by the numerical vector $q \in \Re^n$ the coordinates on the configuration manifold $\mathbb{C}$ and by the pair $\{q\,, p\} \in \Re^n \times \Re^n$ the induced coordinates on the cotangent bundle with respect to the natural basis dq. Moreover, the Liouville one-form is written in components as $\boldsymbol{\theta} = p\, dq$ and its exterior derivative as $d\boldsymbol{\theta} = dp \wedge dq$, [11] p. 545, [4] p. 179, [3] p. 199, [29] p. 268. We claim that this erroneous notation should lead to the wrong conclusion that $\boldsymbol{\theta}$ is a one-form and $d\boldsymbol{\theta}$ a two-form on $\mathbb{C}$. Moreover, an unpleasant coincidence of notations arises since the action one-form $\boldsymbol{\theta}(\mathbf{v}^*) - H_t(\mathbf{v}^*)\, dt$ and the exact one-form $\mathbf{v}^* - H_t(\mathbf{v}^*)\, dt = dJ(\boldsymbol{\tau}^*_{\mathbb{C}}(\mathbf{v}^*), t)$, where $J \in \mathrm{C}^1(\mathbb{C} \times I\,; \Re)$ is the eikonal functional in the Hamilton–Jacobi theory, are both denoted by $p\, dq - H\, dt$, [3] p. 233 and p. 250. We cannot explain why the notation came in use.

8. Conclusions

At a first sight, the constrained action principle of Theorem 2 could look like a nonholonomic principle under the linear constraint of energy conservation on the bi-velocity fields. In fact it is not so, because this constraint is imposed only on virtual bi-velocities and not on the speed of the trajectory in the phase space. As we have proved, the property of energy conservation along the trajectory is a result and not an assumption of the principle. If the force-forms and the energy functional are time independent, the constraint of energy conservation on virtual bi-velocities is independent of time and the constrained action principle directly leads to Maupertuis principle for the trajectory in the velocity phase-space. The time schedule can be fixed by imposing that the rate of change of the energy along the trajectory be equal to the power expended by the force system. For conservative systems the constraint of energy conservation is described by an integrable distribution on the tangent bundle over the configuration manifold and becomes the classical constraint of constant energy. Anyway, according to our statement of the Maupertuis principle, the constraint of energy conservation is imposed only on the virtual bi-velocity field, while conservation of energy is a natural consequence of the variational principle. This result modifies the usual statement, concerning the standard Maupertuis principle in the configuration manifold and for conservative systems, that conservation of energy along the trajectory must be assumed *a priori*, see e.g. [3, 4, 8, 28]. Indeed, to get rid of the energy term, the assumption that the dynamical evolution takes place in a leaf of constant energy is self-proposing in the classical context. The formulation in the velocity phase-space has the merit of revealing that, to get a homogeneous Euler equation, it suffices to consider virtual bi-velocities which fulfil the constraint of energy conservation, thus leading to the general statement of Maupertuis principle contributed in this paper.

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