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On formulation of nonlocal elasticity problems

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Abstract Nonlocal elasticity models are tackled with a general formulation in terms of source and target fields belonging to dual Hilbert spaces. The analysis is declaredly focused on small movements, so that a geometrically linearised approximation is assumed to be feasible. A linear, symmetric and positive definite relation between dual fields, with the physical interpretation of stress and elastic states, is assumed for the local elastic law which is thus governed by a strictly convex, quadratic energy functional. Genesis and developments of most referenced theoretical models of nonlocal elasticity are then illustrated and commented upon. The purpose is to enlighten main assumptions, to detect comparative merits and limitations of the nonlocal models and to focus on still open problems. Integral convolutions with symmetric averaging kernels, according to both strain-driven and stress-driven perspectives, homogeneous and non-homogeneous elasticity models, together with stress gradient, strain gradient, peridynamic models and nonlocal interactions between beams and elastic foundations, are included in the analysis.

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1 Introduction

Nonlocal elastic models were early proposed in literature as a viable alternative for investigating problems involving dynamical properties of atomic lattices.

In the 2002 review article by Bazant and Jirasek [1] motivations for nonlocal constitutive theories of continua were so summarised:

1. the need to capture small-scale deviations from local continuum models caused by material heterogeneity;
2. the need to achieve objective and properly convergent numerical solutions for localised damage;
3. the need to regularise the boundary value problem to prevent ill posedness.
4. the need to capture size effects observed in experiments and in discrete simulations.

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An early proposal of a nonlocal model was made in [2, 3] by Rogula who later provided in [4] an exposition of the theory up to that time, together with a comprehensive list of references.

Today, widely referenced is a paper by Eringen [5] where integral convolutions were investigated in connection with spatial acoustic dispersion and dynamical properties of dislocations.

Nonlocal models were there formulated according to the *strain-driven* constitutive perspective, a nomenclature recently introduced by the author and coworkers in [6, 7] to label constitutive laws in which the stress is the output of an integral convolution between an averaging kernel and the elastic strain field.

Equivalence to a differential formulation was illustrated in [5] by taking into account linearity of the convolution operator, under the further assumption that the kernel appearing in the convolution is the fundamental solution of a differential equation. This equivalence is a well-known result of potential theory in linear spaces, for problems in unbounded domains where the involved fields are rapidly vanishing towards infinity. For problems in $\Re^n$ fundamental solutions of differential operators are available [8]. On the other hand, for bounded domains fundamental solutions are not available, in general.

In more recent times, the differential equation associated with a strain-driven model formulated according to an integral convolution with an Helmholtz kernel, has been diffusely applied in static and dynamic investigations of nano-structures, as exposed in the reviews [9, 10]. The starting point was the treatment by Peddieson et al. [11]. These authors were however not aware of the fact that, for bounded domains, to close the constitutive problem, constitutive boundary conditions must be imposed to the differential formulation. In this way equivalence of the differential constitutive law to the integral convolution constitutive law is assured [6, 12, 13, 15].

A paradoxical result, was detected in [11], by observing that, for simple beams with a strain-driven integral nonlocal elasticity model, equilibrium requirements impose that the the bending field is coincident with the one of the standard local elastic problem, in evident contrast with the constitutive requirements of the integral convolution law.

This phenomenon was eventually explained in [16] by observing that stress fields, output by the *strain-driven* nonlocal models, are not able to fulfil also the equilibrium conditions, so that in fact the elastostatic problem admits no solution.

For simple beams when constitutive boundary conditions are compared with the requirements of equilibrium, the conflict between equilibrium and constitutive requirements is manifest.

This unexpected failure emerges as an essential difficulty of nonlocal constitutive problems where a relation between fields of state variables, and not just between their point values as in local constitutive cases, is involved.

The literature pertinent to the differential formulation associated with *strain-driven* nonlocal models, see e.g. [32], amounts nowadays to a huge collection and is ever more increasing, notwithstanding the serious obstructions outlined above and evidenced by recent contributions [6, 7, 14–19].

Several skilful modifications have been proposed to overcome the inherent ill-posedness of nonlocal elastic beam problems based on strain-driven convolution models.

A first expedient was the formulation of a mixture of local-nonlocal elastic behaviour [15, 21–26].

This modification, although effectively able to ensure existence of a solution to the structural problem, leads to singularities when the percentage of local elasticity tends to vanish.

In [27] Pijaudier-Cabot and Bazant set forth another proposal by introducing a normalisation of the kernel to get a compensation of boundary effects. The ensuing response operator is however non-symmetric, cannot be derived a potential and therefore could hardly be considered an elastic stiffness.

An alternative proposal was contributed by Polizzotto [28] and Borino [29] where a modified response operator was designed to attain locality recovery in the case of a homogeneous local elasticity.

This means that the response to a uniform input field generates a local target field. In fact a trivial modification of the nonlocal response yields the proposed design by subtracting the response pertaining to a uniform source field and adding a term equal to the local response. The former term trivially vanishes for uniform input fields. The corresponding response operator is symmetric, as linear combination of symmetric terms, and existence of a potential is assured.

Another approach was undertaken in the papers by Khodabakhshi and Reddy [30] and in the subsequent one by Fernández Sáez et al. [31]. There discrete formulations according to the finite element method (FEM) were adopted to transform continuum mechanics problems into solvable algebraic ones.

An overview of theories of continuum mechanics with nonlocal elastic response prior to 2017 has been contributed by Reddy and Srinivasa [33].

In discrete formulations of nonlocal elastic problems, the difficulties inherent to nonlocal strain-driven continuum formulations of beam models are hidden since equilibrium requirements are drastically relaxed by discretisation.

Moreover equilibrium properties of stress distributions emerging from numerical computations are difficult to be checked. Most often the stress distribution is simply not checked at all, with only displacement solutions explicitly displayed and commented upon.

This is an interesting situation in which, while the continuous problem does not admit solution, all interpolating discrete problems are solvable.

What is then the meaning of an approximation?

A merging of nonlocal elasticity and strain gradient model was proposed by Lim et al. [34] and developed by Barretta and Marotti de Sciarra [35, 36].

Computational costs for the discrete solution of nonlocal elastic beams, according to Eringen local/nonlocal mixture have been recently compared in [37].

The finite element (FEM), element-free Galerkin (EFG) and finite point methods (FPM), were chosen for the comparison.

In order to get a consistent theory, a different nonlocal model for elastic beams was contributed by Romano and Barretta in [18] by envisaging a *stress-driven* nonlocal elastic model.

The peculiar obstruction of *strain-driven* nonlocal models previously evidenced was thus overcome.

The idea underlying the proposal consists in swapping the roles of the involved constitutive fields, with the stress as input and the elastic strain as output.

This new nonlocal model follows the track paved by a general revisitation of elasticity theory which led to a *stress-driven* theory of elasticity based on an integrable incremental formulation [38–40].

While in local models the elasticity law is invertible, in nonlocal models the response operator is not such. Consequently strain-driven and stress-driven formulations lead to well distinct constitutive and structural problems [6, 7].

The elastic stiffness operator is positive definite in the local elastic model, but persistence of this property in the nonlocal model is presently an open problem.

In the sequel several proposals of nonlocal elasticity models will be critically revisited and compared.

The treatment includes local/nonlocal mixtures, peridynamics, and stress and strain gradient models, with a general formulation which, avoiding needless repetitions, puts into evidence comparative merits, limitations and also severe incongruences.

In this respect basic geometric notions are worth to be put into evidence, as discussed in the sequel.

All nonlocal constitutive models are based either on operations of differentiation or on integrations to be performed in the body manifold.

The former operation requires that a parallel transport is defined to perform the derivatives while the latter needs in addition that the parallel transport is distant. This means that the point-values of the tensor field to be integrated are parallel transported to a fixed evaluation point in a way independent of the path tracked to join the base-points.

The choice of a parallel transport is usually passed under silence, due to existence of an obvious candidate: the metric preserving and distant transport induced in the body manifold by the translation in Euclid space.

When dealing with non flat body manifolds whose geometric dimension is lower than the one of the container Euclid space, such as in the case of curved beams and shells, the practical adoption of the translation induced parallel transport is however challenging.

A feasible and convenient choice is provided by the parallel transport associated with a system of coordinates in the body manifold, by imposing invariance of the components with respect to the corresponding frame, with or without normalisation of base vectors.

Ultimately, the choice of a parallel transport in the body manifold is rather a matter of convenience which enters in the very definition of the nonlocal constitutive relation.

Nonlocal constitutive relations are therefore not natural, in the sense that knowledge of the metric field in the Euclid space is a geometric tool not sufficient for developing the theory [41].

Consequently, nonlocal constitutive laws cannot assurge to the role of properly defined material descriptions. Indeed they include in an essential way the distant parallel transport (that is one independent of the curve chosen to join start and target points) needed to perform linear operations between tensors based at distinct points, but the choice of a parallel transport is a geometric affair which the material is not aware of.

For this reason nonlocal relations are to be intended as simulations whose validity can only be assessed on the basis of their mathematical and physical consistency and of the purported usefulness in predicting experimental evidence.

Although in last instance this feature is common to any constitutive law, arbitrariness of the involved parallel transport is a feature, seemingly not clearly evidenced before, which is peculiar to nonlocal models.

2 Preliminaries

Given a normed linear space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and the dual space $\mathcal{X}'$, the *right polar* (or *annihilator*) $\mathcal{A}^{\circ} \subset \mathcal{X}'$ of a set $\mathcal{A} \subset \mathcal{X}$, is the linear subspace of the elements in the dual space which have a null duality interaction with all elements of $\mathcal{A} \subset \mathcal{X}$:[1]

$$\mathcal{A}^{\circ} := \{\mathbf{x}' \in \mathcal{X}' \,:\, \langle \mathbf{x}', \mathbf{x} \rangle = 0\,, \quad \forall\, \mathbf{x} \in \mathcal{A}\}\,. \tag{1}$$

Similarly, the *left polar* ${}^{\circ}\mathcal{B} \subset \mathcal{X}$ of $\mathcal{B} \subset \mathcal{X}'$ is defined by:

$$^{\circ}\mathcal{B} := \{\mathbf{x} \in \mathcal{X} \,:\, \langle \mathbf{x}', \mathbf{x} \rangle = 0\,, \quad \forall\, \mathbf{x}' \in \mathcal{B}\}\,. \tag{2}$$

The quotient $\mathcal{X}/\mathcal{A}$ is the linear space whose elements are the cosets $\mathbf{x} + \mathcal{A} \subset \mathcal{X}$ with $\mathbf{x} \in \mathcal{X}$, normed by:

$$\|\mathbf{x} + \mathcal{A}\|_{\mathcal{X}/\mathcal{A}} := \inf_{\mathbf{y} \in \mathcal{A}} \|\mathbf{x} - \mathbf{y}\|_{\mathcal{X}}\,. \tag{3}$$

If $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is complete (a Banach space) and $\mathcal{A}$ is closed in $\mathcal{X}$, then $\mathcal{X}/\mathcal{A}$ is Banach too.

If $\mathcal{X}$ is Hilbert[2] with inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, and $\mathcal{A}$ is closed in $\mathcal{X}$, then $\mathcal{X}/\mathcal{A}$ is Hilbert too, with inner product $\langle \mathbf{x} + \mathcal{A}, \mathbf{y} + \mathcal{A} \rangle = \langle \mathbf{P}\mathbf{x}, \mathbf{P}\mathbf{y} \rangle$ where $\mathbf{P} : \mathbf{X} \mapsto \mathbf{X}$ is the projector on the orthogonal complement $\mathcal{A}^{\perp}$.

[1] The dual space $\mathcal{X}'$ of a normed linear space $\mathcal{X}$ is composed of the continuous linear functionals $f : \mathcal{X} \mapsto \Re$. Their values are denoted by means of the duality pairing $\langle f, \mathbf{v} \rangle$ with $\mathbf{v} \in \mathcal{X}$. In this context, continuity is equivalent to boundedness: $|\langle f, \mathbf{v} \rangle| \le c\, \|\mathbf{v}\|_{\mathcal{X}}$.

[2] A Hilbert space is Banach with norm fulfilling the parallelogram law and therefore derivable from a symmetric, positive definite bilinear form [39, 42].

3 Structural problems

Let us consider a geometrically linearised structural model consisting of a body occupying a configuration $\boldsymbol{\Omega}$ in the Euclid space E at a given time $t \in \mathcal{Z}$.

The kinematic space $\mathcal{V}$ of spatial velocity fields $\mathbf{v} : \boldsymbol{\Omega} \mapsto T_{\boldsymbol{\Omega}}\mathrm{E}$ is assumed to be Hilbert with dual force space $F = \mathcal{V}'$, so that the pairing $\langle f, \mathbf{v} \rangle$ evaluates the virtual power performed by the force $f \in F$ for the velocity $\mathbf{v} \in \mathcal{V}$.

The kinematic operator $\mathbf{B} : \mathcal{V} \mapsto \mathcal{D}$, linear and continuous, evaluates the straining $\mathbf{B}(\mathbf{v}) \in \mathcal{D}$, a tensor-valued field on $\boldsymbol{\Omega}$, corresponding to the velocity field $\mathbf{v} \in \mathcal{V}$.

Classical 3D continuum mechanics considers motions in the Euclid space-time. The metric tensor field $\mathbf{g} : T_{\boldsymbol{\Omega}}\mathrm{E} \mapsto (T_{\boldsymbol{\Omega}}\mathrm{E})^*$ is time independepent and the kinematic operator is the mixed alteration of its Lie-derivative along the spatial velocity field:

$$\mathbf{g} \cdot \mathbf{B}(\mathbf{v}) = \mathcal{L}_{\mathbf{v}}(\mathbf{g})\,. \tag{4}$$

The kinematic operator $\mathbf{B} : \mathcal{V} \mapsto \mathcal{D}$ is a differential operator and its kernel $\mathrm{Ker}(\mathbf{B}) \subset \mathcal{V}$ is exactly the subspace of spatial velocities that are infinitesimal isometries, that is the ones along which the Lie derivatives of the metric field do vanish.

The stress space Σ and the stretching space $\mathcal{D}$ are dual Hilbert spaces, with $\Sigma = \mathcal{D}'$.

In a geometrically linearised theory, the configuration $\boldsymbol{\Omega}$ is assumed to be invariant during the involved mechanical processes. Accordingly, a small displacement is treated as a velocity field and a small strain as a straining.

In the same way, a small stress variation is treated as a stressing, that is as a Lie derivatives of the stress field along the motion.

The dual operator $\mathbf{B}' : \Sigma \mapsto F$ yields the force $\mathbf{B}'(\boldsymbol{\sigma}) \in F$ in equilibrium with the stress field $\boldsymbol{\sigma} \in \Sigma$ and is uniquely defined by the virtual power identity:

$$\langle \boldsymbol{\sigma}, \mathbf{B}(\mathbf{u}) \rangle = \langle \mathbf{B}'(\boldsymbol{\sigma}), \mathbf{u} \rangle\,, \quad \forall\, \boldsymbol{\sigma} \in \Sigma\,, \quad \forall\, \mathbf{u} \in \mathcal{V}\,. \tag{5}$$

The body is subject to linear kinematic constraints, that are assumed firm and smooth.

The mathematical representation of these constraints consists in assuming that *conforming* (constraint respecting) kinematic fields $\mathbf{v} : \boldsymbol{\Omega} \mapsto \mathrm{E}$, belong to a closed linear subspace $\mathcal{L} \subset \mathcal{V}$.

The multivalued constitutive relation of firm and smooth constraints is expressed by a graph in the product space $\mathcal{V} \times F$ composed by the pairs $\{\mathbf{v}, \mathbf{r}\} \in \mathcal{V} \times F$ such that:

$$\mathbf{v} \in \mathcal{L}, \quad \mathbf{r} \in \mathcal{L}^{\circ}. \tag{6}$$

This amounts to state that reactions of smooth constrains and conforming small displacements do have a null mutual interaction. Elastic constraints will be dealt with in Sect. 4.

In the geometrically linearised theory the assigned data are:

- a prescribed small displacement field $\mathbf{w} \in \mathcal{V}$,
- a prescribed small strain field $\boldsymbol{\eta} \in \mathcal{D}$,
- a *load* (also named *active* force) functional:

  $$\ell \in \mathcal{L}' \equiv F/\mathcal{L}^{\circ}. \tag{7}$$

  The linear subspace of *reactive* force functionals is in duality with the quotient space $\mathcal{V}/\mathcal{L}$ of constraint velocity fields:

  $$\mathcal{L}^{\circ} \equiv (\mathcal{V}/\mathcal{L})'. \tag{8}$$

  Reactive forces $\mathbf{r} \in \mathcal{L}^{\circ}$ perform no virtual power for any *conforming* virtual velocity field:

  $$\mathbf{r} \in \mathcal{L}^{\circ} \subset F \iff \langle \mathbf{r}, \delta\mathbf{v} \rangle = 0, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}. \tag{9}$$

Small displacement fields belonging to the coset $\mathbf{w} + \mathcal{L} \in \mathcal{V}/\mathcal{L}$ are said to be *admissible*.

The kinematic operator $\mathbf{B}_{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{D}$, restriction of the operator $\mathbf{B} : \mathcal{V} \mapsto \mathcal{D}$ to the conformity subspace $\mathcal{L} \subset \mathcal{V}$, is related to the dual static operator $\mathbf{B}'_{\mathcal{L}} : \Sigma \mapsto \mathcal{L}'$ by the virtual power identity:

$$\langle \boldsymbol{\sigma}, \mathbf{B}_{\mathcal{L}}(\mathbf{v}) \rangle = \langle \mathbf{B}'_{\mathcal{L}}(\boldsymbol{\sigma}), \mathbf{v} \rangle, \quad \forall\, \boldsymbol{\sigma} \in \Sigma, \forall \mathbf{v} \in \mathcal{L}. \tag{10}$$

Structural analysis is based on the assumption that, for any given subspace $\mathcal{L} \subset \mathcal{V}$ of *conforming* kinematic fields, the kinematic operator $\mathbf{B}_{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{D}$ has a closed range [39], as expressed by the inequality:

$$\| \mathbf{B}_{\mathcal{L}}(\mathbf{v}) \|_{\mathcal{D}} \geq c\, \| \mathbf{v} + \mathcal{L} \|_{\mathcal{V}/\mathrm{Ker}(\mathbf{B}_{\mathcal{L}})}, \quad \forall\, \mathbf{v} \in \mathcal{L}. \tag{11}$$

If $\dim(\mathrm{Ker}(\mathbf{B})) < \infty$, being:

$$\mathrm{Ker}(\mathbf{B}_{\mathcal{L}}) = \mathrm{Ker}(\mathbf{B}) \cap \mathcal{L}, \tag{12}$$

fulfilment of the inequality Eq. (11), for any closed subspace $\mathcal{L} \subset \mathcal{V}$ of conforming displacements, may be inferred form an inequality of Korn's type [43–45]:

$$\| \mathbf{B}_{\mathcal{L}}(\mathbf{v}) \|_{\mathcal{D}} + \| \mathbf{G}(\mathbf{v}) \|_{G} \geq c \| \mathbf{v} \|_{\mathcal{V}}, \quad \forall\, \mathbf{v} \in \mathcal{L}, \tag{13}$$

with $\mathbf{G} : \mathcal{V} \mapsto G$ continuous and compact linear operator with codomain in a Hilbert space G.

From the duality relation in Eq. (10), two polarity properties are directly deduced:

$$\begin{cases} \mathrm{Ker}(\mathbf{B}_{\mathcal{L}}) = {}^{\circ}\mathrm{Im}(\mathbf{B}'_{\mathcal{L}}), \\ \mathrm{Ker}(\mathbf{B}'_{\mathcal{L}}) = \mathrm{Im}(\mathbf{B}_{\mathcal{L}})^{\circ}. \end{cases} \tag{14}$$

By Banach closed range theorem [42, 46], closedness of the range $\mathrm{Im}(\mathbf{B}_{\mathcal{L}}) = \mathbf{B}\mathcal{L}$ of the kinematic operator $\mathbf{B}_{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{D}$, Eq. (11), implies that the equilibrium operator $\mathbf{B}'_{\mathcal{L}} : \Sigma \mapsto \mathcal{L}' = F/\mathcal{L}^{\circ}$ has closed range too, as expressed by the inequality:

$$\| \mathbf{B}'_{\mathcal{L}}(\boldsymbol{\sigma}) \|_{F/\mathcal{L}^{\circ}} \geq c\, \| \boldsymbol{\sigma} \|_{\mathcal{S}/(\mathbf{B}\mathcal{L})^{\circ}}, \quad \forall\, \boldsymbol{\sigma} \in \mathcal{S}. \tag{15}$$

and that the following complementary polarity properties hold true:

$$\begin{cases} \mathrm{Im}(\mathbf{B}'_{\mathcal{L}}) = \mathrm{Ker}(\mathbf{B}_{\mathcal{L}})^{\circ}, \\ \mathrm{Im}(\mathbf{B}_{\mathcal{L}}) = {}^{\circ}\mathrm{Ker}(\mathbf{B}'_{\mathcal{L}}). \end{cases} \tag{16}$$

The polarity relations in Eq. (16) provide the basic existence results in continuum mechanics [39].

Let a load functional $\ell \in \mathcal{L}'$ be in equilibrium:

$$\ell \in \mathrm{Ker}(\mathbf{B}_{\mathcal{L}})^{\circ}. \tag{17}$$

The virtual power principle states that there exists a stressing field $\boldsymbol{\sigma} \in \Sigma$ such that:

$$\langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v}) \rangle = \langle \ell, \delta\mathbf{v} \rangle, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}. \tag{18}$$

The basic equality in Eq. $(16)_1$ assures that the affine variety $\Sigma_{\ell} \subset \Sigma$ of stressing fields statically compatible with an equilibrated applied load fulfilling Eq. (18), is not empty.

Then we can write

$$\Sigma_\ell = \boldsymbol{\sigma}_\ell + \Sigma_0 \,, \tag{19}$$

with Σ_0 linear subspace of self-equilibrated stressing fields:

$$\Sigma_0 := \mathrm{Ker}(\mathbf{B}'_{\mathcal{L}}) = \left(\mathrm{Im}(\mathbf{B}_{\mathcal{L}})\right)^\circ = (\mathbf{B}\mathcal{L})^\circ \,. \tag{20}$$

If $\Sigma_0 = \{\mathbf{0}\}$, the structural problem is qualified as *statically determinate*.

Denoting by $\mathbf{w} \in \mathcal{V}$ a prescribed small displacement field, with the meaning of a constraint small displacement, the solution of the structural problem will be an admissible small displacement field $\mathbf{u} \in \mathbf{w} + \mathcal{L}$.

Let us now introduce the dual operators:

$$\begin{aligned} &\bar{\mathbf{B}} : \mathcal{V}/\mathcal{L} \mapsto \mathcal{D}/(\mathbf{B}\mathcal{L}) \,, \\ &\bar{\mathbf{B}}' : (\mathbf{B}\mathcal{L})^\circ \mapsto \mathcal{L}^\circ \,, \end{aligned} \tag{21}$$

with the former defined by:

$$\bar{\mathbf{B}}(\mathbf{w} + \mathcal{L}) := \mathbf{B}(\mathbf{w} + \mathcal{L}) = \mathbf{B}(\mathbf{w}) + \mathbf{B}\mathcal{L} \,, \tag{22}$$

so that $\mathrm{Ker}(\bar{\mathbf{B}}) = \mathcal{L}$, and the latter by:

$$\begin{aligned} \langle \bar{\mathbf{B}}'(\boldsymbol{\sigma}_0), \mathbf{w} + \mathcal{L} \rangle &= \langle \boldsymbol{\sigma}_0, \bar{\mathbf{B}}(\mathbf{w} + \mathcal{L}) \rangle \\ &= \langle \boldsymbol{\sigma}_0, \mathbf{B}(\mathbf{w} + \mathcal{L}) \rangle = \langle \mathbf{r}, \mathbf{w} \rangle \,. \end{aligned} \tag{23}$$

The internal virtual power of self-stress fields, for the small strain corresponding to a constraint velocity field, is then equal to the external virtual power of the emerging constraint reactions, for the constraint velocity field.

4 Local elastostatic problems

The standard local elastic relation is governed by an invertible, linear, symmetric and positive definite stiffness operator $E : \mathcal{D} \mapsto \Sigma$ with inverse compliance $E^{-1} : \Sigma \mapsto \mathcal{D}$:

$$\mathbf{e} = E^{-1}(\boldsymbol{\sigma}) \iff \boldsymbol{\sigma} = E(\mathbf{e}) \,. \tag{24}$$

Here $\mathbf{e} \in \mathcal{D}$ is the elastic state, a notion recently introduced in [38, 39] in the general context of nonlinear elastic processes involving large movements.

In the small movements framework, considered in the present paper, by virtue of geometric linearisation and relying upon linearity of the constitutive law, the elastic relation can be expressed in terms of *stress state variations* and *elastic state variations*, usually referred to as *elastic strains*.

For its relevance in applications, we consider also elastic contact interactions, between the body under investigation and external elastic constraints, such as the supporting elastic medium in a foundation problem, whose stiffness operator $K : \mathcal{L} \mapsto \mathcal{L}'$, denoting by $\mathbf{w} \in \mathcal{V}$ a given small displacement, is defined by:

$$-\mathbf{f} = K(\mathbf{u} - \mathbf{w}) \,, \quad \mathbf{u} \in \mathbf{w} + \mathcal{L} \,, \quad \mathbf{f} \in F \,. \tag{25}$$

The stiffness of the elastic constraints is a continuous, symmetric and positive semidefinite linear operator $K : \mathcal{L} \mapsto \mathcal{L}'$, so that:

$$\begin{cases} \|K(\mathbf{u})\|_F \le c\, \|\mathbf{u}\|_{\mathcal{V}} \,, \quad c > 0 \,, \quad \forall\, \mathbf{u} \in \mathcal{V} \,, \\ \langle K(\mathbf{u}_1), \mathbf{u}_2 \rangle = \langle K(\mathbf{u}_2), \mathbf{u}_1 \rangle \,, \qquad \forall\, \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{V} \,, \\ \langle K(\mathbf{u}), \mathbf{u} \rangle \ge 0 \,, \qquad \forall\, \mathbf{u} \in \mathcal{V} \,. \end{cases} \tag{26}$$

In terms of the adjoint stiffness operator $K^A : \mathcal{L} \mapsto \mathcal{L}'$:

$$\langle K^A(\mathbf{u}_1), \mathbf{u}_2 \rangle = \langle K(\mathbf{u}_2), \mathbf{u}_1 \rangle \,, \quad \forall\, \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{L} \,, \tag{27}$$

symmetry is conveniently expressed by the equality:

$$K^A = K \,. \tag{28}$$

The subspace of elastic interactions $\mathrm{Im}(K) \subset \mathcal{L}'$ is assumed to be closed in F.

Consequently, by virtue of the polarity relations Eqs. (14) and (16), the subspace of elastic interactions:

$$\mathrm{Im}(K) \subset \mathcal{L}' \,, \tag{29}$$

fulfils, with the subspace of elastically ineffective and conforming small displacements:

$$\mathrm{Ker}(K) \subset \mathcal{L} \,, \tag{30}$$

the annihilation rules:

$$\begin{cases} \mathrm{Ker}(K) = \mathrm{Ker}(K^A) = {}^\circ\mathrm{Im}(K) \,, \\ \mathrm{Im}(K) = \mathrm{Im}(K^A) = \mathrm{Ker}(K)^\circ \,. \end{cases} \tag{31}$$

In local elasticity, the data set is made of:

1. an imposed small displacement $\mathbf{w} \in \mathcal{V}$,
2. an impressed small strain $\boldsymbol{\eta} \in \mathcal{D}$,

3. a prescribed load functional $\ell \in \mathcal{F}$ in equilibrium:

$$\ell \in \Big(\mathrm{Ker}(\mathbf{B}_{\mathcal{L}}) \cap \mathrm{Ker}(K)\Big)^{\circ} . \tag{32}$$

The linear elastostatic problem is a set of three variational conditions expressing equilibrium, kinematic compatibility and constitutive relation:

$$\begin{cases} \langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v})\rangle = \langle \ell, \delta\mathbf{v}\rangle - \langle K(\mathbf{u}-\mathbf{w}), \delta\mathbf{v}\rangle \,, \\ \langle \mathbf{B}(\mathbf{u}), \delta\boldsymbol{\sigma}\rangle - \langle \mathbf{e}, \delta\boldsymbol{\sigma}\rangle = \langle \boldsymbol{\eta}, \delta\boldsymbol{\sigma}\rangle \,, \\ -\langle \boldsymbol{\sigma}, \delta\mathbf{e}\rangle + \langle E(\mathbf{e}), \delta\mathbf{e}\rangle = 0 \,. \end{cases} \tag{33}$$

Trial fields are:

1. admissible small displacements $\mathbf{u} \in \mathbf{w} + \mathcal{L}$,
2. stress variations $\boldsymbol{\sigma} \in \mathcal{S}$,
3. elastic states variations $\mathbf{e} \in \mathcal{D}$.

and test fields are:

1. conforming small displacements $\delta\mathbf{v} \in \mathcal{L}$,
2. stress variations $\delta\boldsymbol{\sigma} \in \mathcal{S}$,
3. elastic states variations $\delta\mathbf{e} \in \mathcal{D}$.

For computational purposes it is expedient to formulate the linear elastostatic problem in terms of conforming displacement fields $\mathbf{v} \in \mathcal{L}$ by expressing the displacement field as:

$$\mathbf{u} = \mathbf{v} + \mathbf{w} \,, \quad \mathbf{v} \in \mathcal{L} \,, \quad \mathbf{w} \in \mathcal{V} \,, \tag{34}$$

Test and trial fields have thus the same linear domain spaces. It is convenient to introduce the *datum strain*:

$$\mathbf{d} := \boldsymbol{\eta} - \mathbf{B}(\mathbf{w}) \in \mathcal{D} \,, \tag{35}$$

so that the linear elastostatic problem writes as follows:

$$\begin{cases} \langle K(\mathbf{v}), \delta\mathbf{v}\rangle + \langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v})\rangle = \langle \ell, \delta\mathbf{v}\rangle \,, \\ \langle \mathbf{B}(\mathbf{v}), \delta\boldsymbol{\sigma}\rangle - \langle \mathbf{e}, \delta\boldsymbol{\sigma}\rangle = \langle \mathbf{d}, \delta\boldsymbol{\sigma}\rangle \,, \\ -\langle \boldsymbol{\sigma}, \delta\mathbf{e}\rangle + \langle E(\mathbf{e}), \delta\mathbf{e}\rangle = 0 \,, \end{cases} \tag{36}$$

and, in block-matrix form with $\mathbf{I}_{\mathcal{D}} : \mathcal{D} \mapsto \mathcal{D}$ and $\mathbf{I}_{\mathcal{S}} : \mathcal{S} \mapsto \mathcal{S}$ identity maps:

$$\begin{bmatrix} K & \mathbf{B}' & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & -\mathbf{I}_{\mathcal{D}} \\ \mathbf{0} & -\mathbf{I}_{\mathcal{S}} & E \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \ell \\ \mathbf{d} \\ \mathbf{0} \end{bmatrix} \tag{37}$$

References [55, 56] provide a discussion about existence and uniqueness of the solution:

$$(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{e}) \in \mathcal{L} \times \Sigma \times \mathcal{D} \,, \tag{38}$$

and about continuous dependence on the data:

$$(\mathbf{w}, \boldsymbol{\eta}, \ell) \in (\mathcal{V}, \mathcal{D}, F) \,. \tag{39}$$

A solution of the local elastic problem Eq. (33) is a stationary point for the three-field functional $\Xi : \mathcal{L} \times \Sigma \times \mathcal{D} \mapsto \Re$:[3]

$$\begin{aligned} \Xi(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{e}) := &\tfrac{1}{2}\langle K(\mathbf{v}), \mathbf{v}\rangle + \tfrac{1}{2}\langle E(\mathbf{e}), \mathbf{e}\rangle \\ &+ \langle \boldsymbol{\sigma}, \mathbf{B}(\mathbf{v}) - \mathbf{e} - \mathbf{d}\rangle - \langle \ell, \mathbf{v}\rangle \,. \end{aligned} \tag{40}$$

By imposing a priori fulfilment of the constitutive law Eq. $(33)_3$ we get the two-field functional:[4]

$$\begin{aligned} \Xi(\mathbf{v}, \boldsymbol{\sigma}) := &\tfrac{1}{2}\langle K(\mathbf{v}), \mathbf{v}\rangle - \tfrac{1}{2}\langle E^{-1}(\boldsymbol{\sigma}), \boldsymbol{\sigma}\rangle \\ &+ \langle \boldsymbol{\sigma}, \mathbf{B}(\mathbf{v}) - \mathbf{d}\rangle - \langle \ell, \mathbf{v}\rangle \,. \end{aligned} \tag{41}$$

whose saddle point is solution of the problem:

$$\begin{bmatrix} K & \mathbf{B}'_{\mathcal{L}} \\ \mathbf{B}_{\mathcal{L}} & -E^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \ell \\ \mathbf{d} \end{bmatrix} \tag{42}$$

Mathematical well-posedness results are available for the two-field formulation, due to symmetry and coerciveness of the linear operators $K : \mathcal{V} \mapsto F$ and $E^{-1} : \Sigma \mapsto \mathcal{D}$.[5]

By imposing fulfilment *a priori* of the kinematic compatibility law Eq. $(33)_2$, we get the one-field functional:[6]

$$\begin{aligned} \Xi(\mathbf{v}) := &\tfrac{1}{2}\langle K(\mathbf{v}), \mathbf{v}\rangle \\ &+ \tfrac{1}{2}\langle E(\mathbf{B}(\mathbf{v}) - \mathbf{d}), \mathbf{B}(\mathbf{v}) - \mathbf{d}\rangle \\ &- \langle \ell, \mathbf{v}\rangle \,, \end{aligned} \tag{43}$$

whose minimum point is solution of the variational problem:

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3 Hu-Washizu-Fraeijs de Veubeke functional [57–59].

4 Hellinger-Prange-Reissner functional [61–63].

5 In non-finite dimensional Hilbert spaces, positive definiteness is to be replaced by the stronger assumption of coerciveness. Existence proofs in local elasticity are based on closedness of the kinematic operator [55].

6 The *displacement functional* $\Xi : \mathcal{L} \mapsto \Re$ is customarily named *total potential energy*, with an improper and misleading terminology, since no potential of the load ℓ is required to exist and the term $\langle \ell, \delta\mathbf{v}\rangle$ is a virtual work.

$$\langle K(\mathbf{v}), \delta\mathbf{v} \rangle + \langle E\mathbf{B}(\mathbf{v}), \mathbf{B}(\delta\mathbf{v}) \rangle = \langle \ell, \delta\mathbf{v} \rangle + \langle E(\mathbf{d}), \mathbf{B}(\delta\mathbf{v}) \rangle \,, \tag{44}$$

for all $\delta\mathbf{v} \in \mathcal{L}$, equivalent to the linear problem in $\mathcal{L}$:

$$\Big(\mathbf{B}'_{\mathcal{L}} E \mathbf{B}_{\mathcal{L}} + K \Big)(\mathbf{v}) = \ell + \mathbf{B}'_{\mathcal{L}} E(\mathbf{d}) \,. \tag{45}$$

Well-posedness of the problem Eq. (45) is a standard result of the linear theory of local elastostatics [60].

Existence of a solution of the elastic problem Eq. (45) is directly inferred from the property that the continuous governing operator:

$$\mathbf{A}_{\mathcal{L}} := \Big(\mathbf{B}'_{\mathcal{L}} E \mathbf{B}_{\mathcal{L}} + K \Big) : \mathcal{L} \mapsto \mathcal{L}' \,, \tag{46}$$

fulfils the coerciveness inequality:

$$\langle \mathbf{A}_{\mathcal{L}}(\mathbf{v}), \mathbf{v} \rangle \geq c \, \| \mathbf{v} \|^2_{\mathcal{V}/\mathrm{Ker}(\mathbf{A}_{\mathcal{L}})} \,, \quad \forall\, \mathbf{v} \in \mathcal{L} \,. \tag{47}$$

Then the range of $\mathbf{A}_{\mathcal{L}} = \mathbf{A}'_{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{L}'$ is closed and

$$\begin{cases} \mathrm{Im}(\mathbf{A}_{\mathcal{L}}) = \mathrm{Ker}(\mathbf{A}'_{\mathcal{L}})^{\circ} \,, \\ \mathrm{Ker}(\mathbf{A}_{\mathcal{L}}) = {}^{\circ}\mathrm{Im}(\mathbf{A}_{\mathcal{L}}) \,. \end{cases} \tag{48}$$

Uniqueness holds to within rigid, conforming and elastically ineffective displacements,[7] since by symmetry and positive definiteness of $E : \mathcal{D} \mapsto \Sigma$: and $K : \mathcal{L} \mapsto \mathcal{L}'$:

$$\mathrm{Ker}(\mathbf{A}_{\mathcal{L}}) = \mathrm{Ker}(\mathbf{B}_{\mathcal{L}}) \cap \mathrm{Ker}(\mathbf{K}) \,. \tag{49}$$

Existence holds for any load in equilibrium under firm and elastic constraints, that is fulfilling the condition in Eq. (32).

5 Nonlocal constitutive response

A nonlocal constitutive relation is expressed by a variational rule involving a source field $s \in \mathcal{S}$, an output field $f \in \mathcal{F}$ over the domain $\boldsymbol{\Omega}$, a continuous response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ and a class of test fields $\delta s \in \delta\mathcal{S}$ belonging to a suitable linear test space $\delta\mathcal{S} \subseteq \mathcal{S}$:[8]

$$\langle f, \delta s \rangle = \langle \mathcal{R}(s), \delta s \rangle \,, \quad \forall\, \delta s \in \delta\mathcal{S} \,. \tag{50}$$

The response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ is an injective mapping between Hilbert spaces $\mathcal{S}$ and $\mathcal{F} = \mathcal{S}'$ in topological duality.

The fields in $\mathcal{S}$ are square integrable from $\boldsymbol{\Omega}$ to a target finite dimensional Hilbert space $\mathcal{H}$.

Denoting by $\boldsymbol{\mu}$ the standard volume form in $\boldsymbol{\Omega}$ and by $(\bullet, \bullet)_{\mathcal{H}}$ the inner product in $\mathcal{H}$, the induced inner product in the source space $\mathcal{S} := L^2(\boldsymbol{\Omega}; \mathcal{H})$ reads:

$$(s_1, s_2)_{\mathcal{S}} := \int_{\boldsymbol{\Omega}} (s_{1\mathbf{x}}, s_{2\mathbf{x}})_{\mathcal{H}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \,. \tag{51}$$

The response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ is not required to be invertible. Even when the inverse $\mathcal{R}^{-1} : \mathcal{F} \mapsto \mathcal{S}$ exists, it may not be explicitly available.

By Riesz-Fréchet theorem, between an Hilbert space $\mathcal{S}$ and its topological dual $\mathcal{F} = \mathcal{S}'$ there exists a symmetric and positive definite isometric isomorphism $\mathbf{J} : \mathcal{S} \mapsto \mathcal{F}$ defined by:

$$\begin{aligned} &\langle \mathbf{J}(s_1), s_2 \rangle = \langle \mathbf{J}(s_2), s_1 \rangle = (s_1, s_2)_{\mathcal{S}} \,, \\ &s \neq \mathbf{0} \Rightarrow \langle \mathbf{J}(s), s \rangle > 0 \,. \end{aligned} \tag{52}$$

When the space $\mathcal{S}$ is formed by square integrable fields on $\boldsymbol{\Omega}$, the dual Hilbert spaces $\mathcal{S}$ and $\mathcal{F}$ are usually both identified with a pivot space and the isometric isomorphism $\mathbf{J} : \mathcal{S} \mapsto \mathcal{F}$ is treated as if it were the identity.

In continuum mechanics this identification is however neither feasible nor advisable due to distinct physical dimensions of source and target fields, $s \in \mathcal{S}$ and $f \in \mathcal{F}$, which should then be kept well separated.

If the injection $\delta\mathcal{S} \hookrightarrow \mathcal{S}$ is continuous and dense, the variational condition Eq. (50) can be written as an equality:

$$f = \mathcal{R}(s) \,. \tag{53}$$

For applications to nonlocal elasticity, the response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ is required to be the (Gâteaux) gradient of a scalar potential $\mathcal{U} : \mathcal{S} \mapsto \Re$:

$$\langle \mathcal{R}(s), \delta s \rangle = \langle d\mathcal{U}(s), \delta s \rangle \,, \quad \forall\, \delta s \in \delta\mathcal{S} \,, \tag{54}$$

where the operator d is the Gâteaux directional derivative:

[7] Here *elastically ineffective* means that no interaction with the elastic constraint is activated.

[8] In this paper the symbol δ has no special meaning by itself. Adopted as a prefix, it denotes test fields belonging to linear test spaces.

$$\langle d\mathcal{U}(s), \delta s\rangle = \lim_{\lambda\to 0} \frac{\mathcal{U}(s+\lambda\cdot\delta s) - \mathcal{U}(s)}{\lambda} = \partial_{\lambda=0}\,\mathcal{U}(s+\lambda\cdot\delta s)\,. \tag{55}$$

Necessary and sufficient for potentiality Eq. (54) to hold for every $\delta s \in \mathcal{S}$ is that the Gâteaux derivative of $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ be linear and symmetric [64]:[9]

$$\langle d\mathcal{R}(s)\cdot s_1, s_2\rangle = \langle d\mathcal{R}(s)\cdot s_2, s_1\rangle\,. \tag{56}$$

In particular, if the response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ is linear, we have that:

$$d\mathcal{R}(s) = \mathcal{R}\,, \tag{57}$$

and the symmetry condition Eq. (56) is tantamount to symmetry of the linear operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$:

$$\langle \mathcal{R}(s_1), s_2\rangle = \langle \mathcal{R}(s_2), s_1\rangle\,, \tag{58}$$

which admits then the quadratic potential $\mathcal{U} : \mathcal{S} \mapsto \Re$:

$$\mathcal{U}(s) = \tfrac{1}{2}\langle \mathcal{R}(s), s\rangle\,. \tag{59}$$

Under validity of Eq. (54) with the potential in Eq. (59) the law in Eq. (50) is equivalent to:

$$\langle f, \delta s\rangle = \langle d\mathcal{U}(s), \delta s\rangle\,, \quad \forall\, \delta s \in \delta\mathcal{S}\,. \tag{60}$$

In this general framework, the treatments exposed in the sequel may be applied to all models of nonlocal elasticity proposed in literature.

6 Integral convolution

Nonlocal elasticity involving an integral convolution with a smoothing kernel was first proposed in [2, 5] by adopting the strain-driven bias of Sect. 11.

This choice leads however to non-solvable nonlocal elastic problems, as pointed out in [16].

An innovative method has been recently proposed in [6, 7] by adopting the stress-driven bias of Sect. 11.

The new paradigm has the merit that, by its adoption, the drawbacks of the strain-driven model are manifestly overcome.

To properly describe integral convolutions, we consider a source space $\mathcal{S}$ of tensor fields $s : \boldsymbol{\Omega} \mapsto \mathcal{H}$, so that $s_{\mathbf{x}} = s(\mathbf{x}) \in \mathcal{H}_{\mathbf{x}}$, the finite dimensional linear space of tensors based at $\mathbf{x} \in \boldsymbol{\Omega}$.[10]

The dual tensor space of $\mathcal{H}$ is denoted by $\mathcal{H}'$.[11]

In nonlocal constitutive relations the point value at $\mathbf{x} \in \boldsymbol{\Omega}$ of the output field $f : \boldsymbol{\Omega} \mapsto \mathcal{H}'$ is got by an integral convolution, on the domain $\boldsymbol{\Omega}$, between the source field $s : \boldsymbol{\Omega} \mapsto \mathcal{H}$ and a smooth kernel operator:

$$\boldsymbol{\phi}(\mathbf{x},\mathbf{y}) : \mathcal{H}_{\mathbf{y}} \mapsto \mathcal{H}'_{\mathbf{x}}\,. \tag{61}$$

The law in Eq. (53) may then be written as:

$$f_{\mathbf{x}} = \mathcal{R}(s)_{\mathbf{x}} = (\boldsymbol{\phi} * s)(\mathbf{x}) := \int_{\boldsymbol{\Omega}} \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) \cdot s(\mathbf{y}) \cdot \boldsymbol{\mu}_{\mathbf{y}}\,. \tag{62}$$

Here $\boldsymbol{\mu}$ is the involved volume form and the lower index in $\boldsymbol{\mu}_{\mathbf{y}}$ specifies that integration is performed with respect to the $\mathbf{y}$ variable. Moreover, the asterisk $*$ denotes an integral convolution and the dot $\cdot$ means linear dependence. Accordingly, the source field is a density *per unit volume*.

We also emphasise that integration is intended to be performed by means of an assumed distant parallel transport.

This is a peculiar and delicate conceptual point in the theory of nonlocal continua, never enlightened in the relevant literature.

Symmetry of linear response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$, expressed by Eq. (58), is implied by following symmetry properties of the kernel field [1]:

$$\begin{cases} \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) = \boldsymbol{\phi}(\mathbf{y},\mathbf{x})\,, & \forall\, \mathbf{x},\mathbf{y} \in \boldsymbol{\Omega}\,, \\ \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) = \boldsymbol{\phi}^A(\mathbf{x},\mathbf{y})\,, & \end{cases} \tag{63}$$

where the adjoint

$$\boldsymbol{\phi}^A(\mathbf{x},\mathbf{y}) : \mathcal{H}_{\mathbf{x}} \mapsto \mathcal{H}'_{\mathbf{y}}\,, \tag{64}$$

is intended with respect to duality between $\mathcal{H}$ and $\mathcal{H}'$:

[9] This result is a direct corollary of Volterra's theorem [65] usually improperly attributed and quoted in literature as Poincaré lemma [66].

[10] The choice of the compact manifold $\boldsymbol{\Omega}$ is a challenging point in the theory of nonlocal elasticity since it plays a basic role in the whole treatment.

[11] Strictly speaking, $\mathcal{H}$ is a tensor bundle over the manifold $\boldsymbol{\Omega}$ and $\mathcal{H}'$ is the dual bundle, with the fiber $\mathcal{H}'_{\mathbf{x}}$ dual to the fiber $\mathcal{H}_{\mathbf{x}}$, for all $\mathbf{x} \in \boldsymbol{\Omega}$.

$$\langle \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) \cdot s_{1\mathbf{y}}, s_{2\mathbf{x}} \rangle = \langle \boldsymbol{\phi}^A(\mathbf{x},\mathbf{y}) \cdot s_{2\mathbf{x}}, s_{1\mathbf{y}} \rangle \,. \tag{65}$$

Under fulfilment of the symmetry in Eq. (58), the linear response $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ can be expressed as a gradient by Eq. (54) with the scalar quadratic potential $\mathcal{U} : \mathcal{S} \mapsto \Re$ given by Eq. (59), and explicitly [1]:

$$\mathcal{U}(s) := \frac{1}{2} \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) \cdot s(\mathbf{y}), s(\mathbf{x}) \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \,. \tag{66}$$

To prove existence of a quadratic elastic potential $\mathcal{U} : \mathcal{S} \mapsto \Re$ and to detect its expression Eq. (66), we rely on the symmetry properties Eq. (63) and on Fubini's theorem on exchange of iterated integrals [42, p.18]:

$$\begin{aligned} \langle \mathcal{R}(s_1), s_2 \rangle &= \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi} * s_1, s_2 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) \cdot s_{1\mathbf{y}}, s_{2\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}^A(\mathbf{x},\mathbf{y}) \cdot s_{2\mathbf{x}}, s_{1\mathbf{y}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{y},\mathbf{x}) \cdot s_{2\mathbf{y}}, s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{y},\mathbf{x}) \cdot s_{2\mathbf{y}}, s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{x},\mathbf{y}) \cdot s_{2\mathbf{y}}, s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi} * s_2, s_1 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} = \langle \mathcal{R}(s_2), s_1 \rangle \,. \end{aligned} \tag{67}$$

7 Nonlocal homogeneous elasticity

In local elasticity the constitutive operator $C : \mathcal{S} \mapsto \mathcal{F}$, is pointwise defined:

$$C_{\mathbf{x}} : \mathcal{S}_{\mathbf{x}} \mapsto \mathcal{F}_{\mathbf{x}} \,, \tag{68}$$

and is symmetric and positive definite:

$$\begin{cases} C = C^A \iff \langle C(s_{1\mathbf{x}}), s_{2\mathbf{x}} \rangle = \langle C(s_{2\mathbf{x}}), s_{1\mathbf{x}} \rangle \,, \\ s_{\mathbf{x}} \neq \mathbf{0} \Rightarrow \langle C(s_{\mathbf{x}}), s_{\mathbf{x}} \rangle > 0 \,. \end{cases} \tag{69}$$

Elasticity is termed homogeneous if the constitutive operator $C : \mathcal{S} \mapsto \mathcal{F}$ is uniform in $\boldsymbol{\Omega}$:

$$C_{\mathbf{x}} = C_{\mathbf{y}} = C \,, \quad \forall \, \mathbf{x}, \mathbf{y} \in \boldsymbol{\Omega} \,, \tag{70}$$

where equality is meant to be evaluated by means of the chosen distant parallel transport from $\mathbf{x} \in \boldsymbol{\Omega}$ to $\mathbf{y} \in \boldsymbol{\Omega}$ or vice versa.

In integral convolution models of homogeneous nonlocal elasticity, the kernel is taken to be given by the composition:

$$\boldsymbol{\phi}(\mathbf{x},\mathbf{y}) = C \cdot \varphi(\mathbf{x},\mathbf{y}) \,, \tag{71}$$

of the constitutive operator $C : \mathcal{S} \mapsto \mathcal{F}$ with a scalar kernel $\varphi : \boldsymbol{\Omega} \times \boldsymbol{\Omega} \mapsto \Re$, symmetric and positive:

$$\varphi(\mathbf{x},\mathbf{y}) = \varphi(\mathbf{y},\mathbf{x}) > 0 \,, \quad \forall \, \mathbf{x}, \mathbf{y} \in \boldsymbol{\Omega} \,. \tag{72}$$

The nonlocal response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ takes then the expression:

$$\mathcal{R} := \varphi * C \,. \tag{73}$$

and explicitly

$$\mathcal{R}(s) := \varphi * C(s) \,. \tag{74}$$

The standard local elasticity model can be included as an asymptotic trend by assuming that the kernel depends on a scale parameter $\lambda > 0$ and that the following *impulsivity condition* (IC) holds for any source field $s \in \mathcal{S}$ which is continuous at $\mathbf{x} \in \boldsymbol{\Omega}$:[12]

$$\begin{aligned} \lim_{\lambda \to 0^+} (\varphi_\lambda * s)_{\mathbf{x}} &= \lim_{\lambda \to 0^+} \int_{\boldsymbol{\Omega}} \varphi_\lambda(\mathbf{x},\mathbf{y}) \cdot s_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= \Theta \, s_{\mathbf{x}} \,, \quad \forall \, \mathbf{x} \in \boldsymbol{\Omega} \,. \end{aligned} \tag{76}$$

A direct investigation on boundary effects [6, 19] reveals that at inner points in $\boldsymbol{\Omega}$:

- $\Theta = 1$,

while on the boundary $\partial\boldsymbol{\Omega}$ of $\boldsymbol{\Omega}$:

- $\Theta = 1/2$ at regular points,
- and $\Theta < 1/2$ and equal to the fraction of inward solid angle at singular points.

This means that for $\lambda \to 0^+$ the response operator tends to a Dirac impulse at interior points and to a

[12] The limit property in Eq. (76) provides the correct formulation of the usual kernel normalising condition in which integration is improperly extended over a phantom reference unbounded domain $\boldsymbol{\Omega}_\infty$, with the output equalled to $s_{\mathbf{x}}$:

$$\int_{\boldsymbol{\Omega}_\infty} \varphi(\mathbf{x},\mathbf{y}) \cdot s_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{y}} = s_{\mathbf{x}} \,. \tag{75}$$

fraction of it at boundary points, with a reduction factor at least one half.

The symmetry conditions Eq. (69) and Eq. (72) imply symmetry of the nonlocal response, according to Eq. (58).

Indeed the local response at $\mathbf{x} \in \boldsymbol{\Omega}$ is given by:

$$\mathcal{R}(s)_{\mathbf{x}} = \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot C(s_{\mathbf{y}}) \cdot \boldsymbol{\mu}_{\mathbf{y}} \,. \tag{77}$$

By Fubini's theorem for iterated integrals, we get:

$$\begin{aligned}
\langle \mathcal{R}(s_1), s_2 \rangle &:= \int_{\boldsymbol{\Omega}} \langle \varphi * C(s_1), s_2 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot \langle C(s_{1\mathbf{y}}), s_{2\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot \langle C(s_{2\mathbf{x}}), s_{1\mathbf{y}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{y}, \mathbf{x}) \cdot \langle C(s_{2\mathbf{y}}), s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{y}} \\
&= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{y}, \mathbf{x}) \cdot \langle C(s_{2\mathbf{y}}), s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot \langle C(s_{2\mathbf{y}}), s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&= \int_{\boldsymbol{\Omega}} \langle \varphi * C(s_2), s_1 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} = \langle \mathcal{R}(s_2), s_1 \rangle \,.
\end{aligned} \tag{78}$$

The uniform constitutive operator $C : \mathcal{S} \mapsto \mathcal{F}$ is taken equal to the appropriate symmetric and positive definite operator of the classical local theory of elasticity, depending on whether the strain-driven or the stress-driven model is adopted:

– strain-driven : $f = \boldsymbol{\sigma}$, $s = \mathbf{e}$ so that:

$$C = E : \mathcal{D} \mapsto \Sigma \,, \tag{79}$$

local elastic stiffness,

– stress-driven : $f = \mathbf{e}$, $s = \boldsymbol{\sigma}$ so that:

$$C = E^{-1} : \Sigma \mapsto \mathcal{D} \,, \tag{80}$$

local elastic compliance.

Existence of a quadratic elastic potential

$$\mathcal{U} : \mathcal{S} \mapsto \Re \,, \tag{81}$$

given by Eq. (59) is assured by symmetry of the bounded linear response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$, proven by Eq. (78) under fulfilment of properties in Eqs. (71), (72), (69).

No general result is however available, to our knowledge, to detect positive definiteness of the elastic potential. Uniformity in $\boldsymbol{\Omega}$ of the constitutive operator:

$$C : \mathcal{S} \mapsto \mathcal{F} \,, \tag{82}$$

expressed by Eq. (70) , is an essential property for carrying out the symmetry proof exposed in Eq. (78).

Therefore the nonlocal model set forth in Eq. (77) can only be applied to homogeneous elasticity problems if existence of a scalar potential is to be ensured.

For what concerns the impulsivity condition Eq. (76), we put the following observation.

Let $\xi : \boldsymbol{\Omega} \mapsto \mathrm{E}$ be a transformation whose co-restriction $\xi : \boldsymbol{\Omega} \mapsto \xi(\boldsymbol{\Omega})$ is a diffeomorphism.

The theory of integral transformation and the notion of push-forward $\xi\!\uparrow$ by $\xi : \boldsymbol{\Omega} \mapsto \xi(\boldsymbol{\Omega})$, give:[13]

$$\begin{aligned}
&\int_{\boldsymbol{\Omega}} \boldsymbol{\phi}(\mathbf{x}, \mathbf{y}) \cdot s(\mathbf{y}) \cdot \boldsymbol{\mu}_{\mathbf{y}} \\
&\quad = \int_{\xi(\boldsymbol{\Omega})} (\xi\!\uparrow\!\boldsymbol{\phi})(\xi(\mathbf{x}), \xi(\mathbf{y})) \cdot (\xi\!\uparrow\! s)(\xi(\mathbf{y})) \cdot (\xi\!\uparrow\!\boldsymbol{\mu})_{\xi(\mathbf{y})} \,.
\end{aligned} \tag{83}$$

By the transformation formula in Eq. (84), we infer that, for the impulsivity condition Eq. (76) to hold in a distorted domain, it is necessary that all involved fields are pushed forward.

8 Locality recovery

The assumption that, in homogeneous elastic bodies, the nonlocal response to a uniform source field should be equal to the output of the standard local model, was set forth in [27–29, 49, 50] in the context of a strain-

[13] The push forward of a vector field from $\boldsymbol{\Omega}$ to $\xi(\boldsymbol{\Omega})$ is its image through the tangent map, i.e. if $\mathbf{v}_{\mathbf{x}}$ is the velocity of a curve through $\mathbf{x} \in \boldsymbol{\Omega}$, the push forward is the velocity of the pushed curve at $\xi(\mathbf{x}) \in \xi(\boldsymbol{\Omega})$:

$$(\xi\!\uparrow\!\mathbf{v})_{\xi(\mathbf{x})} = (T_{\mathbf{x}}\xi) \cdot \mathbf{v}_{\mathbf{x}} \,, \quad \forall \mathbf{x} \in \boldsymbol{\Omega} \,. \tag{84}$$

The push-forward of a scalar field is defined by invariance and all other tensor fields are pushed accordingly.

driven perspective, and referred to as *locality recovery* (LR).[14]

Let us provide a synthetic abstract formulation of the proposals exposed in literature to get fulfilment of the LR property, expressed by the condition:

$$s_0 \in \mathcal{S} \quad \text{uniform in } \boldsymbol{\Omega} \Rightarrow \mathcal{R}(s_0) = C(s_0)\,. \tag{85}$$

Preliminarily it is expedient to observe that, for a kernel fulfilling Eq. (71), the integral $\Phi : \boldsymbol{\Omega} \mapsto \Re$ defined by:

$$\Phi_\lambda(\mathbf{x}) := \int_{\boldsymbol{\Omega}} \varphi_\lambda(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\mu}_{\mathbf{y}}\,, \tag{86}$$

is not uniform in the whole bounded domain $\boldsymbol{\Omega}$, due boundary effects.

Indeed, even when the kernel $\varphi_\lambda(\mathbf{x}, \mathbf{y})$ depends only on the distance $\|\mathbf{y} - \mathbf{x}\|$ and is rapidly decreasing to zero for $\mathbf{y} \in \boldsymbol{\Omega}$ away from the evaluation point $\mathbf{x} \in \boldsymbol{\Omega}$, nearby to the boundary a part of the effective domain is lost in the integration.

Moreover, the impulsivity condition Eq. (76) implies that:

$$\lim_{\lambda \to 0^+} \int_{\boldsymbol{\Omega}} \varphi_\lambda(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\mu}_{\mathbf{y}} = \lim_{\lambda \to 0^+} \Phi_\lambda(\mathbf{x}) = \Theta\,. \tag{87}$$

From Eq. (76) we infer that $\Theta = 1$ at inner points of $\boldsymbol{\Omega}$, while on the boundary $\Theta = 1/2$ at regular points, and $\Theta < 1/2$ and equal to the fraction of inward solid angle at singular points [19].

By positivity of φ_λ, assumed in Eq. (72), we have that:

$$0 < \Phi_\lambda(\mathbf{x}) \leq 1\,, \quad \forall\, \lambda > 0\,, \quad \forall\, \mathbf{x} \in \boldsymbol{\Omega}\,. \tag{88}$$

1. **Modified kernel**: A first proposal was set forth in [27] by assuming that:

   $$\mathcal{R}(s) = \boldsymbol{\psi}_\lambda * s\,, \tag{89}$$

   with the modified kernel $\boldsymbol{\psi}(\mathbf{x}, \mathbf{y}) : \mathcal{H} \mapsto \mathcal{H}'$ given by:

   $$\boldsymbol{\psi}_\lambda(\mathbf{x}, \mathbf{y}) := \frac{\boldsymbol{\phi}_\lambda(\mathbf{x}, \mathbf{y})}{\Phi_\lambda(\mathbf{x})} = \frac{\varphi_\lambda(\mathbf{x}, \mathbf{y})}{\Phi_\lambda(\mathbf{x})} \cdot C\,. \tag{90}$$

   The LR condition in Eq. (85) is checked by observing that, for any uniform source field $s_0 \in \mathcal{S}$, Eq. (86) gives:

   $$\begin{aligned} \mathcal{A}_\psi(s)_{\mathbf{x}} &= \int_{\boldsymbol{\Omega}} \boldsymbol{\psi}_\lambda(\mathbf{x}, \mathbf{y}) \cdot s_0 \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= \int_{\boldsymbol{\Omega}} \frac{\varphi_\lambda(\mathbf{x}, \mathbf{y})}{\Phi_\lambda(\mathbf{x})} \cdot C(s_0) \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= C(s_0)\,. \end{aligned} \tag{91}$$

   The modified kernel proposed in [27], expressed by Eqs. (86) and (90), does not fulfil the symmetry property:

   $$\boldsymbol{\psi}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\psi}(\mathbf{y}, \mathbf{x})\,. \tag{92}$$

   We confirm here the concern expressed in [1, 29], since this lack of symmetry breaks also the symmetry of the response operator.
   Indeed:

   $$\begin{aligned} \langle \mathcal{R}(s_1), s_2 \rangle &= \int_{\boldsymbol{\Omega}} \langle \mathcal{A}_\psi(s_1), s_2 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{x}, \mathbf{y}) \cdot s_{1\mathbf{y}}, \frac{s_{2\mathbf{x}}}{\Phi(\mathbf{x})} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}^A(\mathbf{x}, \mathbf{y}) \cdot \frac{s_{2\mathbf{x}}}{\Phi(\mathbf{x})}, s_{1\mathbf{y}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{y}, \mathbf{x}) \cdot \frac{s_{2\mathbf{y}}}{\Phi(\mathbf{y})}, s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{y}, \mathbf{x}) \cdot \frac{s_{2\mathbf{y}}}{\Phi(\mathbf{y})}, s_{1\mathbf{x}} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{x}, \mathbf{y}) \cdot s_{2\mathbf{y}}, \frac{s_{1\mathbf{x}}}{\Phi(\mathbf{y})} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}}\,. \end{aligned} \tag{93}$$

   Symmetry would instead require equality to:

   $$\begin{aligned} \langle \mathcal{R}(s_2), s_1 \rangle &= \int_{\boldsymbol{\Omega}} \langle \mathcal{R}(s_2), s_1 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\phi}(\mathbf{x}, \mathbf{y}) \cdot s_{2\mathbf{y}}, \frac{s_{1\mathbf{x}}}{\Phi(\mathbf{x})} \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}}\,. \end{aligned} \tag{94}$$

   Consequently, the response in Eq. (89) cannot be expressed as gradient of a potential.

2. **Modified response**: Another proposal to attain fulfilment of the LR condition of Eq. (85), originally set forth in [28, 29], consists in evaluating

[14] The *locality recovery* considered in [50] includes also the condition of a vanishing *energy residual*, see also [48, Eq. (4)]. We do not comment on this modification of the first principle of thermodynamics but just observe that the additional term therein is rather a *power residual*.

the special expression taken by the nonlocal response operator when acting on a uniform source field:

$$\begin{aligned} \mathcal{R}_{\text{NLOC}}(s_0)_{\mathbf{x}} &= \int_{\boldsymbol{\Omega}} \varphi_\lambda(\mathbf{x}, \mathbf{y}) \cdot C(s_0) \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= \Phi_\lambda(\mathbf{x}) \cdot C(s_0) \,. \end{aligned} \tag{95}$$

where we put $\boldsymbol{\phi}(\mathbf{x}, \mathbf{y}) = C \cdot \varphi_\lambda(\mathbf{x}, \mathbf{y})$. It follows that the LR property may be fulfilled by setting, for any $s \in \mathcal{S}$:[15]

$$\mathcal{R}(s) = (1 - \Phi_\lambda) \cdot C(s) + \mathcal{R}_{\text{NLOC}}(s) \,. \tag{96}$$

In fact, substituting Eq. (95) into Eq. (96) we get:

$$\mathcal{R}(s_0) = (1 - \Phi_\lambda) \cdot C(s_0) + \mathcal{R}_{\text{NLOC}}(s_0) = C(s_0) \,. \tag{97}$$

The modified response Eq. (96) fulfils the symmetry property Eq. (58) since the component operators C and $\mathcal{R}$ are both symmetric. Accordingly, the response may be expressed in terms of a potential, as in Eqs. (54), (59).
As claimed in [1, 28, 29, 49, 50], this is an improvement over the one based on Eq. (89).
The formulation in Eq. (96) is confined to homogeneous elastic bodies and consists in expressing the nonlocal response as sum of the local response plus a term vanishing when the source is uniform.
Therefore it is no more than a trivial escamotage. Moreover, the general scheme that will be exposed in Eq. (107) reveals that the model in Eq. (96) is essentially undetermined.

Strain-driven and stress-driven perspectives lead to two alternative models with well-distinct features also for the response modified to ensure fulfilment of the LR condition of Eq. (85) for homogeneous elastic problems.

As will be detailedly discussed in Sects. 15.1 and 15.2, when $\lambda \to 0^+$ the modified strain-driven model leads to ill-posed beam problems, while the modified stress-driven model leads to well-posed ones.

[15] This proposal was set forth in [28, 29] for a pure strain-driven model.

9 Combinations and mixtures

A combination of local/nonlocal responses, with positive parameters $0 \le \alpha, \beta : \boldsymbol{\Omega} \mapsto \Re$, is generated by setting:

$$\mathcal{R} = \alpha \cdot \mathcal{R}_{\text{LOC}} + \beta \cdot \mathcal{R}_{\text{NLOC}} \,. \tag{98}$$

In Sect. 9.3 a special combination will be considered by leaving the former parameter to be free to vary in $\boldsymbol{\Omega}$ so that $0 \le \alpha : \boldsymbol{\Omega} \mapsto \Re$, while requiring the latter to be uniform in $\boldsymbol{\Omega}$ so that $0 \le \beta : \boldsymbol{\Omega} \mapsto A \in \Re$. A mixture is a convex combination with uniform parameters $0 \le \alpha : \boldsymbol{\Omega} \mapsto m \in \Re$ and $0 \le \beta : \boldsymbol{\Omega} \mapsto 1 - m \in \Re$ so that:

$$\begin{aligned} &0 \le \alpha = m \le 1 \,, \\ &0 \le \beta = 1 - m \le 1 \,. \end{aligned} \tag{99}$$

For $m = 0$ or $m = 1$, the nonlocal integral convolution of Eq. (62) or the local law Eq. (24) are respectively recovered.

Usefulness of local/nonlocal combination or mixtures consists in the fact that well-posedness of the elastic equilibrium problem for strain-driven models is assured for $\alpha > 0$ and $\beta = 1$.

On the contrary, pure strain-driven models as a rule admit no solution, as discussed in [6, 7] and Sect. 15.1.

Weakness of mixture strain-driven models is that in nonlocal elastic problems singular behaviours are detected when the local fraction is quite small.

9.1 Local/nonlocal combination

A general combination of local/nonlocal responses is got by setting:

$$\begin{aligned} &\mathcal{R} = \alpha \cdot \mathcal{R}_{\text{LOC}} + \beta \cdot \mathcal{R}_{\text{NLOC}} \,, \quad \alpha, \beta \in \Re \,, \\ &\mathcal{R}_{\text{LOC}}(s) := C(s) \,, \\ &\mathcal{R}_{\text{NLOC}}(s) := \boldsymbol{\phi} * s \,. \end{aligned} \tag{100}$$

9.2 Local/nonlocal mixture

A mixture is a convex combination of local/nonlocal models and is got by setting $\alpha = m$ and $\beta = 1 - m$, with $0 \le m \le 1$, so that the response is given by:

$$\begin{aligned}
&\mathcal{R} = m \cdot \mathcal{R}_{\text{LOC}} + (1-m) \cdot \mathcal{R}_{\text{NLOC}} \,, \\
&\mathcal{R}_{\text{LOC}}(s) := C(s) \,, \qquad (101)\\
&\mathcal{R}_{\text{NLOC}}(s) := \boldsymbol{\phi} * s \,.
\end{aligned}$$

9.3 A special combination

In view of the model that will be discussed in Eq. (96), let us consider a more general combination characterised, as in Sect. 9, by a variable parameter:

$$0 \leq \alpha : \boldsymbol{\Omega} \mapsto \Re \,, \qquad (102)$$

and a uniform one:

$$0 \leq \beta : \boldsymbol{\Omega} \mapsto A \in \Re \,, \qquad (103)$$

so that:

$$\begin{aligned}
&\mathcal{R} = \alpha \cdot \mathcal{R}_{\text{LOC}} + \beta \cdot \mathcal{R}_{\text{NLOC}} \,, \\
&\mathcal{R}_{\text{LOC}}(s) := C(s) \,, \qquad (104)\\
&\mathcal{R}_{\text{NLOC}}(s) := \boldsymbol{\phi} * s = C(\varphi * s) \,.
\end{aligned}$$

The modified response, envisaged in Eq. (96) to fulfil the LR condition of Eq. (85), can be got from the special combination in Eq. (104) by setting:

$$\begin{cases}
\alpha = 1 - \Phi_\lambda \,, \\
\beta = 1 \,, \\
\mathcal{R}_{\text{LOC}} = C \,, \\
\mathcal{R}_{\text{NLOC}} = \boldsymbol{\phi}_\lambda * s \,.
\end{cases} \qquad (105)$$

The model in Eq. (96) is essentially undetermined since the LR condition of Eq. (85) can be still fulfilled by amplifying the nonlocal component with any uniform real factor $A \in \Re$:

$$\begin{cases}
\alpha = 1 - A\Phi_\lambda \,, \\
\beta = A \,, \\
\mathcal{R}_{\text{LOC}} = C \,, \\
\mathcal{R}_{\text{NLOC}} = \boldsymbol{\phi}_\lambda * s \,,
\end{cases} \qquad (106)$$

so that:

$$\mathcal{R}(s) = C(s) + A\left(\boldsymbol{\phi}_\lambda * s - \Phi_\lambda \cdot C(s)\right) . \qquad (107)$$

For instance, setting:

$$A = 0 \,,\; A = 1 \;\text{ and }\; A = -1 \,, \qquad (108)$$

the local elastic law, the nonlocal positively and the negatively modified response, are respectively recovered.

10 Nonlocal non-homogeneous elasticity

Nonlocal non-homogeneous elasticity problems cannot be properly treated by the model set forth in Eq. (71) since, as is clear from the symmetry proof exposed in Eq. (78), the non-uniformity of the constitutive operator $C : \mathcal{S} \mapsto \mathcal{F}$ destroys symmetry of the response operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ of Eq. (77).[16]

The obstruction can be circumvented by a proper definition of averaging kernel in the non-homogeneous case, according to the following original proposal set forth here by the author.

In order to introduce the new model, we preliminarily observe that the symmetric and positive definite local constitutive operator C, has a square-root $\sqrt{C}$ still symmetric and positive definite.

In the integral convolution models of nonlocal non-homogeneous elasticity, the kernel operator can be taken to be given by the composition of the symmetric and positive scalar kernel:

$$\varphi : \boldsymbol{\Omega} \times \boldsymbol{\Omega} \mapsto \Re \,, \qquad (109)$$

and of square roots of the constitutive operator $C : \mathcal{S} \mapsto \mathcal{F}$, as described below:[17]

$$\boldsymbol{\phi}(\mathbf{x}, \mathbf{y}) = \sqrt{C_{\mathbf{x}}} \cdot \varphi(\mathbf{x}, \mathbf{y}) \cdot \sqrt{C_{\mathbf{y}}} \,. \qquad (110)$$

This expression extends Eq. (71), relevant to homogeneity, and reduces to it when Eq. (70) holds true.

Accordingly, the response operator is given by:

$$\mathcal{R} := \sqrt{C} \cdot \left(\varphi * \sqrt{C}\right) , \qquad (111)$$

and explicitly:

[16] This fact is likely to motivate the choice in [1] where it is said: “For simplicity, we will restrict our attention to macroscopically homogeneous bodies”.

[17] After having independently envisaged this way of getting a symmetric kernel, the authors became aware of the fact that a similar trick, involving the non-uniform mass density of a rotating shaft, was adopted by Tricomi in [12, p.3, Eq. (4)].

$$\mathcal{R}(s)_{\mathbf{x}} = \int_{\boldsymbol{\Omega}} \left(\sqrt{C_{\mathbf{x}}} \cdot \varphi(\mathbf{x}, \mathbf{y}) \cdot \sqrt{C_{\mathbf{y}}} \right) \cdot s_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{y}} \,. \tag{112}$$

An evaluation similar to the one in Eq. (78) yields symmetry of the nonlocal response Eq. (111) :

$$\begin{aligned} \langle \mathcal{R}(s_1), s_2 \rangle &= \int_{\boldsymbol{\Omega}} \langle \mathcal{R}(s_1), s_2 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot \langle \sqrt{C_{\mathbf{y}}}(s_{1\mathbf{y}}), \sqrt{C_{\mathbf{x}}}(s_{2\mathbf{x}}) \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot \langle \sqrt{C_{\mathbf{x}}}(s_{2\mathbf{x}}), \sqrt{C_{\mathbf{y}}}(s_{1\mathbf{y}}) \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{y}, \mathbf{x}) \cdot \langle \sqrt{C_{\mathbf{y}}}(s_{2\mathbf{y}}), \sqrt{C_{\mathbf{x}}}(s_{1\mathbf{x}}) \rangle \cdot \boldsymbol{\mu}_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{y}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{y}, \mathbf{x}) \cdot \langle \sqrt{C_{\mathbf{y}}}(s_{2\mathbf{y}}), \sqrt{C_{\mathbf{x}}}(s_{1\mathbf{x}}) \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \int_{\boldsymbol{\Omega}} \varphi(\mathbf{x}, \mathbf{y}) \cdot \langle \sqrt{C_{\mathbf{y}}}(s_{2\mathbf{y}}), \sqrt{C_{\mathbf{x}}}(s_{1\mathbf{x}}) \rangle \cdot \boldsymbol{\mu}_{\mathbf{y}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \langle \mathcal{R}(s_2), s_1 \rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} = \langle \mathcal{R}(s_2), s_1 \rangle \,, \end{aligned} \tag{113}$$

so that existence of a quadratic elastic potential is assured:

$$\mathcal{U}(s) := \tfrac{1}{2} \langle \mathcal{R}(s), s \rangle \,. \tag{114}$$

When the constitutive operator is uniform, the formulae in Eqs. (112) and (113) boil down to the usual ones in Eqs. (77) and (78) valid for the homogeneous case.

The locality recovery condition cannot be extended to non-homogeneous nonlocal elastic models.

10.1 Another nonlocal non-homogeneous model

To deal with non-homogeneous nonlocal elasticity problems, the proposal in [49, 50] consists in expressing the total elastic potential $\mathcal{U} : \mathcal{S} \mapsto \Re$ as sum of two positive definite contributions, a local and a nonlocal quadratic potential $\mathcal{U}_{\text{LOC}} : \mathcal{S} \mapsto \Re$ and a nonlocal quadratic potential:

$$\begin{aligned} &\mathcal{U}_{\text{LOC}} : \mathcal{S} \mapsto \Re \,, \\ &\mathcal{U}_{\text{NLOC}} : \mathcal{S} \mapsto \Re \,, \end{aligned} \tag{115}$$

generated by the symmetric bilinear forms:

$$\begin{aligned} &\mathcal{E} : \mathcal{S} \times \mathcal{S} \mapsto \Re \,, \\ &\mathcal{W} : \mathcal{F} \times \mathcal{F} \mapsto \Re \,, \end{aligned} \tag{116}$$

according to the laws:

$$\begin{aligned} \mathcal{U}(s) &:= \mathcal{U}_{\text{LOC}}(s) + \mathcal{U}_{\text{NLOC}}(s) \,, \\ \mathcal{U}_{\text{LOC}}(s) &:= \tfrac{1}{2} \mathcal{E}\Big(s, s\Big) \,, \\ \mathcal{U}_{\text{NLOC}}(s) &:= \tfrac{1}{2} \mathcal{W}\Big(\mathcal{R}(s), \mathcal{R}(s)\Big) \,. \end{aligned} \tag{117}$$

The gradient of the potential $\mathcal{U}_{\text{NLOC}} : \mathcal{S} \mapsto \Re$ in Eq. $(117)_3$ is expressed by:

$$\begin{aligned} \langle d\mathcal{U}_{\text{NLOC}}(s), \delta s \rangle &= \langle d\mathcal{W}(\mathcal{R}(s)), d\mathcal{R}(s) \cdot \delta s \rangle \\ &= \langle d\mathcal{W}(\mathcal{R}(s)), \mathcal{R}(\delta s) \rangle \,. \end{aligned} \tag{118}$$

Then, by symmetry of the linear response $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ stated in Eq. (67), we have that $\mathcal{R}^A = \mathcal{R}$ and hence:

$$\begin{aligned} \mathcal{R}(s) = d\mathcal{U}(s) &= \mathcal{R}^A \cdot d\mathcal{W}(\mathcal{R}(s)) \\ &= \Big(\mathcal{R} \cdot d\mathcal{W} \cdot \mathcal{R}\Big)(s) \,. \end{aligned} \tag{119}$$

This expression is in agreement with the one given in [49, Eq. (18)] and [50, Eq. (13)] in the context of strain-driven models of nonlocal elasticity.

The nonlocal elasticity operator:

$$(\mathcal{R} \cdot d\mathcal{W} \cdot \mathcal{R}) : \mathcal{S} \mapsto \mathcal{F} \,, \tag{120}$$

is able to describe non-homogeneous problems, since the fields $\mathcal{E}$ and $d\mathcal{W}$ are not required to be uniform in the configuration $\boldsymbol{\Omega}$.

11 Strain-driven and stress-driven nonlocal elasticity

When the overall constitutive response $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ is not invertible, two distinct constitutive models are associated with the nonlocal law in Eq. (50), depending on which one of the following choices is made:

strain-driven :

$$\mathcal{R}_{\mathcal{D}} : \mathcal{D} \mapsto \Sigma \,, \quad s = \mathbf{e} \in \mathcal{D} \,, \quad f = \boldsymbol{\sigma} = \mathcal{R}_{\mathcal{D}}(\mathbf{e}) \,,$$

stress-driven :

$$\mathcal{R}_{\Sigma} : \Sigma \mapsto \mathcal{D} \,, \quad s = \boldsymbol{\sigma} \in \Sigma \,, \quad f = \mathbf{e} = \mathcal{R}_{\Sigma}(\boldsymbol{\sigma}) \,.$$

Adoption of the stress-driven perspective, introduced in [6, 7] as a paradigm for local and nonlocal elasticity, was prompted by an epistemological argument consisting in the following ansatz based on physical and mathematical motivations.

In elastic constitutive relations, the basic role of governing state variable is played by the stress field, while the elastic state field is just the output of the elastic constitutive law and therefore is not apt to play the role of a driving variable [38].

In the approximate context of a small-displacement theory, where all configurations are assumed to be coincident with a given fixed one, the elastic strain can be defined as an increment of elastic state due to an increment of stress state. It is convenient and natural to assume that the elastic state is vanishing when the stress state is such.

In the general framework of large dynamical processes, constitutive laws can only involve tensor fields representative, on the current configuration, of state variables and of their convective time derivatives (Lie derivatives) along the motion [40].

In fact, the difference between tensors pertaining, at different time instants, to the same particle along the spacetime dynamical trajectory, are performable only after a suitable pull-back or push forward along the motion is carried out to bring both to have the same base point on the trajectory.

In a dynamical process along a nonlinear trajectory manifold, stress fields and elastic state fields are well-defined state variables in the current configuration.

On the contrary, an elastic strain field is defined as increment of elastic state and makes naturally reference to a pair of distinct source and target configurations, with the increment evaluated after a pull back to a common configuration along the motion is carried out. Therefore an elastic strain field cannot be compared with stress state field since the latter pertains just to one configuration [39].

Elastic strains are not state variables.

The notion of elastic strain can usefully be introduced just for computational purposes when the nonlinear trajectory manifold is mapped, by means of a diffeomorphic space-time transformation, onto a straightened trajectory where all configurations mapped in an interval of time are identified with a given fixed reference manifold.

This is indeed the main tool in computational algorithms based on the *finite element method* (FEM), or similar ones, since there each element of a material mesh drawn on the moving body is mapped onto a fixed simplex or parallelepiped in a reference *alien* manifold, with the time variable playing the role of an evolution parameter.

The adjective *alien* means here that the reference manifold is not assumed to belong to the dynamical trajectory [40].

For all these reasons, the input of the rate elastic relation is the stress rate field while the output is the rate of elastic state.

In the context of a small-displacement theory, the output can be assumed to be the elastic strain, defined as finite increment of the elastic state.

12 Gradient models

Gradient models are based on the assumed existence on the configuration $\boldsymbol{\Omega}$ of a global potential $\mathcal{U}$ defined by integrating a local potential, according to the formula:

$$\mathcal{U}(s) := \int_{\boldsymbol{\Omega}} \mathcal{U}_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s) \cdot \boldsymbol{\mu}_{\mathbf{x}} \,. \tag{121}$$

The integrand is composed of the addition of two positive definite quadratic potentials:

$$\mathcal{U}_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s) = \mathcal{U}_{1\mathbf{x}}(s_{\mathbf{x}}) + \mathcal{U}_{2\mathbf{x}}(s) \,, \tag{122}$$

whose expressions are:

$$\begin{aligned} \mathcal{U}_{1\mathbf{x}}(s_{\mathbf{x}}) &= \frac{1}{2} \langle C_{\mathbf{x}} \cdot s_{\mathbf{x}}, s_{\mathbf{x}} \rangle \,, \\ \mathcal{U}_{2\mathbf{x}}(s_{\mathbf{x}}) &= \frac{1}{2} \alpha^2 \, \langle C_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} s, \nabla_{\mathbf{x}} s \rangle \,. \end{aligned} \tag{123}$$

Here $\alpha \geq 0$ is a scale parameter uniform over $\boldsymbol{\Omega}$, with the physical dimension of a length.

The linear constitutive operator:

$$C : \mathcal{S} \mapsto \mathcal{F} \,, \tag{124}$$

is symmetric and positive definite.

Applying integration by parts, and denoting ∇^A the formal adjoint of ∇, the constitutive law writes:

$$
\begin{aligned}
\langle f, \delta s\rangle &= \langle d\mathcal{U}(s), \delta s\rangle \\
&= \int_{\boldsymbol{\Omega}} \Big(\langle C \cdot s, \delta s\rangle_{\mathbf{x}} + \alpha^2 \, \langle C \cdot \nabla s, \nabla \delta s\rangle_{\mathbf{x}} \Big) \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&= \int_{\boldsymbol{\Omega}} \langle C \cdot s, \delta s\rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&\quad - \alpha^2 \int_{\boldsymbol{\Omega}} \langle \nabla^A (C \cdot \nabla s), \delta s\rangle_{\mathbf{x}} \cdot \boldsymbol{\mu}_{\mathbf{x}} \\
&\quad + \alpha^2 \oint_{\partial\boldsymbol{\Omega}} \langle C \cdot \nabla s \cdot \mathbf{n}, \delta s\rangle_{\mathbf{x}} \cdot \partial\boldsymbol{\mu}_{\mathbf{x}} \,,
\end{aligned}
\tag{125}
$$

The linear subspace $\delta\mathcal{S} \subset \mathcal{L}^2(\boldsymbol{\Omega}\,;\mathcal{H})$ of test fields is taken to be such to include the subspace $\mathrm{C}^\infty(\boldsymbol{\Omega}\,;\mathcal{H})$ of indefinitely smooth fields. The subspace $\mathrm{C}_0^\infty(\boldsymbol{\Omega}\,;\mathcal{H}) \subset \mathrm{C}^\infty(\boldsymbol{\Omega}\,;\mathcal{H})$, of smooth fields vanishing in a boundary layer, is dense in $\mathcal{L}^2(\boldsymbol{\Omega}\,;\mathcal{H})$.

Localisation of Eq. (125) is then performed by taking $\delta s \in \mathrm{C}_0^\infty(\boldsymbol{\Omega}\,;\mathcal{H})$ to infer validity of the constitutive differential law:

$$
f = C \cdot s - \alpha^2 \cdot \nabla^A (C \cdot \nabla s)\,. \tag{126}
$$

Then, taking $\delta\mathcal{S} = \mathrm{C}^\infty(\boldsymbol{\Omega}\,;\mathcal{H})$, from localisation of Eq. (125) we infer the constitutive boundary condition:

$$
\alpha^2 \cdot \nabla (C \cdot s) \cdot \mathbf{n} = \mathbf{0}\,, \tag{127}
$$

which, for $\alpha > 0$, expresses vanishing of the flux $\nabla (C \cdot s) \cdot \mathbf{n}$ of the local response across the boundary.

The previous treatment may be applied to both strain-gradient or stress-gradient models proposed in literature.

However, as soon as the theory is applied to elastostatic problems formulated according to one or the other of these models, a drastic difference becomes manifest.

12.1 Stress gradient models

Elastostatic problems based on stress-gradient constitutive models are affected by drawbacks similar to those of models based on the strain-driven Eringen's model.

Indeed, setting $s = \boldsymbol{\sigma}$, $f = \mathbf{e}$ and $C = E^{-1}$ in Eqs. (125) and (127), the differential constitutive condition Eqs. (126) becomes:

$$
\mathbf{e} = E^{-1} \cdot \Big(\boldsymbol{\sigma} - \alpha^2 \cdot \nabla^A \nabla \boldsymbol{\sigma} \Big)\,, \tag{128}
$$

and the constitutive boundary condition Eq. (127) writes:

$$
\alpha^2 \cdot \nabla \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}\,. \tag{129}
$$

This boundary condition is likely to conflict with equilibrium requirement on the stress field, thus leading to lack of solution.

Indeed, in statically determinate structural models, the stress field is univocally determined by the equilibrium condition.

In statically indeterminate models, the indeterminacy of elastic strain fields is fixed by the kinematic compatibility conditions on the geometric strain field.

Kinematic compatibility is conveniently imposed by recalling Eq. $(16)_2$ and Eq. $(36)_2$ and the definition in Eq. (35) of the *datum strain* $\mathbf{d} \in \mathcal{D}$:

$$
\mathbf{d} := \boldsymbol{\eta} - \mathbf{B}(\mathbf{w})\,. \tag{130}
$$

The kinematic compatibility is expressed by the following condition on the sum of the elastic and datum strains:

$$
\mathbf{e} + \mathbf{d} \in \mathbf{B}\mathcal{L}\,, \tag{131}
$$

and hence by the equivalent polarity condition:

$$
\langle \delta\boldsymbol{\sigma}_0, \mathbf{e} + \mathbf{d}\rangle = 0\,, \quad \forall\, \delta\boldsymbol{\sigma}_0 \in \Sigma_0 = (\mathbf{B}\mathcal{L})^\circ\,. \tag{132}
$$

Substituting Eq. (128) into Eq. (131) gives a linear algebraic system which yields the elastic strain field at solution.

The constitutive boundary condition Eq. (129) is then always redundant and unlikely to be fulfilled by equilibrated stress, which by definition belong to the linear variety $\boldsymbol{\sigma}_\ell + \Sigma_0$.

This obstruction may be circumvented by taking the test fields in the space $\delta\mathcal{S} = \mathrm{C}_0^\infty(\boldsymbol{\Omega}\,;\mathcal{H})$ of indefinitely differentiable fields with compact support in the open set $\boldsymbol{\Omega}$, so that the boundary integral in Eq. (125) vanishes and no constitutive boundary condition will emerge from localisation.

The adoption of this remedy leads to a simplest scheme of nonlocality that has been widely adopted in literature. However the elastic response based on this stress gradient model reproduces the standard elastic one for stress fields such that $\nabla^A \nabla \boldsymbol{\sigma} = \mathbf{0}$.

This feature was first evidenced in [11] with reference to simple beam problems. In the successive literature this fact was quoted as a *paradoxical* result since the governing differential relation Eq. (128) was erroneously interpreted as stemming from the strain-driven Eringen's model, expressed by Eq. (167) but ignoring the essential constitutive boundary conditions Eq. (168). If these conditions are taken into account the right conclusion is that the strain-driven Eringen's model for beam problems doesn't admit solution [16].

12.2 Strain gradient models

On the other hand, strain-gradient elastostatic problems, are formulated by setting $s = \mathbf{e}, f = \boldsymbol{\sigma}$, and $C = E$ in Eq. (125). The differential constitutive condition Eq. (126) becomes:

$$\boldsymbol{\sigma} = E \cdot \left(\mathbf{e} - \alpha^2 \cdot \nabla^A \nabla \mathbf{e} \right), \tag{133}$$

and the constitutive boundary condition Eq. (127) writes:

$$\alpha^2 \cdot \nabla \mathbf{e} \cdot \mathbf{n} = \mathbf{0}. \tag{134}$$

Contrary to the constitutive boundary condition in Eq. (129), the condition in Eq. (134) cannot be neglected since it is needed to detect a unique elastic strain field corresponding to an equilibrated stress field by means of Eqs. (133) and (134).

As in the case of stress gradient models, in statically indeterminate models, the indeterminacy of elastic strain fields is fixed by the kinematic compatibility conditions on the geometric strain field.

We note here the similarity between Eqs. (128) and (133), respectively expressing the differential constitutive conditions pertaining to stress-driven and strain-gradient models of elasticity. However a full constitutive analogy breaks down due to essential differences between the relevant constitutive boundary conditions, respectively given by Eqs. (129) and (134).

Strain gradient elasticity models, early proposed and investigated by Mindlin [51], have been recently commented upon in [52, 53] by considering an energy functional depending only on first and second gradients of the displacement field.

In all models quoted above, the strain field and its gradients should in fact be replaced with the elastic strain field and its gradients, since non-elastic strain fields (i.e. thermal ones) are not to be taken into account in modelling elasticity.

More properly, elastic state fields should be considered in place of elastic strain fields, in full conformity with the theory of elasticity developed in [38, 40].

In this respect, it should be underlined that elastic state fields do not have a their own physical definition as state variables other than the one coming from their appearance as output of an explicit constitutive relation expressed in terms of stress states.

It is disappointing that the formulation of strain gradient models exposed by Aifantis [67] is based on the assumption that the elastic potential depends on the elastic strain and on its image through the Laplace operator Δ:

$$\mathcal{U}(s)_{\mathbf{x}} = \mathcal{U}_{\mathbf{x}}(s, \Delta s), \quad \text{with} \quad s = \mathbf{e}. \tag{135}$$

The differential relation Eq. (126) was thence claimed to hold, with no clear derivation, and no statement was contributed concerning constitutive boundary conditions.[18]

13 Peridynamic models

A *relative displacement-driven* approach was first envisaged by Silling [68] and named *peridynamic* model, to propose a treatment of discontinuities in displacements and cracks [69]. When inertia effects are neglected, the ensuing models are termed *peri-static* [70].

In these models, the elastic energy is assumed to be composed of a standard *contact* energy depending on the elastic strain field and of a microelastic *long-range* interaction energy $w : \boldsymbol{\Omega} \times \boldsymbol{\Omega} \mapsto \Re$ assumed to depend on the relative position and displacement of each pair of particles $\mathbf{x}, \mathbf{y} \in \boldsymbol{\Omega}$, as expressed by $w(\mathbf{y} - \mathbf{x}, \mathbf{u}_{\mathbf{y}} - \mathbf{u}_{\mathbf{x}})$.

The density of the macroelastic energy at $\mathbf{x} \in \boldsymbol{\Omega}$ is then given by the resultant potential field:

[18] This assumption was there motivated by the statement that "*first gradients are suppressed as this would lead, in general, to third order tensors that previous linear models of gradient elasticity do not usually consider.*" We can see from Eq. (123) that it suffices to consider the first gradient as argument of the potential, to get the differential condition Eq. (126).

$$W_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}}) := \int_{\Omega} w(\mathbf{x}-\mathbf{y}\,,\mathbf{u}_{\mathbf{x}}-\mathbf{u}_{\mathbf{y}})\cdot\boldsymbol{\mu}_{\mathbf{y}}\,. \tag{136}$$

The global macroelastic energy is evaluated by the integral:

$$W(\mathbf{u}) := \int_{\Omega} W_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}})\cdot\boldsymbol{\mu}_{\mathbf{x}}\,. \tag{137}$$

We will not pursue here a detailed presentation of peridynamic models, but just point out a serious difficulty inherent to the continuum model.

In fact, when adopting the expression in Eqs. (136), (137), for bodies undergoing non-elastic processes (e.g. those involving thermal variations), displacement fields would improperly participate to the evaluation of the *long-range* elastic energy.

Until a proper answer (if any) will be given to these serious conceptual troubles, the peridynamic models cannot be considered as conceived and set up in a satisfactory manner for use in continuum mechanics.

14 Nonlocal elastic equilibrium

Let us discuss the strain-driven and the stress-driven nonlocal elasticity problem separately since the two models differ significantly in properties and in computational approaches.

14.1 Strain-driven nonlocal elasticity

The *strain-driven* nonlocal elastic problem is formulated, in terms of trial fields $\mathbf{v}\in\mathcal{L}$, $\boldsymbol{\sigma}\in\Sigma$, $\mathbf{e}\in\mathcal{D}$, and of test fields $\delta\mathbf{v}\in\mathcal{L}$, $\delta\boldsymbol{\sigma}\in\Sigma$, $\delta\mathbf{e}\in\mathcal{D}$, by simply replacing, in Eq. (36), the local elastic operator $E:\mathcal{D}\mapsto\Sigma$ with the response operator $\mathcal{R}_{\mathcal{D}}:\mathcal{D}\mapsto\Sigma$.

The elastic stiffness of the external constraints is for the moment assumed to be a linear, symmetric and positive definite operator $K:\mathcal{L}\mapsto\mathcal{L}'$, as in the local case described by Eqs. (25) and (26).

The elastostatic problem is then formulated by:

$$\begin{cases} \langle K(\mathbf{v}),\delta\mathbf{v}\rangle + \langle\boldsymbol{\sigma},\mathbf{B}(\delta\mathbf{v})\rangle = \langle\ell,\delta\mathbf{v}\rangle\,,\\ \langle\mathbf{B}(\mathbf{v}),\delta\boldsymbol{\sigma}\rangle - \langle\mathbf{e},\delta\boldsymbol{\sigma}\rangle = \langle\mathbf{d},\delta\boldsymbol{\sigma}\rangle\,,\\ \quad -\langle\boldsymbol{\sigma},\delta\mathbf{e}\rangle + \langle\mathcal{R}_{\mathcal{D}}(\mathbf{e}),\delta\mathbf{e}\rangle = 0\,. \end{cases} \tag{138}$$

where $\mathbf{d} := \boldsymbol{\eta} - \mathbf{B}(\mathbf{w})$.

In terms of the conforming displacement field $\mathbf{v}\in\mathcal{L}$ the nonlocal elastostatic problem can then be written as:

$$\begin{aligned} \langle K(\mathbf{v}),\delta\mathbf{v}\rangle &+ \langle\mathcal{R}_{\mathcal{D}}(\mathbf{B}(\mathbf{v})),\mathbf{B}(\delta\mathbf{v})\rangle \\ &= \langle\ell,\delta\mathbf{v}\rangle + \langle\mathcal{R}_{\mathcal{D}}(\mathbf{d}),\mathbf{B}(\delta\mathbf{v})\rangle\,, \end{aligned} \tag{139}$$

for all $\delta\mathbf{v}\in\mathcal{L}$.

An essential difficulty is however sneakily hidden therein.

Indeed, for $K=\mathbf{0}$ the variational problem in Eq. (138) admits, as a rule, no solution for pure strain-driven nonlocal models, where $\mathcal{R}_{\mathcal{D}} = \boldsymbol{\phi}\,*$.

This obstruction to existence of a solution is due to the fact that the output fields of the constitutive law $\boldsymbol{\sigma} = \mathcal{R}_{\mathcal{D}}(\mathbf{e})$, with $\mathbf{e} = \mathbf{B}(\mathbf{u})\in\mathcal{D}$ and $\mathbf{u}\in\mathbf{w}+\mathcal{L}$ may not be able to fulfil the equilibrium condition Eq. $(138)_1$ with $K=\mathbf{0}$, see [16].

As discussed in Sect. 15.1 with reference to simple beams, this obstruction may be partially overcome by adopting a local/nonlocal mixture model as in Eq. (101), with with $m>0$, or a modified response as in Eq. (96).

14.2 Stress-driven nonlocal elasticity

The *stress-driven* nonlocal elastic problem may be formulated in variational terms as a two field problem, with $\delta\mathbf{v}\in\mathcal{L}$ and $\delta\boldsymbol{\sigma}\in\Sigma$:[19]

$$\begin{cases} \langle K(\mathbf{v}),\delta\mathbf{v}\rangle + \langle\boldsymbol{\sigma},\mathbf{B}(\delta\mathbf{v})\rangle = \langle\ell,\delta\mathbf{v}\rangle\,,\\ \langle\mathbf{B}(\mathbf{v}),\delta\boldsymbol{\sigma}\rangle - \langle\mathcal{R}_{\Sigma}(\boldsymbol{\sigma}),\delta\boldsymbol{\sigma}\rangle = \langle\mathbf{d},\delta\boldsymbol{\sigma}\rangle\,. \end{cases} \tag{140}$$

The corresponding block-matrix expression is:

$$\begin{bmatrix} K & \mathbf{B}' \\ \mathbf{B} & -\mathcal{R}_{\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \ell \\ \mathbf{d} \end{bmatrix}. \tag{141}$$

Note that, by symmetry of K and $\mathcal{R}_{\Sigma}$, the structural operator at the l.h.s. of Eq. (141) is symmetric.

The problem can then be stated as stationarity property on the product space $\mathcal{V}\times\Sigma$ of the mixed complete quadratic functional extension of the Hellinger-Prange-Reissner functional to the nonlocal stress-driven context:

[19] A three-field formulation is not feasible, unless an explicit inverse of the nonlocal response operator is available.

$$\mathcal{H}(\mathbf{v},\boldsymbol{\sigma}) := \tfrac{1}{2}\langle K(\mathbf{v}),\mathbf{v}\rangle - \tfrac{1}{2}\langle \mathcal{R}_\Sigma(\boldsymbol{\sigma}),\boldsymbol{\sigma}\rangle + \langle \boldsymbol{\sigma},\mathbf{B}(\mathbf{v})\rangle - \langle \ell,\mathbf{v}\rangle - \langle \boldsymbol{\sigma},\mathbf{d}\rangle\,. \tag{142}$$

When $K=\mathbf{0}$, an equivalent formulation in terms of self-equilibrated stress fields $\boldsymbol{\sigma}_\mathbf{0} \in \Sigma_0$ and of a particular equilibrium stress $\boldsymbol{\sigma}_\ell \in \Sigma_\ell$ stems from Eq. (131) and is expressed by the condition that, for all $\delta\boldsymbol{\sigma}_0 \in \Sigma_0$:

$$\langle \mathcal{R}_\Sigma(\boldsymbol{\sigma}_\mathbf{0}),\delta\boldsymbol{\sigma}_0\rangle + \langle \mathbf{d}+\mathcal{R}_\Sigma(\boldsymbol{\sigma}_\ell),\delta\boldsymbol{\sigma}_0\rangle = 0\,. \tag{143}$$

The geometric interpretation of the problem in Eq. (143) is that the orthogonal projection on the subspace Σ_0 of the unknown vector $\mathcal{R}_\Sigma(\boldsymbol{\sigma}_\mathbf{0}) \in \mathcal{D}$ must be opposite to the one of the given vector:

$$\mathbf{d}+\mathcal{R}_\Sigma(\boldsymbol{\sigma}_\ell) \in \mathcal{D}\,. \tag{144}$$

Note that the special choice made for $\boldsymbol{\sigma}_\ell \in \Sigma_\ell$ is irrelevant.

When detection of a particular equilibrium stress $\boldsymbol{\sigma}_\ell \in \Sigma_\ell$ and a parametric description of the self-stress subspace Σ_0 are not available, or when $K \neq \mathbf{0}$, a two-field formulation (stress and small displacement) is compelling.

In nonlocal elasticity problems an applicable criterion able to assure coerciveness of the response operator $\mathcal{R}$ is presently lacking. If coerciveness holds, a solution will be a saddle-point of the convex-concave functional in Eq. (142).

14.3 Further considerations

Let us compare the formal structure of the elastostatic problems associated with stress and strain driven nonlocal models.

Preliminarily we recall from Eqs. (11) and (20) that:

$$\begin{cases} \Sigma_0 = (\mathbf{B}\mathcal{L})^\circ\,, \\ \mathbf{B}\mathcal{L} = {}^\circ\Sigma_0\,. \end{cases} \tag{145}$$

a. In a pure stress-driven nonlocal elastic model we have:

$$\mathbf{e} = \mathcal{R}_\Sigma(\boldsymbol{\sigma})\,, \tag{146}$$

and hence the conformity condition, according to Eq. (131), is given by:

$$\mathcal{R}_\Sigma(\boldsymbol{\sigma}) + \mathbf{d} \in \mathbf{B}\mathcal{L}\,. \tag{147}$$

Since equilibrium requires that

$$\boldsymbol{\sigma} \in \boldsymbol{\sigma}_\ell + \Sigma_0\,, \tag{148}$$

the condition of existence of a strain solution is expressed by:

$$\Big(\mathcal{R}_\Sigma(\boldsymbol{\sigma}_\ell) + \mathbf{d} + \mathcal{R}_\Sigma(\Sigma_0)\Big) \cap {}^\circ\Sigma_0 \neq \emptyset\,, \tag{149}$$

also written as:

$$\Big(\mathcal{R}_\Sigma(\boldsymbol{\sigma}_\ell) + \mathbf{d} + \mathcal{R}_\Sigma((\mathbf{B}\mathcal{L})^\circ)\Big) \cap \mathbf{B}\mathcal{L} \neq \emptyset\,. \tag{150}$$

Uniqueness of the strain solution requires that:

$$\mathcal{R}_\Sigma(\Sigma_0) \cap {}^\circ\Sigma_0 = \{\mathbf{0}\}\,, \tag{151}$$

which may also be written as:

$$\mathcal{R}_\Sigma((\mathbf{B}\mathcal{L})^\circ) \cap \mathbf{B}\mathcal{L} = \{\mathbf{0}\}\,. \tag{152}$$

b. In a pure strain-driven nonlocal elastic model, we have:

$$\boldsymbol{\sigma} = \mathcal{R}_\mathcal{D}(\mathbf{e})\,, \tag{153}$$

and the conformity condition, according to Eq. (131), requires that:

$$\mathbf{e} \in -\mathbf{d} + \mathbf{B}\mathcal{L}\,. \tag{154}$$

the condition of existence of a stress solution, recalling Eq. (148), is expressed by:

$$\Big(-\mathcal{R}_\mathcal{D}(\mathbf{d}) + \mathcal{R}_\mathcal{D}(\mathbf{B}\mathcal{L})\Big) \cap \Big(\boldsymbol{\sigma}_\ell + (\mathbf{B}\mathcal{L})^\circ\Big) \neq \emptyset\,. \tag{155}$$

Uniqueness of the stress solution requires that:

$$\mathcal{R}_\mathcal{D}(\mathbf{B}\mathcal{L}) \cap (\mathbf{B}\mathcal{L})^\circ = \{\mathbf{0}\}\,. \tag{156}$$

A full comprehension of intimate differences responsible for ill-posedness of pure strain-driven nonlocal elastic models versus the well-posedness pure stress-driven ones, as evidenced by computations in simple beam problems, is still lacking and certainly worth of further investigation.

In this context, we limit ourselves to observe that, in 1D beam theory, the subspace $\Sigma_0 \subset \Sigma$ of selfequilibrated stress fields is finite dimensional. This fact makes the condition in Eq. (155) much more stringent than the one in Eq. (150).

15 One-dimensional beam problems

In 1-D beam bending problems with axial abscissa:

$$a \leq x \leq b \,, \tag{157}$$

the integral convolution is conveniently set up by adopting the kernel defined by the bi-exponential map:

$$h_\lambda(x) := \frac{1}{2\,\lambda} \exp\left(-\frac{|x|}{\lambda}\right), \tag{158}$$

which fulfils the normalisation condition:

$$\lim_{\lambda \to 0^+} \int_a^b h_\lambda(x) \cdot dx = 1 \,. \tag{159}$$

The map in Eq. (158) is the fundamental solution associated with the linear differential operator:[20]

$$\frac{\nabla}{\lambda^2} - \nabla^2 \,. \tag{160}$$

The kernel of the integral convolution fulfils the symmetry property Eq. (72) being defined for all $a \leq x, y \leq b$ by the Green function:

$$\varphi_\lambda(x,y) := h_\lambda(x-y) = h_\lambda(y-x) \,. \tag{161}$$

The integral convolution model is then given by:

$$f(x) = \int_a^b \varphi(x,y) \cdot C \cdot s_y \, dy \,. \tag{162}$$

The constitutive law corresponding to local/nonlocal mixtures of Eq. (101) is equivalent to the differential equation:

$$\frac{f}{\lambda^2} - f'' = \frac{Cs}{\lambda^2} - m\,Cs'' \,, \tag{163}$$

with the boundary condition:

$$\begin{cases} f'(a) - \dfrac{f(a)}{\lambda} = m\left(Cs'(a) - \dfrac{Cs(a)}{\lambda}\right), \\[2ex] f'(b) + \dfrac{f(b)}{\lambda} = m\left(Cs'(b) + \dfrac{Cs(b)}{\lambda}\right). \end{cases} \tag{164}$$

15.1 Strain driven convolution

For a straight beam, a pure strain-driven convolution law is got by setting:

$$m = 0 \,, \quad s = \mathbf{e} \,, \quad f = \boldsymbol{\sigma} \,, \quad C = K \,, \tag{165}$$

where $\boldsymbol{\sigma} = M$ bending interaction, $\mathbf{e} = \chi$ elastic curvature and K uniform elastic bending stiffness, so that:

$$M(x) = \int_{\boldsymbol{\Omega}} \varphi_\lambda(x,y) \cdot K \cdot \chi_y \, dy \,. \tag{166}$$

Therefore the constitutive differential equation is:

$$\frac{M}{\lambda^2} - M'' = \frac{K\chi}{\lambda^2} \,, \tag{167}$$

and the constitutive boundary condition are:

$$\begin{cases} M'(a) - \dfrac{M(a)}{\lambda} = 0 \,, \\[2ex] M'(b) + \dfrac{M(b)}{\lambda} = 0 \,. \end{cases} \tag{168}$$

The constitutive differential law Eq. (167) with the constitutive boundary condition (168), provide an equivalent formulation of the convolution law Eq. (166).

In elastostatics the kinematic compatibility condition requires that:

$$\chi + \eta = u'' \,, \tag{169}$$

where η is an imposed (e.g. thermal) curvature and u is an admissible small displacement.

The differential problem Eqs. (167) and (168) yields the bending interaction corresponding to an elastic curvature fulfilling the kinematic compatibility condition:

Since bending interaction must also fulfil the equilibrium conditions, the constitutive boundary condition Eq. (168), which depend on the nonlocal parameter λ, are likely to conflict with the equilibrium boundary conditions, which are independent of the parameter λ.

[20] The nabla ∇ and the apex $'$ both denote differentiation with respect to x.

Therefore an unavoidable contrast arises in practice, so that no solution exists, as a rule, to the nonlocal elastostatic problem governed by the strain-driven integral law Eq. (166).

This obstruction may be overcome by adopting a local/nonlocal mixture model as in Eqs. (163) and (164) but with $m > 0$, to get the equivalent differential equation [6, 23]:

$$\frac{M}{\lambda^2} - M'' = \frac{K\chi}{\lambda^2} - m\,K\chi''\,, \tag{170}$$

with the boundary condition:

$$\begin{cases} M'(a) - \dfrac{M(a)}{\lambda} = m\left(K\chi'(a) - \dfrac{K\chi(a)}{\lambda}\right), \\ M'(b) + \dfrac{M(b)}{\lambda} = m\left(K\chi'(b) + \dfrac{K\chi(b)}{\lambda}\right). \end{cases} \tag{171}$$

15.2 Stress driven convolution

For a straight beam, a pure stress-driven convolution law is got by setting $m = 0$ (no mixture), $s = M$ bending interaction, $f = \chi$ elastic curvature, $C = K$ uniform elastic bending stiffness, $\boldsymbol{\sigma} = M$ bending interaction, so that:

$$\chi(x) = \int_{\Omega} \varphi_\lambda(x,y)\cdot K^{-1}\cdot M_y\,dy\,. \tag{172}$$

The equivalent constitutive differential equation is:

$$\frac{\chi}{\lambda^2} - \chi'' = \frac{K^{-1}M}{\lambda^2}\,, \tag{173}$$

with the constitutive boundary condition:

$$\begin{cases} \chi'(a) - \dfrac{\chi(a)}{\lambda} = 0\,, \\ \chi'(b) + \dfrac{\chi(b)}{\lambda} = 0\,. \end{cases} \tag{174}$$

The differential law Eqs. (173) and (174) yields the unique elastic curvature corresponding to a bending interaction fulfilling the equilibrium condition.

The constitutive law Eq. (104) is equivalent to the differential equation:

$$\frac{f}{\lambda^2} - f'' = C\left(\frac{\alpha+\beta}{\alpha\lambda^2}\,(\alpha s) - (\alpha s)''\right), \tag{175}$$

with the boundary condition:

$$\begin{cases} f'(a) - \dfrac{f(a)}{\lambda} = C\left((\alpha\, s)'(a) - \dfrac{(\alpha\, s)(a)}{\lambda}\right), \\ f'(b) + \dfrac{f(b)}{\lambda} = C\left((\alpha\, s)'(b) + \dfrac{(\alpha\, s)(b)}{\lambda}\right). \end{cases} \tag{176}$$

16 Nonlocal external elasticity

In some structural applications, the external elastic law relating displacement and constraint reaction fields can be conveniently assumed to be of a nonlocal type.

A classical example is provided by beams and plates resting on an elastic foundation.

A prototype local model of elastic foundation was proposed in the second half of the nineteenth century by Winkler [73] and by Zimmermann [74], assuming a symmetric, positive definite and local linear relation between continuous fields of displacements and constraint reactions in the domain $[a,b]$ of the beam [81]:

$$r(x) = K(x)\cdot u(x)\,, \quad x \in [a,b]\,. \tag{177}$$

16.1 Reaction-driven nonlocal external elasticity

A nonlocal model for an inflected straight beam resting on an elastic foundation was first introduced by Wieghardt [75] who criticised the presence of discontinuities in the displacement of the elastic foundation at the boundary of the support domain, due to vanishing of the soil displacement field outside the domain $[a,b]$, according to the local Winkler-Zimmermann model Eq. (177).

This criticiscìm is however improper because the elastic behaviour model in [73, 74] was concerned just with the beam-foundation interface in the domain $[a,b]$ and not with the whole elastic soil foundation.

The nonlocal model proposed in [75] adopts the kernel in Eq. (158), as suggested by Föppl in 1909 [76]. It can be named *reaction-driven*, being given by:

$$u(x) = \int_a^b \varphi_\lambda(x,y)\cdot K^{-1}\cdot r_y\,dy\,. \tag{178}$$

However the elastostatic beam problem ensuing from Eq. (178) is not well-posed.

Indeed the displacement field u, being common to the beam axis and to the supporting foundation, is required to be the output of the *reaction-driven* nonlocal law Eq. (178) and to fulfil the fourth-order differential equation of the beam elastic equilibrium in terms of the displacement field, under the action of the imposed loading and of the foundation reaction.

This is an impossible task in general.

The underlying obstruction is confirmed by the fact that when the kernel is assumed to be defined by Eqs. (158)–(161), the equivalent constitutive differential equation is:

$$\frac{u}{\lambda^2} - u'' = \frac{K^{-1} r}{\lambda^2}\,, \tag{179}$$

with the constitutive boundary condition:

$$\begin{cases} u'(a) - \dfrac{u(a)}{\lambda} = 0\,, \\ u'(b) + \dfrac{u(b)}{\lambda} = 0\,. \end{cases} \tag{180}$$

When the expression in Eq. (179) is substituted in the beam equation of equilibrium, the differential order remains four, but two more boundary conditions Eq. (180) do appear.

To overcome this obstruction, fictitious concentrated reactions to be added at the beam ends were proposed by Telemaco Langendonck [77] and by Alfredo Sollazzo [78].

An extension to shear deformable foundation beams, according to Timoshenko model, was contributed by Ylinen and Mikkola [79].

A remarkable extension to 2D foundations was contributed by Michele Capurso in the same year [80].

The presence of concentrated reactions at the boundary of the supporting elastic foundation remained however a questionable assumption.

16.2 Displacement-driven nonlocal external elasticity

An alternative nonlocal model for the interaction between an inflected beam and supporting elastic foundation has been recently proposed by Barretta [82].

The proposal consists in a swap akin to the one introduced in [18] and permits to overcome the obstructions resident in the Wieghardt [75] nonlocal model, without advocating the presence of concentrated support reactions.

The scheme, dual to the one in Eq. (178), adopts a *displacement-driven* nonlocal integral law:

$$r(x) = \int_a^b \varphi_\lambda(x, y) \cdot K \cdot u_y \, dy\,. \tag{181}$$

The equivalent constitutive differential equation is:

$$\frac{r}{\lambda^2} - r'' = \frac{Ku}{\lambda^2}\,, \tag{182}$$

with the constitutive boundary condition:

$$\begin{cases} r'(a) - \dfrac{r(a)}{\lambda} = 0\,, \\ r'(b) + \dfrac{r(b)}{\lambda} = 0\,. \end{cases} \tag{183}$$

The expression of the displacement field u, given in terms of the reaction r by Eq. (182), can then be placed in the equilibrium equation of the beam to get a six order differential equation in the unknown reaction field, with six boundary conditions, four kinematic-static plus two constitutive given by Eq. (183). The resulting nonlocal problem is thus well-posed [82].

17 Comments and conclusions

By the given description of the various adopted models for nonlocal elasticity problems, a critical examination is made available to validate, reject or improve various proposals in literature and to compare merits and difficulties.

Presently, we are not aware of mathematical statements providing effective criteria and operative tests concerning existence and uniqueness of the solution of nonlocal elastic problems that are not local.

Formulations of variational principles for both strain-driven and stress-driven models of nonlocal elasticity are therefore slightly more than formal exercises until these basic questions are not properly answered.

Although nonlocal formulation are especially challenging from the conceptual and the operative points of view, it is to be said that this kind of difficulty is common to many other engineering models of complex structural problems.

This fact makes engineers confident in that, physical insight and successive modifications suggested by manifest obstructions, may lead to solvable problems and to results that can be useful in applications.

In local and nonlocal elasticity problems, stress-driven constitutive models are to be considered as basic ones, since increments of elastic states are properly induced by increments of stress states [38–40].

In spite of the lack of a theoretical assessment, numerical evidence shows that a unique solution exists for nonlocal problems of applicative interest if the response operator is of the stress-driven kind and also for strain-driven models which are suitable mixtures of local/nonlocal laws.

Analytical expressions of the solution have been evaluated in simple cases, see e.g. [23].

Evidence of existence and uniqueness of a solution holds both for constitutive models formulated as integral convolution with homogeneous elasticity moduli, and for models deduced from the quadratic potential in Eq. (117), for non-homogeneous elasticity.

Iterative schemes of solution were early suggested by Polizzotto [71] and have been recently revised, thoroughly investigated, and reformulated in [72].

Both stress-driven and (mixture) strain-driven models were there considered, with effective applications to simple nonlocal elasticity problems, by proving equivalence between nonlocal problems and fixed points of suitable algorithms.

The computational tests on iterative schemes show remarkable convergence properties and in particular especially fast rates for stress-driven nonlocal models.

Iterative schemes of solution provide a valuable computational tool for nonlocal problems since at each step only standard local elasticity problems, with an imposed distortion field, need to be considered. In this way, standard computational tools can be resorted to.

We comment no further on currently most adopted models of nonlocal elasticity, leaving to the readers the task of taking a cue to perform additional considerations and draw their own meditated conclusions.

Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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