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ORIGINAL PAPER

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Solid–fluid interaction: a continuum mechanics assessment

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Abstract The dynamical interaction between solids and fluids is a subject of paramount importance in Mechanics with a wide range of applications to engineering problems. It is, however, still a challenging topic of theoretical investigation. With a view to case studies of dynamical behaviour of rockets, turbines, jets and sprinklers, we develop here a treatment that, in the full respect of the principle of conservation of mass and under suitable simplifying assumptions, leads to evaluate the thrusting force exerted by the fluid on the solid. The goal is reached by applying the Euler–d'Alembert law of continuum dynamics to the trajectory of a skeleton whose motion is an extension of the one of the solid. It is shown that the formulation in the context of continuum mechanics is essential to get a full understanding of the dynamical problem and for grasping the meaning and range of validity of the results. This is a distinctive feature from treatments in literature where particles or control windows with variable mass are considered. The statement of the von Buquoy–Meshchersky law as a governing principle in the dynamics of particles with variable mass, in substitution of Newton's second law, is critically addressed. Under the assumption of low mass and high momentum time rate, the formula for the thrusting force is validated as a simplified expression fulfilling Galilei's principle of relativity.

1 Introduction

Dynamical problems, in which the interaction between a fluid and a solid case in relative motion plays an essential role, have quite a long history, the most popular example being a rocket burning the conveyed propellant.

One of the earliest texts mentioning the use of rockets is the *Huolongjing* (Fire Dragon Manual), a fourteenth-century military treatise by Jiao Yu and Liu Bowen of the early Ming Dynasty (1368–1644) in China.

A comprehensive treatise on rocketry, the *Artis Magnae Artilleriae*, was elaborated in Europe as far back as 1650 by the Polish–Lithuanian general of artillery Kazimierz Siemienowicz.

Dynamical interactions between fluids and solids were investigated by Daniel Bernoulli in 1727 [1] and inserted in his celebrated treatise *Hydrodynamica* [2] of 1738. The British mathematician and military engineer

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Benjamin Robins invented the ballistic pendulum in 1742 and published the *New Principles of Gunnery*. Leonhard Euler, under commission of Frederick the Great, translated into German, commented and enlarged three-times Robins's treatise [3], soon after the publication of his masterpiece [4]. Euler's *Principia Motus Fluidorum* was published in 1761 [5]. Later on, specific contributions to rocketry were made by the British mathematician William Moore in his *Treatise on the Motion of Rockets* to which is added *An Essay on Naval Gunnery in Theory and in Practise*, G. & S. Robinson, London (1813) [6,7].

Our interest is here centred on the investigation about the dynamical behaviour of systems involving an interaction between a solid case and a fluid in relative motion, and more precisely in determining the law of motion of the solid case, as induced by the thrust exerted by the interacting fluid.

In such systems, the involved mass is possibly varying in time and therefore these systems are often referred to as *variable mass systems*.

However, many instances of interactive phenomena of interest in engineering applications do not belong to the category of *variable mass systems* since mass variation is not the significant characteristic of the dynamical system.

All these problems are conveniently investigated in the more general and appropriate framework of the dynamical interactions between fluid and solid trajectories (or possibly also solid–solid or fluid–fluid interactions).[1]

Priority in the treatment of mechanical systems of *atom*[2] particles with variable mass is attributed to the Czech Count Georg Franz von Buquoy who formulated the relevant equation of motion in 1812 [8, p. 66]. In August 1815, he presented his results at the Paris Academy of Sciences to Laplace, Poisson, Ampère, Delambre, Arago, Cauchy, Fourier and other savants [9,10]. Nevertheless, apart from a single short article by Poisson [11], his ideas did not attract attention, and gradually became forgotten [12].

Treatments of dynamical problems involving variable mass make also reference to the contributions by Konstantin Eduardovich Tsiolkovsky, founding father of Russian rocketry and astronautics, with his well-known *formula of aviation* [13,14], and to the equation for dynamics of *atom* particles with variable mass published in Russian, in the same year 1897, by Ivan Vsevolodovich Meshchersky [15], who cites [16,17].

A recent revival of interest in the topic is witnessed by many contributions in physics and engineering literature starting from the second half of the twentieth century [18–32].

Meshchersky equation is referred to in several treatments in the literature, see, e.g., Hadjidemetriou [33], McIver [34, p. 256, Eq. (41)], Mikhailov [35], Oates [36, p. 64, Eq. (3.3)], Irschik and Holl [37, p. 245, Eq. (6.10)] and Zhao and Yu [38, p. 713, Eqs. (1), (2)]. Significant contributions to variable mass problems were reviewed in [39–42].

Most treatments still adopt, in the spirit of Newtonian mechanics, a formulation in terms of atom particles with variable mass. We will see in Sect. 6 that this kind of approach may, however, lead to unphysical descriptions [43,44], with Galilei's principle of relativity [45] not obeyed.

In recent times, a special attention has been devoted to extending the formulation of classical action principles of dynamics to systems with variable mass [26,27,38,46–48], and valuable investigations have been contributed to the formulation of dynamical laws by means of control windows moving along a discretised dynamical trajectory [29,32,37].

The main motivation of the present contribution is to show that the equation governing the dynamics of a solid case, interacting with a fluid in relative motion, may be deduced, under peculiar simplifying assumptions, from the general equation of classical dynamics, see Sect. 5.

The point of departure consists in the application of Euler's law of motion to the complex system compound by the solid case and the interacting fluid. It is assumed that the motion of the solid can be extended to the motion of a larger *skeleton* which at each instant includes the portion of fluid interacting with the solid.

The analysis adopts the space-time formulation of classical dynamics, the one providing the theoretical framework suitable for a proper treatment.

The expression of the thrust exerted on the solid case by the fluid interacting with it is deduced while keeping validity of the basic principle of conservation of mass. Fulfilment of Galilei's principle of relativity is thus ensured. The spotlight of the formulation is pointed on the motion of the solid case, which is at the centre of the scene, while the motion of the fluid relative to the solid case is considered to be significant only for the evaluation of the dynamical thrust exerted on the solid case.

[1] The description of the dynamics of turbines, jets and sprinklers does not involve significant rate of mass change. Even in the dynamics of rockets, mass changes are only taken into account to evaluate the effect of propellant consumption resulting in a major loss of the rocket's weight, even more than 80 % during an entire flight.

[2] Here and in the sequel *atom* stands for *concentrated mass*.

This is the central interest in most important engineering applications of the theory including the dynamical behaviour of rockets, sprinklers and jets that can be directly dealt with by the continuum mechanics formulation.

A revisitation of basics of continuum kinematics is performed in Sect. 2 to introduce essential notions and notations in the suitable space-time framework [49–56].

The extension of the Euler law to continuum dynamics is illustrated in detail in Sect. 4, and the relevant d'Alembert formulation is inferred by relying on conservation of mass, so that Galilei's principle of relativity is fulfilled.

Solid–fluid interaction is investigated in Sect. 5. The dynamical problem as it stands is quite complex, and to get a sufficiently manageable description and a tractable governing law, peculiar simplifying assumptions must be made.

The essential step towards this simplified analysis, which leads to the proper assessment of the thrusting force, is based on fulfilment of the following peculiar conditions:

1. The domain of the solid motion is assumed to be extendable to a larger trajectory pertaining to a *skeleton* which includes the portion of the fluid interacting with the solid.
2. Conservation of mass is assumed to hold to a sufficient extent for the solid and for the fluid moving according to the skeleton motion and conceived so to make the thrusting force exerted by the fluid evaluable by a practicable computation.

Under these assumptions, the motion of the interacting fluid is expressed as left-composition of the skeleton motion with a relative motion, see Eq. (33).

The *skeleton* is a special *control window*, with a definite physical interpretation of its motion, which is conceived as a natural extension of the motion of the solid. Conservation of mass of the fluid filling the skeleton and fictitiously moving according to the skeleton motion can be assumed without significant loss of generality. Galilei's principle of relativity is then fulfilled by the motion of the complex made of solid case and fluid-filled skeleton, under the actions of dynamical force and thrust exerted by interacting fluid in relative motion.

The formulation of the problem in the context of continuum dynamics is shown to be essential to derive, from well-established general laws, a formula for the evaluation of the *thrust* which keeps its validity also when neither local nor global variations of mass are involved.

At first sight, the new expression for the thrust could appear to be the proper generalisation of the one introduced by von Buquoy, and later independently by Meshchersky, as governing rule of dynamics of particles with variable mass, in replacement of Newton's second law. Meaning and range of applicability of the new formula are, however, drastically different. The new formula is derived as an expression ensuing from standard dynamics by appealing to the simplifying assumptions listed at items 1 and 2 above. On the other hand, the range of applicability is substantially widened to include functioning of important machines not treatable by the particle dynamics formulation.

The new treatment is in accord with the physics of involved phenomena and is well suited for investigating problems of great technical interest, such as rockets, jet, turbines, sprinklers.

In all these applications, evaluation of the thrust does not depend on the rate of mass loss but rather on the rate of variation of kinetic momentum of the interacting fluid.

Accordingly, for these dynamical systems, the usual naming of *system with variable mass* should be modified into *system with low mass and high momentum time rate*.

As a matter of fact, in case of gross rates of mass loss, a satisfactory analysis could only be performed by means of a challenging dynamical treatment of the system compound by the solid case and the interacting fluid, with the specification of proper interface conditions.

Standard treatments carried out in the pertinent physics literature are summarised in the final Sect. 6 and addressed with a critical analysis.

For the readers convenience, the Appendix provides essential definitions and notions of differential geometry referred to in the mathematical treatment. This is in line with the authors' point of view that basic differential geometry provides general, clear and powerful tools for investigating in mechanics.

2 Kinematics in space-time

The theory of dynamics is best developed in the general framework of a 4D manifold of events $\mathbf{e} \in \mathcal{E}$ and of the relevant tangent bundle $T\mathcal{E}$ with projection[3] $\boldsymbol{\tau}_{\mathcal{E}} : T\mathcal{E} \mapsto \mathcal{E}$ which assign to each tangent vector $\mathbf{d}_{\mathcal{E}} \in T\mathcal{E}$ its base point in $\mathcal{E}$.

[3] A projection is a surjective map whose differentials are surjective.

Each observer performs a double foliation of the 4D events manifold $\mathcal{E}$ into complementary 3D *space slices* $\mathcal{S}$ of *isochronous* events (with a same corresponding time instant) and 1D *time lines* of *isotopic* events (with a same corresponding space location).

Time lines do not intersect one another and each time line intersects a space slice just at one point. Analogously, space slices do not intersect one another and each space slice intersects a time line just at one point.

Each *time line* is parametrised by time in such a way that a *time projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ assigns the same time instant $t_{\mathcal{E}}(\mathbf{e}) \in \mathcal{Z}$ to each event in a *space slice*, that is

$$t_{\mathcal{E}}(\overline{\mathbf{e}}) = t_{\mathcal{E}}(\mathbf{e}), \quad \forall\, \overline{\mathbf{e}} \in \mathcal{S}. \tag{1}$$

Velocities of *time lines* define the field of *time arrows* $\mathbf{v}_{\mathcal{Z}} : \mathcal{E} \mapsto T\mathcal{E}$.[4]

The tangent space $T_{\mathbf{e}}\mathcal{E}$ at any event $\mathbf{e} \in \mathcal{E}$ is split into a complementary pair of a 3D time-vertical subspace $V_{\mathbf{e}}\mathcal{E}$ (tangent to a space slice) and a 1D time-horizontal subspace $H_{\mathbf{e}}\mathcal{E}$ (tangent to a time line) generated by the time arrow $\mathbf{v}_{\mathcal{Z}}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$.

The time projection $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ and the time arrow $\mathbf{v}_{\mathcal{Z}}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$ are assumed to be *tuned* so that

$$\langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{Z}} \rangle = 1 \circ t_{\mathcal{E}}. \tag{2}$$

The symbol $\langle\,,\,\rangle$ denotes the pairing between dual fields and the dot $\cdot$ indicates linear dependence. In the tangent bundle $T\mathcal{E}$, the *time-vertical* subbundle $V\mathcal{E}$ (*time-horizontal subbundle* $H\mathcal{E}$) is the disjoint union of all time-vertical (time-horizontal) subspaces. They are, respectively, called *spatial bundle* and *time bundle*.

In the familiar Euclid setting of classical mechanics, the space slices and the *time projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ are the same for all observers (universality of time).

A reference frame $\{\, \mathbf{d}_i \,;\, i = 0, 1, 2, 3 \,\}$ for the event manifold is *adapted* if $\mathbf{d}_0 = \mathbf{v}_{\mathcal{Z}}$ and $\mathbf{d}_i \in V\mathcal{E}$, $i = 1, 2, 3$.

Definition 1 (*Trajectory*) The *trajectory manifold* is the geometric object investigated in mechanics, characterised by an embedding[5] $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$ into the event manifold $\mathcal{E}$ such that the image $\mathcal{T}_{\mathcal{E}} := \mathbf{i}(\mathcal{T})$ is a submanifold.

Definition 2 (*Motion*) The *motion* along the trajectory

$$\{\, \boldsymbol{\varphi}_{\alpha}^{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T},\ \alpha \in \mathcal{Z} \,\} \tag{3}$$

is a simultaneity preserving one-parameter family of maps fulfilling the composition rule

$$\boldsymbol{\varphi}_{\alpha}^{\mathcal{T}} \circ \boldsymbol{\varphi}_{\beta}^{\mathcal{T}} = \boldsymbol{\varphi}_{(\alpha+\beta)}^{\mathcal{T}} \tag{4}$$

for any pair of time-lapses $\alpha, \beta \in \mathcal{Z}$. Each $\boldsymbol{\varphi}_{\alpha}^{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T}$ is a *movement*.

The trajectory will alternatively be considered as a $(1+n)$D manifold $\mathcal{T}$ by itself or as a submanifold $\mathcal{T}_{\mathcal{E}} = \mathbf{i}(\mathcal{T}) \subset \mathcal{E}$ of the event manifold.

Then, a coordinate system is adopted on $\mathcal{T}$ while an adapted 4D space-time coordinate system in $\mathcal{E}$ is adopted on $\mathcal{T}_{\mathcal{E}}$.

The trajectory inherits from the events manifold the time projection $t_{\mathcal{T}} := t_{\mathcal{E}} \circ \mathbf{i} : \mathcal{T} \mapsto \mathcal{Z}$ which defines a time bundle denoted by $V\mathcal{T}$ and called the *material bundle*.

The immersion $V\mathcal{T}_{\mathcal{E}} := \mathbf{i}{\uparrow}(V\mathcal{T})$ is also named *material bundle*, and a fibre of simultaneous events $\boldsymbol{\Omega} \subset \mathcal{T}_{\mathcal{E}}$ is called a *body placement*. The spatial slice including the placement $\boldsymbol{\Omega}$ is denoted by $\mathcal{S}_{\boldsymbol{\Omega}}$.

The space-time movement $\boldsymbol{\varphi}_{\alpha}^{\mathcal{E}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ and the trajectory movement $\boldsymbol{\varphi}_{\alpha}^{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T}$ are related by the commutative diagram

$$\begin{tikzcd}
\mathcal{T}_{\mathcal{E}} \arrow[r, "\boldsymbol{\varphi}_{\alpha}^{\mathcal{E}}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{T}_{\mathcal{E}} \arrow[dd, bend left=60, "t_{\mathcal{E}}"] \\
\mathcal{T} \arrow[u, "\mathbf{i}"] \arrow[r, "\boldsymbol{\varphi}_{\alpha}^{\mathcal{T}}"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[d, "t_{\mathcal{T}}"] \\
\mathcal{Z} \arrow[r, "\theta_{\alpha}"] & \mathcal{Z}
\end{tikzcd}
\iff
\begin{cases}
t_{\mathcal{E}} \circ \boldsymbol{\varphi}_{\alpha}^{\mathcal{E}} = \theta_{\alpha} \circ t_{\mathcal{E}}, \\
t_{\mathcal{T}} \circ \boldsymbol{\varphi}_{\alpha}^{\mathcal{T}} = \theta_{\alpha} \circ t_{\mathcal{T}},
\end{cases} \tag{5}$$

[4] *Zeit* is the German word for *Time*.

[5] An immersion is a map whose differentials are injective. An embedding is an injective immersion whose corestriction is continuous with the inverse.

where the time-translation $\theta_\alpha : \mathcal{Z} \mapsto \mathcal{Z}$ is defined by

$$\theta_\alpha(t) := t + \alpha, \quad t, \alpha \in \mathcal{Z}. \tag{6}$$

Definition 3 (*Material particles and body manifold*) The physical notion of *material particle* corresponds in the geometric view to a time-parametrised curve of events in the trajectory, related by the motion as follows:

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{T} \; : \; \mathbf{e}_2 = \boldsymbol{\varphi}_\alpha^{\mathcal{T}}(\mathbf{e}_1). \tag{7}$$

Accordingly, we will say that a geometrical object is defined *along* (not *at*) a material particle. Events belonging to a *material particle* form a class of equivalence and the quotient manifold so induced in the trajectory is the *body manifold*.

The space-time *velocity* of the motion is defined by the derivative

$$\mathbf{v}_{\mathcal{E}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{E}} \in T\mathcal{T}_{\mathcal{E}}. \tag{8}$$

Taking the time derivative of (5), we have

$$\partial_{\alpha=0}\,(t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{E}}) = \langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{E}} \rangle = (\partial_{\alpha=0}\,\theta_\alpha) \circ t_{\mathcal{E}} = 1 \circ t_{\mathcal{E}}. \tag{9}$$

Comparing with Eq. (2), we get the decomposition into space and time components

$$\boxed{\mathbf{v}_{\mathcal{E}} = \mathbf{v}_{\mathcal{S}} + \mathbf{v}_{\mathcal{Z}}} \tag{10}$$

with $\langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{S}} \rangle = 0$. Due to the space-time splitting performed by an observer, a space-time motion is decomposed into the chain of a space motion and of a time shift,

$$\boldsymbol{\varphi}_\alpha^{\mathcal{E}} = \boldsymbol{\varphi}_\alpha^{\mathcal{S}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{Z}} = \boldsymbol{\varphi}_\alpha^{\mathcal{Z}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{S}}, \tag{11}$$

so that we have

$$\begin{aligned} \mathbf{v}_{\mathcal{S}} &:= \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{S}}, \\ \mathbf{v}_{\mathcal{Z}} &:= \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{Z}}, \end{aligned} \tag{12}$$

as sketched in the commutative diagram (13):

$$\begin{tikzcd}
\cdot \arrow[rr, "\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}"] & & \bullet \\
\bullet \arrow[u, dashed, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"] \arrow[urr, squiggly, "\boldsymbol{\varphi}_\alpha^{\mathcal{E}}"] \arrow[rr, "\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}"'] & & \cdot \arrow[u, dashed, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"']
\end{tikzcd} \tag{13}$$

3 Mass conservation

A basic axiomatic statement in dynamics is conservation of mass.

The mass of a 3D continuous body is represented, in each placement $\boldsymbol{\Omega}$, by a special *volume-form* $\mathbf{m} : T\boldsymbol{\Omega}^3 \mapsto \Re$ called *mass-form*.

This is a field of alternating trilinear and positive functions which evaluate the "generalised volume" of any positively oriented parallelepiped $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ in the space tangent at each $\mathbf{x} \in \boldsymbol{\Omega}$. In the Euclid space, the metric tensor $\mathbf{g}$ induces in each spatial slice a *metric volume-form* $\boldsymbol{\mu}$ such that positively oriented unit cubes do have unit *metric volume*.

As is well known, all volume-forms are proportional. The scalar *mass density* ρ rescales the metric volume-form to give the corresponding mass-form, according to the relation $\mathbf{m} = \rho \cdot \boldsymbol{\mu}$.

Definition 4 (*Mass conservation*) The axiom of mass conservation along the motion is expressed by each one of the equivalent properties

$$\begin{cases} (i) \quad \boldsymbol{\varphi}_\alpha^{\mathcal{E}} \downarrow \mathbf{m} = \mathbf{m}, \\ (ii) \quad \mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\mathbf{m}) := \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha^{\mathcal{E}} \downarrow \mathbf{m}) = \mathbf{0}, \\ (iii) \quad \displaystyle\int_{\boldsymbol{\varphi}_\alpha^{\mathcal{E}}(\boldsymbol{\Omega})} \mathbf{m} = \int_{\boldsymbol{\Omega}} \boldsymbol{\varphi}_\alpha^{\mathcal{E}} \downarrow \mathbf{m} = \int_{\boldsymbol{\Omega}} \mathbf{m}, \\ (iv) \quad \partial_{\alpha=0} \displaystyle\int_{\boldsymbol{\varphi}_\alpha^{\mathcal{E}}(\boldsymbol{\Omega})} \mathbf{m} = \int_{\boldsymbol{\Omega}} \mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\mathbf{m}) = \mathbf{0}. \end{cases} \tag{14}$$

The physical meaning of property (i) in Eq. (14) can be described as follows. Let $\{\, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 \,\}$ be the sides of any parallelepiped in the tanget space $T_{\mathbf{x}}\boldsymbol{\Omega}$ and $\{\, \boldsymbol{\varphi}_\alpha \uparrow \mathbf{d}_1, \boldsymbol{\varphi}_\alpha \uparrow \mathbf{d}_2, \boldsymbol{\varphi}_\alpha \uparrow \mathbf{d}_3 \,\}$ the sides of the parallelepiped in $T_{\boldsymbol{\varphi}_\alpha(\mathbf{x})}\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})$, transformed by the motion. Then, property (i) states that the mass of the transformed parallelepiped, given by the pull-back

$$(\boldsymbol{\varphi}_\alpha^{\mathcal{E}} \downarrow \mathbf{m})(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) := \mathbf{m}(\boldsymbol{\varphi}_\alpha \uparrow \mathbf{d}_1, \boldsymbol{\varphi}_\alpha \uparrow \mathbf{d}_2, \boldsymbol{\varphi}_\alpha \uparrow \mathbf{d}_3) \tag{15}$$

is equal to the mass

$$\mathbf{m}(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) \tag{16}$$

of the original one.

As will be shown in Sect. 4 Eq. (28), the mass conservation property expressed by Eq. (14) is an essential requirement in order to deduce the d'Alembert law of motion in terms of the acceleration field and to ensure in this way the fulfilment of the following basic axiom of dynamics.

Proposition 1 (Galilei principle of relativity) *Motions in Euclid space-time whose relative velocity field is constant, according to parallel transport by translation, are governed by the same law of dynamics.*

4 Continuum dynamics

Definition 5 (*Virtual motions*) A synchronous virtual motion of a placement $\boldsymbol{\Omega}$ is a one-parameter family $\delta\boldsymbol{\varphi}_\lambda : \boldsymbol{\Omega} \mapsto \mathcal{S}_{\boldsymbol{\Omega}}$ of time-preserving morphisms, as described below:

$$\begin{tikzcd} \boldsymbol{\Omega} \arrow[r, "\delta\boldsymbol{\varphi}_\lambda"] \arrow[d, "t_{\mathcal{E}}"'] & \mathcal{S}_{\boldsymbol{\Omega}} \arrow[d, "t_{\mathcal{E}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{ID}_{\mathcal{Z}}"] & \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad t_{\mathcal{E}} \circ \delta\boldsymbol{\varphi}_\lambda = t_{\mathcal{E}}. \tag{17}$$

The map $\delta\boldsymbol{\varphi}_\lambda : \boldsymbol{\Omega} \mapsto \delta\boldsymbol{\varphi}_\lambda(\boldsymbol{\Omega})$ is invertible and differentiable with the inverse. The associated virtual velocity on the placement is the spatial vector field

$$\delta\mathbf{v}_{\mathcal{S}} := \partial_{\lambda=0}\, \delta\boldsymbol{\varphi}_\lambda : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}. \tag{18}$$

Definition 6 (*External force*) An *external force* $\mathbf{f}_{\mathrm{EXT}}$ describes the action on a body placement $\boldsymbol{\Omega}$ of bulk $\mathbf{b}$ and surficial $\mathbf{t}$ force one-forms, according to the expression

$$\langle \mathbf{f}_{\mathrm{EXT}}, \delta\mathbf{v}_{\mathcal{S}} \rangle := \int_{\boldsymbol{\Omega}} \langle \mathbf{b}, \delta\mathbf{v}_{\mathcal{S}} \rangle \cdot \boldsymbol{\mu} + \int_{\partial\boldsymbol{\Omega}} \langle \mathbf{t}, \delta\mathbf{v}_{\mathcal{S}} \rangle \cdot \partial\boldsymbol{\mu}, \tag{19}$$

where $\boldsymbol{\mu}$ is the volume-form induced by the spatial metric field $\mathbf{g}$. The area-form on the boundary $\partial\boldsymbol{\Omega}$ is defined by $\partial\boldsymbol{\mu} := \boldsymbol{\mu} \cdot \mathbf{n}$, with $\mathbf{n}$ the normal versor.

Definition 7 (*Internal force*) The *internal force* $\mathbf{f}_{\mathrm{INT}}(\boldsymbol{\sigma})$ on a body placement $\boldsymbol{\Omega}$, and associated with a stress field $\boldsymbol{\sigma}$, is the one-form expressed by

$$\langle \mathbf{f}_{\mathrm{INT}}(\boldsymbol{\sigma}), \delta\mathbf{v}_{\mathcal{S}} \rangle := \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}_{\mathcal{S}}) \rangle \cdot \mathbf{m}, \tag{20}$$

where $\boldsymbol{\varepsilon}(\delta\mathbf{v}_{\mathcal{S}}) := \mathrm{sym}(\nabla\delta\mathbf{v}_{\mathcal{S}})$ is the stretching associated with the virtual velocity field $\delta\mathbf{v}_{\mathcal{S}}$.

Definition 8 (*Dynamical force*) The *dynamical force* $\mathbf{f}_{\text{DYN}} \in \mathcal{H}^*_{\mathbf{M}}$ on a body placement $\boldsymbol{\Omega}$ is the difference between external and internal forces expressed by the formula

$$\mathbf{f}_{\text{DYN}} := \mathbf{f}_{\text{EXT}}(\mathbf{b}\,,\mathbf{t}) - \mathbf{f}_{\text{INT}}(\boldsymbol{\sigma})\,. \tag{21}$$

The Euler law of continuum dynamics states that, for piecewise smooth motions, the time rate of variation of the virtual power of the projected kinetic momentum[6] is equal to the virtual power of the dynamical force system acting on the body, at each placement in the event manifold along the trajectory.

Proposition 2 (Euler law of continuum dynamics) *Denoting by* $\mathbf{g}$ *the metric tensor in the spatial bundle* $V\mathcal{E}$ *and by* $\Uparrow$ *the parallel transport by the relevant Levi-Civita connection,*[7] *the Euler differential law writes, for regular motions*

$$\boxed{\partial_{\alpha=0} \int_{\boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m} = \langle \mathbf{f}_{\text{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle,} \tag{22}$$

with the virtual velocity $\delta\hat{\mathbf{v}}_{\mathcal{S}}$ *parallel-transported along the motion:*

$$\delta\hat{\mathbf{v}}_{\mathcal{S}} \circ \boldsymbol{\varphi}^{\mathcal{E}}_{\alpha} := \boldsymbol{\varphi}^{\mathcal{E}}_{\alpha} \Uparrow \delta\mathbf{v}_{\mathcal{S}}, \tag{23}$$

and, at singularities of the kinetic momentum

$$\boxed{\left[\left[\int_{\boldsymbol{\Omega}} \mathbf{g}(\mathbf{v}_{\mathcal{S}}\,, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}\right]\right] = \langle \mathbf{f}_{\text{SING}}, \delta\mathbf{v}_{\mathcal{S}} \rangle,} \tag{24}$$

where the jump $[[\bullet]]$ *is the difference between the limit from the right and the limit from the left, at points of discontinuity. The singular force* $\mathbf{f}_{\text{SING}}$ *is named an* impulse, *in mechanics.*

Lemma 1 (Linear dependence on virtual velocity fields) *The Euler law of dynamics is well posed since the time rate of increase of projected momentum depends in a linear way on virtual velocity fields* $\delta\mathbf{v}_{\mathcal{S}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$.

Proof The rate of variation of the projected kinetic momentum in Eq. (22) may be rewritten by applying the Jacobi pull-back integral transformation and the definition of the Lie derivative along a flow:

$$\begin{aligned} \partial_{\alpha=0} \int_{\boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}(\boldsymbol{\Omega})} \mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m} &= \int_{\boldsymbol{\Omega}} \partial_{\alpha=0}\, \boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}\!\downarrow\! \Big(\mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}\Big) \\ &= \int_{\boldsymbol{\Omega}} \mathcal{L}_{\mathbf{v}_{\mathcal{E}}} \Big((\mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}\Big). \end{aligned} \tag{25}$$

Moreover, the Leibniz rule for the Lie-derivative gives

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\mathcal{E}}} \Big((\mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}}) \cdot \mathbf{m}\Big) &= \mathcal{L}_{\mathbf{v}_{\mathcal{E}}} \Big(\mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}})\Big) \cdot \mathbf{m} \\ &\quad + \mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\mathbf{m}). \end{aligned} \tag{26}$$

By construction of the field $\delta\hat{\mathbf{v}}_{\mathcal{S}}$ in Eq. (23), it follows that $\nabla_{\mathbf{v}_{\mathcal{E}}}(\delta\hat{\mathbf{v}}_{\mathcal{S}}) = \mathbf{0}$. Since parallel transport and push of scalar fields are coincident, applying the Leibniz rule for the parallel derivative we get

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\mathcal{E}}} \Big(\mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}})\Big) &= \nabla_{\mathbf{v}_{\mathcal{E}}} \Big(\mathbf{g}_{\text{SPA}}(\mathbf{v}_{\mathcal{S}}\,, \delta\hat{\mathbf{v}}_{\mathcal{S}})\Big) \\ &= \nabla_{\mathbf{v}_{\mathcal{E}}}(\mathbf{g}_{\text{SPA}})(\mathbf{v}_{\mathcal{S}}\,, \delta\mathbf{v}_{\mathcal{S}}) + \mathbf{g}_{\text{SPA}}(\nabla_{\mathbf{v}_{\mathcal{E}}}(\mathbf{v}_{\mathcal{S}})\,, \delta\mathbf{v}_{\mathcal{S}}). \end{aligned} \tag{27}$$

Substituting into Eq. (26), we get the result. □

[6] The adjective *projected* refers to the inner product between the spatial velocity and the parallel-transported virtual velocity field, by means of the spatial metric.

[7] In classical dynamics the parallel transport is the familiar path-independent operation of translation in the Euclid space.

Proposition 3 (d'Alembert law of continuum dynamics) *By conservation of mass, the Euler law Eq.* (22) *is equivalent to the following:*

$$\boxed{\int_{\boldsymbol{\Omega}} \mathbf{g}(\mathbf{a}_{\mathcal{S}}\,, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m} = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle} \tag{28}$$

where $\mathbf{a}_{\mathcal{S}} : \boldsymbol{\Omega} \mapsto T\mathcal{S}_{\boldsymbol{\Omega}}$ *is the spatial acceleration field, defined as parallel time rate of variation of the spatial velocity along the motion* [8]

$$\boxed{\mathbf{a}_{\mathcal{S}} := \nabla_{\mathbf{v}_{\mathcal{E}}}(\mathbf{v}_{\mathcal{S}}) = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_{\alpha}^{\mathcal{E}} \Downarrow \mathbf{v}_{\mathcal{S}})} \tag{29}$$

Proof Conservation of mass, expressed by the condition in Eq. (14), imposes that $\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\mathbf{m}) = \mathbf{0}$. Moreover $\nabla_{\mathbf{v}_{\mathcal{E}}}(\mathbf{g}_{\mathrm{SPA}}) = \mathbf{0}$ since the connection is Levi-Civita, so that, by Eqs. (26) and (27), the Euler law translates into the d'Alembert law Eq. (28). □

Dynamical forces are Galilei's invariant by assumption. Fulfilment of Galilei's principle of relativity is evident from the expression of the law of motion in Proposition 3. This result underlines the essential role played by conservation of mass in a proper formulation of classical dynamics.

5 Solid–fluid interaction

The study of solid–fluid interaction is extremely valuable for many important engineering applications of dynamics, such as for instance missiles, rockets, jet engines, turbine plants, hydroelectric pipes or lawn sprinklers.

5.1 The skeleton

Fluid trajectory and motion may be very hard to be detected in general, but in many applications, only the motion of the solid body is of central interest.

Investigating the trajectory and the motion of the part of fluid that interacts with the solid body may then suffice and is easily achievable. The idea underlying the procedure proposed here consists in the following items:

(a) The *skeleton* is a control window with motion $\boldsymbol{\varphi}_{\alpha}^{\mathrm{SKE}}$ flying in the trajectory of the interacting fluid, in such a way that contact is kept with the solid case along the fluid–solid interaction boundary.
(b) The geometry of the *skeleton* is chosen so to make the thrusting force exerted by the fluid evaluable by a practicable computation, as exemplified by the schematic diagrams (31) and (32) below.

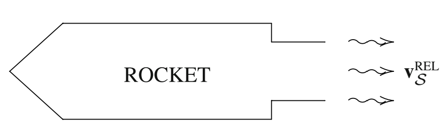
We denote by $\boldsymbol{\Omega}_{\mathrm{SOL}}$ a placement of the solid body, by $\boldsymbol{\Omega}_{\mathrm{FLU}}$ a simultaneous placement of the fluid including the one interacting with the solid.

The placement $\boldsymbol{\Omega}_{\mathrm{SKE}}$ of the *skeleton* is such that

$$\begin{cases} \boldsymbol{\Omega}_{\mathrm{SOL}} \cap \boldsymbol{\Omega}_{\mathrm{SKE}} = \emptyset, \\ \partial\boldsymbol{\Omega}_{\mathrm{SKE}} = \partial\boldsymbol{\Omega}_{\mathrm{SOL}} \cap \partial\boldsymbol{\Omega}_{\mathrm{FLU}} + \boldsymbol{\Sigma}_{\mathrm{OUT}}^{\mathrm{IN}}. \end{cases} \tag{30}$$

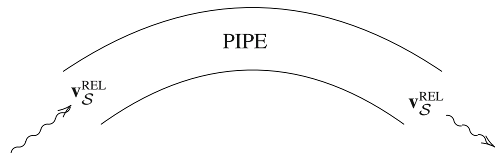
The surface $\boldsymbol{\Sigma}_{\mathrm{OUT}}^{\mathrm{IN}} \subset \partial\boldsymbol{\Omega}_{\mathrm{SKE}}$ is the portion of skeleton boundary through which the fluid is allowed to enter or leave the skeleton.

The models of a rocket and of a fluid conveyance pipe, whose skeletons are sketched in (31) and (32), exemplify dynamical problems involving, respectively, systems with a variable total mass and systems with an invariant total mass along the motion.



(31)

[8] In 3D treatments, the acceleration is often defined by $\mathbf{a}_{\mathcal{S}} = \nabla_{\mathbf{Z}}(\mathbf{v}_{\mathcal{S}}) + \nabla_{\mathbf{v}_{\mathcal{S}}}(\mathbf{v}_{\mathcal{S}})$, see, e.g., [57]. This split formula is not applicable to lower-dimensional trajectories such as the ones pertaining to bullets, wires and membranes.



(32)

5.2 Relative motion of the fluid

We will consider the motions of the two physical components:

(a) the solid $\boldsymbol{\varphi}_\alpha^{\mathrm{SOL}}$ with velocity $\mathbf{v}_{\mathcal{E}}^{\mathrm{SOL}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{SOL}}$,
(b) the fluid $\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}}$ with velocity $\mathbf{v}_{\mathcal{E}}^{\mathrm{FLU}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{FLU}}$.

The idea consists in considering the skeleton motion $\boldsymbol{\varphi}_\alpha^{\mathrm{SKE}}$, which is an extension of the solid-case motion, with velocity $\mathbf{v}_{\mathcal{E}}^{\mathrm{SKE}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}}$.

In the trajectory of the flying skeleton, the relative motion $\boldsymbol{\varphi}_\alpha^{\mathrm{REL}}$ of the fluid with respect to the skeleton is defined by the relation

$$\boxed{\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} = \boldsymbol{\varphi}_\alpha^{\mathrm{REL}} \circ \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}}, \quad \text{with} \quad \begin{cases} t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha^{\mathrm{REL}} = t_{\mathcal{E}} \\[2ex] t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}} = \theta_\alpha \circ t_{\mathcal{E}} \end{cases}} \tag{33}$$

where $\theta_\alpha(t) := t + \alpha$ is the time-translation. The relative motion is then a spatial motion at constant time.

The relevant velocity fields are related by

$$\boxed{\mathbf{v}_{\mathcal{E}}^{\mathrm{FLU}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} = \mathbf{v}_{\mathcal{E}}^{\mathrm{SKE}} + \mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}.} \tag{34}$$

In writing the Euler law of motion (22), the time rate of kinetic momentum of the system composed of solid case and fluid-filled skeleton is expressed by

$$\boxed{\begin{aligned} \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha^{\mathrm{SOL}}(\boldsymbol{\Omega}_{\mathrm{SOL}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{SOL}}, \boldsymbol{\varphi}_\alpha^{\mathrm{SOL}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{SOL}} \\ + \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{FLU}}, \boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}}. \end{aligned}} \tag{35}$$

In evaluating the second integral at the r.h.s. of Eq. (35), we may apply the splitting in Eq. (33) and perform the derivative with respect to the scalar parameter α according to the Leibniz rule, by adding the derivatives pertaining to the following two simpler situations:

1. the relative motion degenerates to the identity, so that the motion of the fluid is equal to the motion of the skeleton. Then
$$\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} = \boldsymbol{\varphi}_\alpha^{\mathrm{SKE}}, \quad \mathbf{v}_{\mathcal{E}}^{\mathrm{FLU}} = \mathbf{v}_{\mathcal{E}}^{\mathrm{SKE}}. \tag{36}$$
2. The motion of the skeleton degenerates to the identity, so that the motion of the fluid is equal to the relative motion. Then
$$\boldsymbol{\varphi}_\alpha^{\mathrm{FLU}} = \boldsymbol{\varphi}_\alpha^{\mathrm{REL}}, \quad \mathbf{v}_{\mathcal{E}}^{\mathrm{FLU}} = \mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}. \tag{37}$$

The first contribution yields the time rate of variation of the kinetic momentum of the fluid filling the skeleton.

The second contribution yields the time rate of variation of the kinetic momentum of the fluid in motion with respect to the skeleton.

The opposite of this second contribution gives the force exerted on the solid case by the interacting fluid in relative motion.

Let us now perform explicitly the relevant calculations.

Adopting the splitting in Eq. (33) and applying the Leibniz rule of differentiation, the second term at the r.h.s. of Eq. (35) may be rewritten as

$$\boxed{\begin{aligned} &\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_{\alpha}^{\mathrm{FLU}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{FLU}}, \boldsymbol{\varphi}_{\alpha}^{\mathrm{FLU}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}} \\ &\quad = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_{\alpha}^{\mathrm{SKE}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{SKE}}, \boldsymbol{\varphi}_{\alpha}^{\mathrm{SKE}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}} \\ &\quad\quad + \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} (\mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}}. \end{aligned}} \tag{38}$$

The second term at the r.h.s. of Eq. (38) may in turn be rewritten by first applying the pull-back Jacobi transformation to the integral and then recalling the definition of the Lie derivative. Extending the virtual velocity $\delta\mathbf{v}_{\mathcal{S}}$ by parallel transport along the relative motion by setting

$$\delta\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}} \circ \boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}} = \boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}, \tag{39}$$

the procedure leads to the integral formula

$$\begin{aligned} &\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}} \\ &\quad = \int_{\boldsymbol{\Omega}_{\mathrm{SKE}}} \partial_{\alpha=0} \Big(\boldsymbol{\varphi}_{\alpha}^{\mathrm{REL}} \downarrow \Big(\mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}) \cdot \mathbf{m}_{\mathrm{FLU}} \Big) \Big) \\ &\quad = \int_{\boldsymbol{\Omega}_{\mathrm{SKE}}} \mathcal{L}_{\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}} \Big(\mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}) \cdot \mathbf{m}_{\mathrm{FLU}} \Big). \end{aligned} \tag{40}$$

We may then resort to the homotopy formula for the Lie derivative and to the Stokes formula for the exterior derivative.

In this way, the expression in Eq. (40) can be conveniently transformed into a boundary integral:

$$\begin{aligned} &\int_{\boldsymbol{\Omega}_{\mathrm{SKE}}} \mathcal{L}_{\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}} \Big(\mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}) \cdot \mathbf{m}_{\mathrm{FLU}} \Big) \\ &\quad = \int_{\boldsymbol{\Omega}_{\mathrm{SKE}}} d \Big(\mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\mathrm{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}) \Big) \\ &\quad = \oint_{\partial\boldsymbol{\Omega}_{\mathrm{SKE}}} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\mathrm{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}). \end{aligned} \tag{41}$$

Substituting the result of Eqs. (38), (40), (41) into the Euler law of motion Eq. (35), we get the final result

$$\boxed{\begin{aligned} \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle = \partial_{\alpha=0} &\int_{\boldsymbol{\varphi}_{\alpha}^{\mathrm{SOL}}(\boldsymbol{\Omega}_{\mathrm{SOL}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{SOL}}, \boldsymbol{\varphi}_{\alpha}^{\mathrm{SOL}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{SOL}} \\ &+ \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_{\alpha}^{\mathrm{SKE}}(\boldsymbol{\Omega}_{\mathrm{SKE}})} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{SKE}}, \boldsymbol{\varphi}_{\alpha}^{\mathrm{SKE}} \Uparrow \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\mathrm{FLU}} \\ &+ \int_{\boldsymbol{\Sigma}_{\mathrm{OUT}}^{\mathrm{IN}}} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\mathrm{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}). \end{aligned}} \tag{42}$$

Here $\boldsymbol{\Sigma}_{\mathrm{OUT}}^{\mathrm{IN}} \subset \partial\boldsymbol{\Omega}_{\mathrm{SKE}}$ is the surface where the mass-outflow term $\mathbf{m}_{\mathrm{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\mathrm{REL}}$ is allowed to be non-vanishing, as for instance the nozzle exit surface for the exhaust gas in a rocket.

The last surface integral in Eq. (42) has the physical meaning of flux of the projected kinetic momentum outflowing from the skeleton $\boldsymbol{\Omega}_{\text{SKE}}$ per unit time. More precisely:

– The term $\mathbf{m}_{\text{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\text{REL}}$ provides the fluid mass leaving the skeleton $\boldsymbol{\Omega}_{\text{SKE}}$ per unit time and per unit surficial area of the boundary $\partial\boldsymbol{\Omega}_{\text{SKE}}$.
– The term $\mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\text{REL}}, \delta\hat{\mathbf{v}}_{\mathcal{S}})$ is the projected relative velocity field of the fluid with respect to the skeleton, namely the scalar projection of the relative velocity field on the virtual velocity field.

We assume that no mass variation occurs in the solid case and that the time rate of fluid mass, fictitiously undergoing the skeleton motion, is negligible. This means that this mass rate is either null or not influential for evaluating the thrust, when compared with the effect of the time rate of kinetic momentum of the fluid in relative motion with respect to the skeleton. Under this physically reasonable assumption, we may set

$$\boxed{\mathcal{L}_{\mathbf{v}_{\mathcal{E}}^{\text{SOL}}}(\mathbf{m}_{\text{SOL}}) = \mathcal{L}_{\mathbf{v}_{\mathcal{E}}^{\text{SKE}}}(\mathbf{m}_{\text{FLU}}) = \mathbf{0}.} \tag{43}$$

Then, the first and the second integrals on the l.h.s. of (42) may be transformed by the same procedure adopted to get the d'Alembert law of continuum dynamics Eq. (28), from the Euler law Eq. (22).

Accordingly, the dynamical law Eq. (42) for the *solid case* and the *fluid-filled skeleton* can be reformulated *à* la d'Alembert, as

$$\boxed{\begin{aligned} \langle \mathbf{f}_{\text{DYN}}, \delta\mathbf{v}_{\mathcal{S}} \rangle &= \int_{\boldsymbol{\Omega}_{\text{SOL}}} \mathbf{g}(\mathbf{a}_{\mathcal{S}}^{\text{SOL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\text{SOL}} \\ &\quad + \int_{\boldsymbol{\Omega}_{\text{SKE}}} \mathbf{g}(\mathbf{a}_{\mathcal{S}}^{\text{SKE}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot \mathbf{m}_{\text{FLU}} \\ &\quad + \int_{\boldsymbol{\Sigma}_{\text{OUT}}^{\text{IN}}} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\text{REL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\text{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\text{REL}}). \end{aligned}} \tag{44}$$

The respect of Galilei's principle of relativity is thus put into evidence.

5.3 The thrust

The *thrust* acting on the solid case is the one-form $\mathbf{f}_{\text{THR}} : \boldsymbol{\Omega} \mapsto T^*\mathcal{S}_{\boldsymbol{\Omega}}$ defined by the virtual power equality

$$\boxed{\langle \mathbf{f}_{\text{THR}}, \delta\mathbf{v}_{\mathcal{S}} \rangle := -\int_{\boldsymbol{\Sigma}_{\text{OUT}}^{\text{IN}}} \mathbf{g}(\mathbf{v}_{\mathcal{S}}^{\text{REL}}, \delta\mathbf{v}_{\mathcal{S}}) \cdot (\mathbf{m}_{\text{FLU}} \cdot \mathbf{v}_{\mathcal{S}}^{\text{REL}}).} \tag{45}$$

The one-form $\mathbf{f}_{\text{THR}}$ represents the force system acting on the solid case, along the solid–fluid interface, due to the relative velocity between the interacting fluid and the solid in motion.

Well posedness of definition of the *thrust* is deduced by observing that:

1. The surface integral in Eqs. (42) and (44) is a linear functional of the virtual velocity field $\delta\mathbf{v}_{\mathcal{S}}$.
2. The relative velocity field is invariant under Galilei frame transformations, so that Galilei principle of relativity is fulfilled.

The assumption that the rate of mass loss in the solid case and in fluid-filled skeleton is negligible, as expressed by Eq. (43), is verified with good approximation in the dynamics of sprinklers and rockets, since the thrust is not significantly affected by neglecting the mass-loss rate.

This argument reveals, however, an inherent and unavoidable conflict between the assumption of negligible mass-loss rate, resorted to in getting a simple but dynamical correct formula for the thrust, and the necessity of taking into account the total mass variation to investigate in a physically effective manner the dynamical property of the trajectory of rockets which are subject to a gross mass loss during operative flights, due to the outflow of exhausted propellant. The approach to this conflicting situation can be put into evidence by referring to this simplified formulation as dynamics of

– systems with "low mass and high momentum time rate",
– in place of the usual naming of systems with "variable mass".

As a matter of fact, in case of a gross rate of mass loss, a satisfactory analysis should be performed by means of a challenging dynamical treatment of the system compound by the solid case and the interacting fluid, with the specification of proper interface conditions.

Let us put in comparison the simplified dynamics of rockets and of jets.

– Rocket dynamics can be investigated in the simplest way by assuming that the virtual velocity $\delta \mathbf{v}_{\mathcal{S}}$ in Eq. (44) is a translation in the axial direction along the rocket. Since the exhaust gas is expelled at expenses of the propellant stored in the reservoir, the total mass of the rocket will decrease during the flight. In evaluating the amount of the thrust, however, the overall mass of the skeleton can be considered as invariant since, in this step of the analysis, the fluid-mass outflow does not play a significant role. Variability of the total mass must, however, be taken into account in detecting the overall motion of the rocket, since it will ultimately run out of fuel, with the global mass drastically reduced.
– Jets and sprinklers dynamics may be investigated by assuming that the virtual velocity $\delta \mathbf{v}_{\mathcal{S}}$ in Eq. (44) is, respectively, an act of translation along the jet axis or an act of rotation about the sprinkler axel. Since the rate at which water is expelled at the end nozzle is equal to the rate of water supplied by the irrigation plant, the total mass of the rotating sprinkler skeleton is constant during the motion. The same is true also for the skeleton adopted to investigate the jet motion, due to substantial equality between mass rates of air forward income and of exhausted air backward outcome.

In both cases, the property of *low mass and high momentum time rate* is fulfilled and hence the thrust can be evaluated by the formula in Eq. (45).

Under a constant water supply, the rotating sprinkler accelerates the motion until air and support frictions equilibrates the thrust.

Under a constant fuel burning rate, a rocket's speed *in vacuo* becomes larger and larger as times goes on.[9]

The relative speed of exhaust gas propelling a rocket ranges in the interval between 2 and 4 kilometres per second, while characteristic speeds of rockets are 7.9 (orbital speed) and 11.2 (escape speed) kilometers per second [13].

The formula in Eq. (45) may be directly applied to other engineering problems, as for instance evaluation of the thrust exerted on a sharp bend of a pressure pipe line of a hydroelectric plant, schematically depicted in Eq. (32).

It should be noted that in most presentations, in the wake of treatments of analytical dynamics, where forces are represented by vectors based at material atom particles, the *thrust* is reductively defined as the resultant vector associated with the *force form* defined by Eq. (45).

The resultant moment of the *thrust* is taken into account by the treatment exposed in [57, §15, Eq. 1, p.108].

There, however, a fixed control window is considered, so that the relevant formula cannot be considered as the expression of a thrusting force, being not Galilei invariant.

6 Discussion

A qualitative treatment of dynamics of direct and inverse sprinklers, and of similar systems involving fluid–solid interactions, can be found in Ernst Mach's treatise on *Mechanik* (1883) [58, fig. 153a, p. 300]. The topic has been recently revisited in [59].

An interesting discussion on dynamics of *atom* particles with variable mass is found in the 1952 treatise on mechanics by Arnold Sommerfeld [60, § I.4,"Variable Masses", p. 28], as summarised hereafter.

Classical formulations are founded on Newton second law of dynamics, concerning *atom* particles with invariant mass. Denoting by M the instantaneous, concentrated mass of the particle, the law states that

– "The change in motion is proportional to the force acting and takes place in the direction of the straight line along which the force acts",

[9] On the web one finds the question: Will rockets accelerate until they run out of fuel? (Read more than 11,000 times since 11/05/2010 to the end of 2015). The Naked Scientists forum. http://www.thenakedscientists.com/forum/index.php?topic=31503.0.

and is expressed by [60, Eq. (3) p. 5]

$$\frac{d}{dt}(M \cdot \mathbf{v}_{\mathcal{S}}) = \mathbf{f}_{\text{EXT}}, \tag{46}$$

where $\mathbf{f}_{\text{EXT}}$ is the external force, for instance the gravitational force. The time rate of kinetic momentum along the motion is evaluated by translation of the spatial velocity vector in the Euclid space.

In this original form, the law was conceived by Newton to describe the motion of bullets or celestial bodies (whose mass is considered invariant along the motion) and is therefore unsuitable for the description of the motion of *atom* particles with a variable mass.

In fact, applying the Leibniz rule to evaluate the time derivative in Eq. (46), one would yield the paradoxical result, [60, Eq. (1a) p. 28] :

$$M \cdot \mathbf{a}_{\mathcal{S}} + \dot{M} \cdot \mathbf{v}_{\mathcal{S}} = \mathbf{f}_{\text{EXT}}, \qquad \dot{M} := \frac{d}{dt} M, \tag{47}$$

where $\mathbf{a}_{\mathcal{S}} = \nabla_{\mathbf{v}_{\mathcal{E}}}(\mathbf{v}_{\mathcal{S}})$. The law expressed by Eq. (47) is unphysical since the term $\dot{M} \cdot \mathbf{v}_{\mathcal{S}}$ violates the Galilei principle of relativity, as pointed out also in [61] where the issue was revisited in the context of dynamical astronomy.

The discussion about Eq. (46), when performed in terms of a single atom particle with variable mass, leads to the following dilemma:

1. Either renounce Newton's second law as a pillar of classical dynamics,
2. or judge the notion of *atom particle with variable mass* as unphysical.

The former choice was made in the conclusion drawn in [61] where Eq. (48) below is considered as the result of a new formulation of dynamics, pertaining to atom particles with variable mass, for which Newton's second law fails and must be substituted by a new principle of dynamics.

As a matter of fact, in formulating the rules of dynamics, conservation of mass is a basic assumption to ensure fulfilment of Galilei's principle of relativity.

The notion of a particle with variable mass may lead to unphysical statements such as the one expressed by Eq. (47).[10]

To overcome the difficulty, Sommerfeld [60] turned the attention to a simple case study, so quoting on p. 28: *Let us consider a familiar example: a sprinkler wagon wets the asphalt on a hot summer day*.

It is convenient to write $\mathbf{v}_{\mathcal{S}}^{\text{FLU}} = \mathbf{v}_{\mathcal{S}}^{\text{SOL}} + \mathbf{v}_{\mathcal{S}}^{\text{REL}}$ with $\mathbf{v}_{\mathcal{S}}^{\text{SOL}} = \mathbf{v}_{\mathcal{S}}$ the velocity of the wagon and $\mathbf{v}_{\mathcal{S}}^{\text{FLU}}$ the absolute velocity of the water as evaluated by an inertial observer, standing in the street taken as frame of reference.

Assuming that $\mu = -\dot{M}$ and adding the momentum variation $\mu \cdot \mathbf{v}_{\mathcal{S}}^{\text{FLU}}$, where μ is the time rate of water mass loss, Sommerfeld eventually finds

$$M \cdot \mathbf{a}_{\mathcal{S}}^{\text{SOL}} = \mathbf{f}_{\text{EXT}} - \mu \cdot \mathbf{v}_{\mathcal{S}}^{\text{REL}}, \tag{48}$$

where $\mathbf{v}_{\mathcal{S}}^{\text{REL}}$ is the relative velocity of the water exiting from the sprinkler wagon. This same law of dynamics for a particle with variable mass was previously formulated by von Buquoy and later adopted by Meshchersky, as illustrated in § 1, but their contributions were not cited in [60].

The approach undertaken by Sommerfeld [60] is artful but enlightening. To find the right law Eq. (48), he considered not a *bullet with variable mass* but instead a *sprinkler wagon* (alas, a reactor with very low dynamic efficiency) which, skilfully, is a continuous body. The suggestion of adding the recoil due to the rate of momentum lost in the outward water flow is ultimately equivalent to recover conservation of mass since $\mu = -\dot{M}$. The formula in Eq. (48) can be effectively applied to describe the basic law of rocket motion, but is unsuitable to describe the dynamics of a sprinkler or the thrust on a curved pressure pipe, where the total mass is invariant during the motion. Turning to a continuum dynamics treatment is therefore mandatory to get a comprehensive result.

[10] This incorrect equation is surprisingly adopted, for the formulation of the ideal rocket equation, in the web page of the NASA Glenn Research Center [44].

7 Conclusions

The problem of providing a simple expression for the thrusting force exerted on a solid by an interacting fluid in relative motion has been addressed and solved in a natural way, under suitable simplifying assumptions.

The analysis developed in the present paper clearly indicates that the route towards a satisfactorily formulation is to be drawn in the framework of continuum dynamics.

The fluid is permitted to pass through an ideal skeleton flying in the fluid trajectory, while keeping contact with the solid in motion.

A technically interesting thrusting force is generated if the fluid momentum changes at a sufficiently high rate.

A possible variation of the system total mass is irrelevant in evaluating the thrusting effect. The analysis conceived and detailed in Sect. 5 is decisive to avoid issues about the formulation of the dynamics concerning *atom particles with variable mass*.

It may be concluded that, in dealing with rockets, turbines, jets, sprinklers and similar dynamical problems, the analysis should consider material bodies interacting along their dynamical trajectories, each one fulfilling the pertinent mass conservation law, as required by the basic theory.

These complex systems can be investigated by the Euler–d'Alembert law of continuum dynamics, and to get an applicable simple expression for the thrust, simplifying assumptions consistent with the qualifying property of a low rate of *mass variation* and a high rate of *momentum variation* must be made.

The consideration of *particles with variable mass* is not appropriate to model the dynamical problems involving solid–fluid interactions since physical problems involving effective thrusting forces, but without significant mass variation, cannot be properly dealt with.

To overcome this intrinsic limitation of the standard formulation of von Buquoy–Meshchersky equation, it has been shown that interactions between solids and fluids in relative motion can effectively be analysed in the context of classical continuum mechanics, under simplifying assumptions concerning a suitable skeleton immersed in the fluid trajectory and moving in continuous contact with the solid case.

Mass conservation is assumed to be fulfilled by the fluid fictitiously moving according to the skeleton motion, to an extent sufficient to ensure respect of Galilei's principle of relativity. This assumption is reasonable and well verified when the chain composition Eq. (33) of the fluid motion is considered. The circumstance can be concisely described by the qualification of investigated systems as those with

– "low mass and high momentum time rate".

The mass of the solid–fluid complex subject to the thrusting force is, however, still considered to be possibly time dependent. This occurrence is technically important to get a correct evaluation of the dynamical trajectory of rockets undergoing long-term flights.

Acknowledgements The authors' interest in this classical, but nevertheless still challenging, subject was prompted by the reading of recent valuable contributions by Prof. Hans Irschik and coworkers, as quoted in the paper.

Appendix: Mathematical notes

All notions listed in this section are detailedly illustrated in [62,63]. To a manifold $\mathbf{M}$, there corresponds a tangent bundle $T\mathbf{M}$ whose fibre at $\mathbf{x} \in \mathbf{M}$ is the linear space of velocities of curves through that point.

A tangent vector field $\mathbf{u} : \mathbf{M} \mapsto T\mathbf{M}$ is characterised by the property that $\mathbf{u}(\mathbf{x}) \in T_{\mathbf{x}}\mathbf{M}$, which may be expressed by stating that the projection $\boldsymbol{\tau}_{\mathbf{M}} : T\mathbf{M} \mapsto \mathbf{M}$ on the base manifold is a right inverse of the vector field, i.e. that $\boldsymbol{\tau}_{\mathbf{M}} \circ \mathbf{u} : \mathbf{M} \mapsto \mathbf{M}$ is the identity.

The flow $\mathbf{Fl}^{\mathbf{u}}_{\lambda} : \mathbf{M} \mapsto \mathbf{M}$ is generated by solutions of the differential equation $\mathbf{u} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{u}}_{\lambda}$.

The push forward of a tangent vector field $\mathbf{w} : \mathbf{M} \mapsto T\mathbf{M}$ along the flow $\mathbf{Fl}^{\mathbf{u}}_{\lambda} : \mathbf{M} \mapsto \mathbf{M}$ is defined by the tangent functor[11]

$$(\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\uparrow\!\mathbf{w}) \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda} := T\mathbf{Fl}^{\mathbf{u}}_{\lambda} \cdot \mathbf{w}, \tag{49}$$

where $T\mathbf{Fl}^{\mathbf{u}}_{\lambda} : T\mathbf{M} \mapsto T\mathbf{M}$ is the tangent map.

[11] Applying the tangent functor T to a map $\boldsymbol{\phi} : \mathbf{M} \mapsto \mathbf{N}$ between manifolds, the outcome is the tangent map $T\boldsymbol{\phi} : T\mathbf{M} \mapsto T\mathbf{N}$ which associates, in a linear way, with the velocity of a curve at a given point, the velocity of the image curve at the image point.

A circle $\circ$ means composition of maps, and an interposed dot $\cdot$ denotes linear dependence on subsequent arguments. The pull-back is defined by $\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow := \mathbf{Fl}^{\mathbf{u}}_{-\lambda}\!\uparrow$, and the Lie derivative $\mathcal{L}_{\mathbf{u}}(\mathbf{w}) \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ of a tangent vector field $\mathbf{w} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ along a tangent vector field $\mathbf{u} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ is the derivative of the pull-back by the relevant flow

$$\mathcal{L}_{\mathbf{u}}(\mathbf{w}) := \partial_{\lambda=0}\,(\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\mathbf{w}) = \partial_{\lambda=0}\,T\mathbf{Fl}^{\mathbf{u}}_{-\lambda} \circ \mathbf{w} \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda}. \tag{50}$$

Push–pull of scalar fields are change of base points, and hence the Lie derivative coincides with directional derivative. Push–pull of tensors are defined by invariance.

Adopting the notation $\mathbf{u}f := \mathcal{L}_{\mathbf{u}}(f)$, with $f \in \mathrm{C}^1(\mathbf{M} \mapsto \Re)$ any scalar field, the *commutator* of tangent vector fields $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ is the skew-symmetric tangent vector-valued differential operator defined by

$$[\mathbf{u}\,,\mathbf{w}]f := (\mathbf{u}\mathbf{w} - \mathbf{w}\mathbf{u})f = (\mathcal{L}_{\mathbf{u}}\mathcal{L}_{\mathbf{w}} - \mathcal{L}_{\mathbf{w}}\mathcal{L}_{\mathbf{u}})f. \tag{51}$$

A basic theorem concerning Lie derivatives states that $\mathcal{L}_{\mathbf{u}}(\mathbf{w}) = [\mathbf{u}\,,\mathbf{w}]$ and hence the commutator of tangent vector fields is called the Lie *bracket*. Moreover, for any injective morphism $\boldsymbol{\phi} \in \mathrm{C}^1(\mathbf{M} \mapsto \mathbf{N})$, the following push naturality property $[\boldsymbol{\phi}\!\uparrow\!\mathbf{v}\,,\boldsymbol{\phi}\!\uparrow\!\mathbf{u}] = \boldsymbol{\phi}\!\uparrow\![\mathbf{v}\,,\mathbf{u}]$ holds. A linear connection ∇ in a manifold $\mathbf{M}$ fulfils the characteristic properties of a point derivation

$$\begin{aligned}
\nabla_{\alpha_1\mathbf{w}_1+\alpha_2\mathbf{w}_2}(\mathbf{u}) &= \alpha_1\nabla_{\mathbf{w}_1}(\mathbf{u}) + \alpha_2\nabla_{\mathbf{w}_2}(\mathbf{u}),\\
\nabla_{\mathbf{w}}(\alpha_1(\mathbf{u}_1) + \alpha_2(\mathbf{u}_2)) &= \alpha_1\nabla_{\mathbf{w}}(\mathbf{u}_1) + \alpha_2\nabla_{\mathbf{w}}(\mathbf{u}_2),\\
\nabla_{\mathbf{w}}(f\mathbf{u}) &= f\,\nabla_{\mathbf{w}}(\mathbf{u}) + (\nabla_{\mathbf{w}}f)(\mathbf{u}),
\end{aligned} \tag{52}$$

with $\alpha_1, \alpha_2 \in \mathrm{C}^1(\mathbf{M} \mapsto \Re)$ scalar fields and $\mathbf{u}, \mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2 \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ tangent vector fields. In terms of parallel transport $\Uparrow$ ($\Downarrow$ denotes the backward parallel transport) along a curve $\mathbf{c} \in \mathrm{C}^1(\Re \mapsto \mathbf{M})$ with $\mathbf{w}_{\mathbf{x}} = \partial_{\lambda=0}\,\mathbf{c}(\lambda)$ and $\mathbf{x} = \mathbf{c}(0)$, the parallel derivative of a vector field $\mathbf{w} \in \mathrm{C}^1(\mathbf{M} \mapsto T\mathbf{M})$ according to a connection is defined by

$$\nabla_{\mathbf{w}}(\mathbf{u}) := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\!\Downarrow\!(\mathbf{u} \circ \mathbf{c})(\lambda). \tag{53}$$

Parallel-transported vector fields $(\mathbf{u} \circ \mathbf{c})(\lambda) = \mathbf{c}(\lambda)\!\Uparrow\!\mathbf{u}_{\mathbf{x}}$ are characterised by a null parallel derivative, because

$$\nabla_{\mathbf{w}_{\mathbf{x}}}(\mathbf{u}) := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\!\Downarrow\!(\mathbf{u} \circ \mathbf{c})(\lambda) = \partial_{\lambda=0}\,(\mathbf{c}(\lambda)\!\Downarrow \circ\, \mathbf{c}(\lambda)\!\Uparrow)\mathbf{u}_{\mathbf{x}} = \partial_{\lambda=0}\,\mathbf{u}_{\mathbf{x}} = 0.$$

The parallel transport of a tensor field is defined by invariance, and the parallel derivative fulfils a Leibniz rule, which for a covector field $\boldsymbol{\alpha} : \mathbf{M} \mapsto T^*\mathbf{M}$ writes

$$\langle \nabla_{\mathbf{w}}(\boldsymbol{\alpha}), \mathbf{u}\rangle = \nabla_{\mathbf{w}}\langle \boldsymbol{\alpha}, \mathbf{u}\rangle - \langle \boldsymbol{\alpha}, \nabla_{\mathbf{w}}(\mathbf{u})\rangle. \tag{54}$$

In a Riemann manifold $(\mathbf{M}\,,\mathbf{g})$, a linear connection ∇ is *metric-preserving* if the metric is invariant under parallel transport

$$\mathbf{g}_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}}\,,\mathbf{v}_{\mathbf{x}}) = \mathbf{g}_{\mathbf{c}(\lambda)}(\mathbf{c}(\lambda)\!\Uparrow\!\mathbf{u}_{\mathbf{x}}\,,\mathbf{c}(\lambda)\!\Uparrow\!\mathbf{v}_{\mathbf{x}}), \tag{55}$$

so that its parallel derivative vanishes $\nabla\mathbf{g} = 0$.

The curvature of the connection is the operator CURV which maps tensorially a tangent vector field $\mathbf{s} : \mathbf{M} \mapsto T\mathbf{M}$ to a tangent vector-valued two-form[12] CURV$(\mathbf{s})$ defined by

$$\mathrm{CURV}(\mathbf{s})(\mathbf{u}\,,\mathbf{w}) := ([\nabla_{\mathbf{u}}\,,\nabla_{\mathbf{w}}] - \nabla_{[\mathbf{u}\,,\mathbf{w}]})(\mathbf{s}), \tag{56}$$

and the torsion TORS is the tangent vector-valued two-form defined by

$$\mathrm{TORS}(\mathbf{u}\,,\mathbf{w}) := \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} - [\mathbf{u}\,,\mathbf{w}]. \tag{57}$$

A connection with vanishing torsion is named *torsion-free* or *symmetric*, and a connection with vanishing curvature is said to be *curvature-free* or *flat*. The Levi-Civita connection is the unique one that is metric and symmetric.

[12] Tensoriality of a linear map, acting on vector fields and generating a vector field, means that point values of the image field depend only on the values of the source fields at the same point. An (exterior) form is a vector-valued, tensorial, alternating multilinear map.

The modern way to introduce integral transformations is to consider maximal-forms, that are forms of order equal to the manifold dimension, as geometric objects to be integrated over a (orientable) manifold and to resort to the notion of exterior differential of a form [63,64].

In a m-dimensional manifold $\mathbf{M}$, let $\boldsymbol{\Gamma}$ be any n-dimensional submanifold ($m \geq n$) with $(n-1)$-dimensional boundary manifold $\partial\boldsymbol{\Gamma}$.

The classical Stokes formula, in its modern formulation by Volterra, characterises the exterior derivative of a $(n-1)$-form $\boldsymbol{\omega} : \mathbf{M} \mapsto \mathrm{ALT}^{n-1}(T\mathbf{M})$, defined as the n-form $d\boldsymbol{\omega} : \mathbf{M} \mapsto \mathrm{ALT}^{n}(T\mathbf{M})$ fulfilling the identity

$$\int_{\boldsymbol{\Gamma}} d\boldsymbol{\omega} = \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega}. \tag{58}$$

Being $\partial\partial\boldsymbol{\Gamma} = 0$ for any manifold $\boldsymbol{\Gamma}$, also $dd\boldsymbol{\omega} = 0$ for any form $\boldsymbol{\omega}$ [63,64].

The exterior derivative of differential forms commutes with the pull-back by an injective immersion $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$ between manifolds $\mathbf{M}$ and $\mathbf{N}$:

$$d \circ \boldsymbol{\chi}\!\downarrow = \boldsymbol{\chi}\!\downarrow \circ\, d, \tag{59}$$

a result inferred, from Stokes and integral transformation formulae

$$\begin{aligned} \int_{\boldsymbol{\Gamma}} d(\boldsymbol{\chi}\!\downarrow\!\boldsymbol{\omega}) &= \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\chi}\!\downarrow\!\boldsymbol{\omega} = \oint_{\boldsymbol{\chi}(\partial\boldsymbol{\Gamma})} \boldsymbol{\omega} \\ &= \oint_{\partial\boldsymbol{\chi}(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \int_{\boldsymbol{\chi}(\boldsymbol{\Gamma})} d\boldsymbol{\omega} = \int_{\boldsymbol{\Gamma}} \boldsymbol{\chi}\!\downarrow\!(d\boldsymbol{\omega}). \end{aligned} \tag{60}$$

Then, for $\mathbf{v} := \partial_{\lambda=0}\, \boldsymbol{\chi}_{\lambda}$ we infer that

$$\mathcal{L}_{\mathbf{v}}\,(d\boldsymbol{\omega}) = d\,\mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}). \tag{61}$$

The *geometric homotopy formula* relates the boundary chain generated by the extrusion of a manifold $\boldsymbol{\Gamma}$ and of its boundary $\partial\boldsymbol{\Gamma}$, as follows:

$$\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda)) = \boldsymbol{\chi}_{\lambda}(\boldsymbol{\Gamma}) - \boldsymbol{\Gamma} - J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \lambda),$$

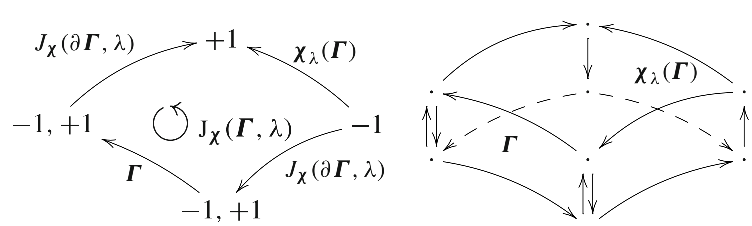
with $\lambda \in \mathcal{Z}$ extrusion parameter and $\boldsymbol{\chi} : \boldsymbol{\Gamma} \times \mathcal{Z} \mapsto \mathbf{M}$ extrusion map fulfilling the commutative diagram

$$\begin{tikzcd} \boldsymbol{\Gamma} \times \mathcal{Z} \arrow[r, "\boldsymbol{\chi}_{\lambda}"] \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z}}"'] & \mathbf{M} \arrow[d, "t_{\mathcal{Z}}"] \\ \mathcal{Z} \arrow[r, "\boldsymbol{\theta}_{\lambda}"] & \mathcal{Z} \end{tikzcd} \iff t_{\mathcal{Z}} \circ \boldsymbol{\chi}_{\lambda} = \boldsymbol{\theta}_{\lambda} \circ \boldsymbol{\pi}_{\mathcal{Z}}, \tag{62}$$

with $\boldsymbol{\theta}_{\lambda} : \mathcal{Z} \mapsto \mathcal{Z}$ the translation defined by $\boldsymbol{\theta}_{\lambda}(\alpha) := \alpha + \lambda$ for $\alpha, \lambda \in \mathcal{Z}$.

The signs in the formula are motivated as follows.

The orientation of the $(n+1)$-dimensional flow tube $J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda)$ induces an orientation on its boundary $\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda))$. In the boundary chain, composed by the elements $\boldsymbol{\chi}_{\lambda}(\boldsymbol{\Gamma})$, $\boldsymbol{\Gamma}$ and $J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \lambda)$, each one with the induced orientation, the element $\boldsymbol{\chi}_{\lambda}(\boldsymbol{\Gamma})$ has orientation opposed to the orientation of $\boldsymbol{\chi}_0(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma}$ and $J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \lambda)$, as depicted in the diagrams (63), for $\dim \boldsymbol{\Gamma} = 1$ and $\dim \boldsymbol{\Gamma} = 2$.



(63)

Let $\boldsymbol{\omega}$ be an n-form defined on the $(n+1)$-manifold $J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda)$ spanned by extrusion of the n-manifold $\boldsymbol{\Gamma}$, so that the geometric homotopy formula gives

$$\int_{\boldsymbol{\chi}_{\lambda}(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \oint_{\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \lambda)} \boldsymbol{\omega} + \int_{\boldsymbol{\Gamma}} \boldsymbol{\omega}. \tag{64}$$

Differentiation with respect to the extrusion-time yields

$$\partial_{\lambda=0} \int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \partial_{\lambda=0} \left(\oint_{\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\lambda))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\lambda)} \boldsymbol{\omega} \right). \tag{65}$$

Then, denoting by $\mathbf{v} := \partial_{\lambda=0}\, \boldsymbol{\chi}_\lambda$ the velocity field of the extrusion, applying Stokes formula and taking into account that by the Fubini theorem [62]

$$\begin{aligned} \partial_{\lambda=0} \int_{J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\lambda)} d\boldsymbol{\omega} &= \int_{\boldsymbol{\Gamma}} (d\boldsymbol{\omega}) \cdot \mathbf{v}, \\ \partial_{\lambda=0} \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\lambda)} \boldsymbol{\omega} &= \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v}, \end{aligned} \tag{66}$$

we get the integral *extrusion formula*

$$\boxed{\partial_{\lambda=0} \int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \int_{\boldsymbol{\Gamma}} (d\boldsymbol{\omega}) \cdot \mathbf{v} + \oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v}.} \tag{67}$$

On the other hand, taking the time rate of the integral transformation formula leads to the Lie-Reynolds formula

$$\partial_{\lambda=0} \int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \partial_{\lambda=0} \int_{\boldsymbol{\Gamma}} (\boldsymbol{\chi}_\lambda{\downarrow}\boldsymbol{\omega}) = \int_{\boldsymbol{\Gamma}} \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}). \tag{68}$$

Comparing the expressions in Eq. (68) and in Eq. (67) and applying Stokes' formula to get the transformation

$$\oint_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v} = \int_{\boldsymbol{\Gamma}} d(\boldsymbol{\omega} \cdot \mathbf{v}), \tag{69}$$

a standard localisation yields the *differential homotopy formula* expressing the Lie derivative of a k-form in terms of exterior derivatives:

$$\boxed{\mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}) = d(\boldsymbol{\omega} \cdot \mathbf{v}) + (d\boldsymbol{\omega}) \cdot \mathbf{v}.} \tag{70}$$

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