The Strain Gap Method
Existence Uniqueness and Convergence Properties

Giovanni Romano, Marina Diaco
University of Naples Federico II
Naples, Italy

Summary

We analyse a three-field method for the approximate solution of structural problems based on a suitable variant of the HU-WASHIZU functional in which the lack of kinematic compatibility, called the strain gap, is assumed as an indipendent variable. A full discussion of wellposedness and convergence properties is provided. A comparison with previous treatments is performed and computational issues are discussed.

Introduction

The strain gap method is a three-field method for the approximate analysis of linear elastostatic problems based on a suitable variant of the standard HU-WASHIZU functional in which the lack of kinematic compatibility, the strain gap, is assumed as an indipendent variable. The analysis of the well posedness of the discrete method is developed in the context of two-field mixed formulations by grouping toghether the displacement and strain gap trial fields and by splitting the discrete problem into a sequence of a reduced problem and of a stress recovery problem. Error bounds estimates are obtained by a suitable specialization of basic results due to F. BREZZI [3], [4]. Recent contributions by the first author and co-workers are also resorted to [7], [8]. On these bases it is possible to provide sufficient criteria to assess the well posedness of the method and asymptotic estimates of the rate of convergence in energy norms. It is shown that the enhanced strain method proposed by J.C. SIMO and M.S. RIFAI [2] is a singular case of the strain gap method since the *a priori* satisfaction of the discrete kinematic compatibility formally eliminates the stresses from the problem. Our analysis reveals that the evaluation of the stress field is a direct consequence of a consistent variational formulation and shows that the stress interpolation plays a basic role in the convergence analysis. Previous convergence treatments [5], [6] were in fact based on the assumption that the enhanced strain shape functions should be orthogonal to a suitable set of complete polynomials, a condition which is here shown to be unnecessary.

Mixed Formulation

Let us consider a structural problem defined on a regular bounded domain $\mathit{\Omega}$ of an euclidean space and governed by a kinematic operator $\mathbf{B}$ which is the regular part of a distributional differential operator $\mathbb{B} : \mathcal{V} \mapsto \mathbb{D}'$ of order m acting on kinematic fields $\mathbf{u} \in \mathcal{V}$ that are square integrable on $\mathit{\Omega}$ and such that the corresponding distributional strain field $\mathbb{B}\mathbf{u} \in \mathbb{D}'$ is square integrable on a finite subdivision $\mathcal{T}_{\mathbf{u}}(\mathit{\Omega})$ of $\mathit{\Omega}$. The kinematic space $\mathcal{V}$ is a pre-HILBERT space when endowed with the topology induced by the norm

$$\| \mathbf{u} \|_{\mathcal{V}}^2 = \| \mathbf{u} \|_H^2 + \| \mathbf{B}\mathbf{u} \|_{\mathcal{H}}^2 \,,$$

where H and $\mathcal{H}$ are the spaces of kinematic fields and linearized strain fields which are square integrable on $\mathit{\Omega}$ [13]. The conforming kinematisms $\mathbf{u} \in \mathcal{L}$ define a closed linear subspace $\mathcal{L} \subset H^m(\mathcal{T}(\mathit{\Omega})) \subset \mathcal{V}$ of the SOBOLEV space $H^m(\mathcal{T}(\mathit{\Omega}))$, where $\mathcal{T}(\mathit{\Omega})$ is a given finite subdivision of $\mathit{\Omega}$.

Thus $\mathcal{L} \subset H^m(\mathcal{T}(\mathit{\Omega}))$ is an HILBERT space and the operator $\mathbf{B} \in \mathrm{Lin}\,\{\mathcal{L}, \mathcal{H}\}$, that defines the linearized regular strain $\mathbf{B}\mathbf{u} \in \mathcal{H}$ associated with the conforming kinematic field $\mathbf{u} \in \mathcal{L}$, is linear and continuous.

The kinematic operator $\mathbf{B} \in \mathrm{Lin}\{\mathcal{L}, \mathcal{H}\}$ is assumed to be regular in the sense that for any $\mathcal{L} \subset \mathcal{V}$ the following conditions are met [13]

$$\begin{cases} \dim\, \mathrm{Ker}\, \mathbf{B} < +\infty\,, \\ \|\, \mathbf{Bu}\, \|_{\mathcal{H}} \geq c_{\mathbf{B}}\, \|\, \mathbf{u}\, \|_{\mathcal{L}/\mathrm{Ker}\, \mathbf{B}} \quad \forall\, \mathbf{u} \in \mathcal{L} \iff \mathrm{Im}\, \mathbf{B} \quad \text{closed in } \mathcal{H}\,. \end{cases}$$

A necessary and sufficient condition in order that the operator $\mathbf{B} \in \mathrm{Lin}\{\mathcal{L}, \mathcal{H}\}$ be regular is that an inequality of the KORN's type be fulfilled in every regular subdomain $\mathcal{P} \subseteq \Omega$ [13]

$$\|\, \mathbf{Bu}\, \|_{\mathcal{H}(\mathcal{P})} + \|\, \mathbf{u}\, \|_{H(\mathcal{P})} \geq \alpha\, \|\, \mathbf{u}\, \|_{H^m(\mathcal{P})} \quad \forall\, \mathbf{u} \in H^m(\mathcal{P})\,.$$

The equilibrium operator $\mathbf{B}' \in \mathrm{Lin}\{\mathcal{H}, \mathcal{L}'\}$ is the continuous operator dual to $\mathbf{B} \in \mathrm{Lin}\{\mathcal{L}, \mathcal{H}\}$. BANACH's closed range theorem [11], [12] ensures that $\mathrm{Im}\, \mathbf{B}' \subset \mathcal{L}'$ is closed too and that the orthogonality properties

$$\mathrm{Im}\, \mathbf{B} = (\,\mathrm{Ker}\, \mathbf{B}'\,)^{\perp}\,, \qquad \mathrm{Im}\, \mathbf{B}' = (\,\mathrm{Ker}\, \mathbf{B})^{\perp}\,,$$

hold true. Let $\boldsymbol{\sigma} \in \mathcal{H}$ and $\boldsymbol{\varepsilon} \in \mathcal{H}$ be the stress and the strain fields, $\ell \in \mathcal{L}'$ be a load functional and $\boldsymbol{\delta} \in \mathcal{H}$ be the an imposed field of distorsions. The linear elasticity operator $\mathcal{E} \in \mathrm{Lin}\{\mathcal{H}, \mathcal{H}\}$ is continuous, symmetric and $\mathcal{H}$-elliptic, that is such that

$$((\, \mathcal{E}\boldsymbol{\varepsilon}\,,\, \boldsymbol{\varepsilon}\,)) \geq c_{\mathcal{E}}\, \|\, \boldsymbol{\varepsilon}\, \|^2_{\mathcal{H}} \quad \forall\, \boldsymbol{\varepsilon} \in \mathcal{H}\,.$$

where $((\,\cdot\,,\,\cdot\,))$ is the inner product in $\mathcal{H}$. The elastostatic problem is then defined by the conditions

$$\begin{cases} ((\, \boldsymbol{\sigma}\,,\, \mathbf{B}\overline{\mathbf{u}}\,)) = \langle\, \ell\,,\, \overline{\mathbf{u}}\,\rangle & \forall\, \overline{\mathbf{u}} \in \mathcal{L} \quad \text{equilibrium}\,, \\ ((\, \mathcal{E}(\boldsymbol{\varepsilon} - \boldsymbol{\delta}) - \boldsymbol{\sigma}\,,\, \overline{\boldsymbol{\varepsilon}}\,)) = 0 & \forall\, \overline{\boldsymbol{\varepsilon}} \in \mathcal{H} \quad \text{elastic law}\,, \\ ((\, \mathbf{Bu} - \boldsymbol{\varepsilon}\,,\, \overline{\boldsymbol{\sigma}}\,)) = 0 & \forall\, \overline{\boldsymbol{\sigma}} \in \mathcal{H} \quad \text{compatibility}\,. \end{cases}$$

The solutions of the elastostatic problem can be characterized as stationarity points of the HU-WASHIZU functional

$$\phi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}) = \tfrac{1}{2}\, ((\, \mathcal{E}(\boldsymbol{\varepsilon} - \boldsymbol{\delta})\,,\, \boldsymbol{\varepsilon} - \boldsymbol{\delta}\,)) + ((\, \boldsymbol{\sigma}\,,\, \mathbf{Bu} - \boldsymbol{\varepsilon}\,)) - \langle\, \ell\,,\, \mathbf{u}\,\rangle\,,$$

where $\mathbf{u} \in \mathcal{L}$, $\boldsymbol{\varepsilon} \in \mathcal{H}$, $\boldsymbol{\sigma} \in \mathcal{H}$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the duality pairing between $\mathcal{L}$ and its dual $\mathcal{L}'$. Defining the strain gap

$$\mathbf{g} := \mathbf{Bu} - \boldsymbol{\varepsilon} \in \mathcal{H}\,,$$

the functional $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u})$ can be re-written as

$$\varphi(\mathbf{u}, \mathbf{g}, \boldsymbol{\sigma}) = \tfrac{1}{2}\, ((\, \mathcal{E}(\mathbf{Bu} - \mathbf{g} - \boldsymbol{\delta})\,,\, \mathbf{Bu} - \mathbf{g} - \boldsymbol{\delta}\,)) + ((\, \boldsymbol{\sigma}\,,\, \mathbf{g}\,)) - \langle\, \ell\,,\, \mathbf{u}\,\rangle\,,$$

with $\mathbf{u} \in \mathcal{L}$, $\mathbf{g} \in \mathcal{H}$, $\boldsymbol{\sigma} \in \mathcal{H}$ and the stationarity condition are given by

$$\begin{cases} ((\, \mathcal{E}(\mathbf{Bu} - \mathbf{g})\,,\, \mathbf{B}\overline{\mathbf{u}}\,)) = \langle\, \ell\,,\, \overline{\mathbf{u}}\,\rangle + ((\, \mathcal{E}\boldsymbol{\delta}\,,\, \mathbf{B}\overline{\mathbf{u}}\,)) & \forall\, \overline{\mathbf{u}} \in \mathcal{L}\,, \\ ((\, \mathcal{E}(\mathbf{Bu} - \mathbf{g}) - \boldsymbol{\sigma}\,,\, \overline{\mathbf{g}}\,)) = ((\, \mathcal{E}\boldsymbol{\delta}\,,\, \overline{\mathbf{g}}\,)) & \forall\, \overline{\mathbf{g}} \in \mathcal{H}\,, \\ ((\, \mathbf{g}\,,\, \overline{\boldsymbol{\sigma}}\,)) = 0 & \forall\, \overline{\boldsymbol{\sigma}} \in \mathcal{H}\,. \end{cases}$$

Note that the last variational condition imposes the kinematic compatibility by requiring the vanishing of the strain gap. It is convenient to rephrase the three-field problem above as a two field problem. To this end we introduce the dual product HILBERT spaces

$$\mathcal{X} := \mathcal{L} \times \mathcal{H}\,, \quad \mathcal{X}' = \mathcal{L}' \times \mathcal{H}\,,$$

with the standard inner product between $\mathbf{x} = \{\mathbf{u}, \mathbf{g}\} \in \mathcal{X}$ and $\mathbf{x}' = \{f, \boldsymbol{\sigma}\} \in \mathcal{X}'$ defined by $\langle\, \mathbf{x}'\,,\, \mathbf{x}\,\rangle = \langle\, f\,,\, \mathbf{u}\,\rangle + ((\, \boldsymbol{\sigma}\,,\, \mathbf{g}\,))$, the continuous bilinear forms

$$\mathbf{a}\,(\mathbf{x}, \overline{\mathbf{x}}) := ((\, \mathcal{E}(\mathbf{Bu} - \mathbf{g})\,,\, \mathbf{B}\overline{\mathbf{u}} - \overline{\mathbf{g}}\,))\,, \qquad \mathbf{j}\,(\boldsymbol{\sigma}, \overline{\mathbf{x}}) := ((\, \boldsymbol{\sigma}\,,\, \overline{\mathbf{g}}\,))\,,$$

and the continuous linear form $\langle\, \mathbf{f}\,,\, \overline{\mathbf{x}}\,\rangle := \langle\, \ell\,,\, \overline{\mathbf{u}}\,\rangle - ((\, \mathcal{E}\boldsymbol{\delta}\,,\, \overline{\mathbf{g}}\,))$.

Strain Gap Method

The strain gap method (**SGM**) provides approximate solutions of the three-field variational problem $\mathbb{M}$ by means of a conforming **FEM** interpolation based on three families of finite dimensional subspaces $\mathcal{L}_h \subset \mathcal{L}$, $\mathcal{D}_h \subset \mathcal{H}$, $\mathcal{S}_h \subset \mathcal{H}$ depending on a parameter h which goes to zero as the finite element mesh is refined ever more. By taking into account the isometric isomorphism between the HILBERT space $\mathcal{X}'_h$, dual to the linear subspace $\mathcal{X}_h \subseteq \mathcal{X}$, and the quotient HILBERT space $\mathcal{X}'/\mathcal{X}_h^{\perp}$ and by setting $\mathcal{X}_h = \mathcal{L}_h \times \mathcal{D}_h$ the approximate mixed problem $\mathbb{M}_h$ can be expressed by

$$\mathbb{M}_h) \quad \begin{cases} \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}}_h) + \mathbf{j}\,(\boldsymbol{\sigma}_h, \overline{\mathbf{x}}_h) = \langle\, \mathbf{f}\,, \overline{\mathbf{x}}_h \,\rangle & \forall \overline{\mathbf{x}}_h \in \mathcal{X}_h\,, \\ \mathbf{j}\,(\overline{\boldsymbol{\sigma}}_h, \mathbf{x}_h) = 0 & \forall \overline{\boldsymbol{\sigma}}_h \in \mathcal{S}_h\,. \end{cases} \iff \begin{cases} \mathbf{A}_h \mathbf{x}_h + \mathbf{J}'_h \boldsymbol{\sigma}_h = \mathbf{f}_h + \mathcal{X}_h^{\perp}\,, \\ \mathbf{J}_h \mathbf{x}_h = \mathcal{S}_h^{\perp}\,. \end{cases}$$

where $\mathbf{A}_h \in \mathrm{Lin}\,\{\mathcal{X}_h, \mathcal{X}'_h\}$, $\mathbf{J}_h \in \mathrm{Lin}\,\{\mathcal{X}_h, \mathcal{S}'_h\}$, $\mathbf{J}'_h \in \mathrm{Lin}\,\{\mathcal{S}_h, \mathcal{X}'_h\}$ are the operators associated with the bilinearformsand $\langle\, \mathbf{f}_h\,, \overline{\mathbf{x}}_h \,\rangle := \langle\, \mathbf{f}\,, \overline{\mathbf{x}}_h \,\rangle \quad \forall \overline{\mathbf{x}}_h \in \mathcal{X}_h$. Weset $\tilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp}$ sothat $\mathrm{Ker}\,\mathbf{J}_h = \mathcal{L}_h \times \tilde{\mathcal{D}}_h$. The mixed problem $\mathbb{M}_h$ can thus be split into a sequence of two problems.

- The *reduced* problem in the product space $\mathrm{Ker}\,\mathbf{J}_h \times \mathrm{Ker}\,\mathbf{J}_h$

$$\mathbb{M}_{oh}) \quad \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}}_h) = \langle\, \mathbf{f}_h\,, \overline{\mathbf{x}}_h \,\rangle \quad \forall \overline{\mathbf{x}}_h \in \mathrm{Ker}\,\mathbf{J}_h\,, \quad \mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h\,,$$

  which can be explicitly written as

$$\mathbb{M}_{oh}) \quad \begin{cases} ((\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)\,, \mathbf{B}\overline{\mathbf{u}}_h \,)) = \langle\, \ell\,, \overline{\mathbf{u}}_h \,\rangle & \forall \overline{\mathbf{u}}_h \in \mathcal{L}_h\,, \quad \mathbf{u}_h \in \mathcal{L}_h\,, \\ ((\, \mathcal{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)\,, \overline{\mathbf{g}}_h \,)) = ((\, \mathcal{E}\boldsymbol{\delta}\,, \overline{\mathbf{g}}_h \,)) & \forall \overline{\mathbf{g}}_h \in \tilde{\mathcal{D}}_h\,, \quad \mathbf{g}_h \in \tilde{\mathcal{D}}_h\,. \end{cases}$$

- The *stress recovery* problem

$$\mathbb{S}_h) \quad \mathbf{j}\,(\boldsymbol{\sigma}_h, \overline{\mathbf{x}}_h) = -\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}}_h) + \langle\, \mathbf{f}_h\,, \overline{\mathbf{x}}_h \,\rangle \quad \forall \overline{\mathbf{x}}_h \in \mathrm{Ker}\,\mathbf{J}_h\,, \quad \boldsymbol{\sigma}_h \in \mathcal{S}_h\,,$$

  where $\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h$ is solution of the problem $\mathbb{M}_{oh}$.

The problem $\mathbb{S}_h$ admits a unique solution for any data if and only if $\mathrm{Ker}\,\mathbf{J}'_h = \mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\mathbf{o}\}$.

By introducing the reduced discrete operator $\mathbf{A}_{oh} \in \mathrm{Lin}\,\{\,\mathrm{Ker}\,\mathbf{J}_h, (\,\mathrm{Ker}\,\mathbf{J}_h)'\}$ and the reduced discrete functional $\mathbf{f}_{oh} \in (\,\mathrm{Ker}\,\mathbf{J}_h)'$ defined by

$$\begin{cases} \langle\, \mathbf{A}_{oh}\,\mathbf{x}_h\,, \overline{\mathbf{x}}_h \,\rangle = \mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}}_h) & \forall \mathbf{x}_h, \overline{\mathbf{x}}_h \in \mathrm{Ker}\,\mathbf{J}_h\,, \\ \langle\, \mathbf{f}_{oh}\,, \overline{\mathbf{x}}_h \,\rangle = \langle\, \mathbf{f}\,, \overline{\mathbf{x}}_h \,\rangle & \forall \overline{\mathbf{x}}_h \in \mathrm{Ker}\,\mathbf{J}_h\,, \end{cases}$$

the problem $\mathbb{M}_{oh}$ can be written in the form

$$\mathbb{M}_{oh}) \quad \mathbf{A}_{oh}\mathbf{x}_h = \mathbf{f}_{oh}\,, \quad \mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h\,,$$

where

$$\begin{cases} \mathbf{A}_{oh}\,\mathbf{x}_h = \mathbf{A}_h\,\mathbf{x}_h + (\,\mathrm{Ker}\,\mathbf{J}_h)^{\perp} & \forall \mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h\,, \\ \mathbf{f}_{oh} = \mathbf{f}_h + (\,\mathrm{Ker}\,\mathbf{J}_h)^{\perp}\,. \end{cases}$$

The problem $\mathbb{M}_{oh}$ admits a unique solution for any data if and only if $\mathrm{Ker}\,\mathbf{A}_{oh} = \mathrm{Ker}\,\mathbf{A}_h \cap \mathrm{Ker}\,\mathbf{J}_h = \{\mathbf{o}\}$.

The bounds of the mean square error of the approximate solution are based on uniform closedness properties concerning the discrete bilinear forms $\mathbf{a} \in \mathrm{Bil}\,\{\,\mathrm{Ker}\,\mathbf{J}_h \times \mathrm{Ker}\,\mathbf{J}_h\}$ and $\mathbf{j} \in \mathrm{Bil}\,\{\mathcal{S}_h \times \mathcal{X}_h\}$ which can be expressed by the uniform inequalities

$$\begin{cases} \inf\limits_{\mathbf{x}_h \in \mathcal{X}_h} \; \sup\limits_{\boldsymbol{\sigma}_h \in \mathcal{S}_h} \dfrac{\mathbf{j}\,(\boldsymbol{\sigma}_h, \mathbf{x}_h)}{\|\, \boldsymbol{\sigma}_h \,\|_{\mathcal{H}/\mathrm{Ker}\,\mathbf{J}'_h}\, \|\, \mathbf{x}_h \,\|_{\mathcal{X}/\mathrm{Ker}\,\mathbf{J}_h}} \geq c_{\mathbf{j}} > 0\,, \\ \inf\limits_{\overline{\mathbf{x}}_h \in \mathrm{Ker}\,\mathbf{J}_h} \; \sup\limits_{\mathbf{x}_h \in \mathrm{Ker}\,\mathbf{J}_h} \dfrac{\mathbf{a}\,(\mathbf{x}_h, \overline{\mathbf{x}}_h)}{\|\, \mathbf{x}_h \,\|_{\mathcal{X}}\, \|\, \overline{\mathbf{x}}_h \,\|_{\mathcal{X}}} \geq c_o > 0\,, \end{cases}$$

whith $c_{\mathbf{j}}$ and c_o independent of the mesh parameter h.

Uniform conditions of this kind are referred to in the literature as *discrete* inf-sup *conditions* or also as **LBB** (LADYZHENSKAYA-BABUŠKA-BREZZI) conditions [4]. The discrete inf-sup condition concerning the bilinear form $\mathbf{a}$ can be deduced from the stronger condition of $\operatorname{Ker}\mathbf{J}_h$-ellipticity of the form $\mathbf{a} \in \operatorname{Bil}\{\operatorname{Ker}\mathbf{J}_h \times \operatorname{Ker}\mathbf{J}_h\}$ since we have [9]

Uniform ellipticity. *Let the properties* $\operatorname{Ker}\mathbf{B} \cap \mathcal{L}_h = \{\mathbf{o}\}_{\mathcal{L}}$, $\mathbf{B}\mathcal{L}_h \cap \tilde{\mathcal{D}}_h^{\perp} = \{\mathbf{o}\}_{\mathcal{H}}$, $\operatorname{Im}\mathbf{B}$ *closed in* $\mathcal{H}$ *and* $\mathbf{B}\mathcal{L}_h + \tilde{\mathcal{D}}_h$ *unoformly closed in* $\mathcal{H}$ *be fulfilled. Then*

$$\mathbf{a}(\mathbf{x}_h, \mathbf{x}_h) \geq c_o \,\|\, \mathbf{x}_h \,\|_{\mathcal{X}}^2 \quad \forall \mathbf{x}_h \in \operatorname{Ker}\mathbf{J}_h\,,$$

that is the symmetric bilinear form $\mathbf{a} \in \operatorname{Bil}\{\operatorname{Ker}\mathbf{J}_h \times \operatorname{Ker}\mathbf{J}_h\}$ *is uniformly elliptic.* □

The previous result, the uniqueness condition $\mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\mathbf{o}\}$ and the uniform closedness of $\operatorname{Im}\mathbf{J}_h = \mathcal{D}_h + \mathcal{S}_h^{\perp}$ or equivalently of $\operatorname{Im}\mathbf{J}'_h = \mathcal{S}_h + \mathcal{D}_h^{\perp}$ lead to the following error bound [10]

$$\|\, \mathbf{x} - \mathbf{x}_h \,\|_{\mathcal{X}} + \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \leq c_{\mathbf{x}} \inf_{\overline{\mathbf{x}}_h \in \mathcal{X}_h} \|\, \mathbf{x} - \overline{\mathbf{x}}_h \,\|_{\mathcal{X}} + c_{\boldsymbol{\sigma}} \inf_{\overline{\boldsymbol{\sigma}}_h \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h \,\|_{\mathcal{H}}\,,$$

which can be written in terms of the three fields $\{\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h\}$ as

$$\|\, \mathbf{u} - \mathbf{u}_h \,\|_{\mathcal{L}} + \|\, \mathbf{g}_h \,\|_{\mathcal{H}} + \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_{\mathcal{H}} \;\leq\; c_{\mathbf{x}} \inf_{\overline{\mathbf{u}}_h \in \mathcal{L}_h} \|\, \mathbf{u} - \overline{\mathbf{u}}_h \,\|_{\mathcal{L}} + c_{\boldsymbol{\sigma}} \inf_{\overline{\boldsymbol{\sigma}}_h \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h \,\|_{\mathcal{H}}\,,$$

with $c_{\mathbf{x}}$ and $c_{\boldsymbol{\sigma}}$ independent of h.

Asymptotic Rate of Convergence

Let us consider a two or three-dimensional elastostatic problem and assume that the bounded domain Ω, the data and the elasticity $\mathcal{E}$ be regular enough to ensure that the displacement and the stress solutions meet the regularity properties $\mathbf{u} \in H^2(\Omega)$ and $\boldsymbol{\sigma} \in H^1(\Omega)$.

We consider isoparametric finite element meshes which enjoy the properties

- the displacement shape functions on the reference element K generate the vectorial polynomial linear subspace $P_1(K)$ whosecomponentsarearbitrarypolynomialsofdegree ≤ 1 orthesubspace $\mathrm{Q}_1(K)$ whose components are arbitrary polynomials of degree ≤ 1 in each variable,
- the stress shape functions generate a tensorial subspace containing the linear subspace $\mathrm{Q}_0(K) = P_0(K)$ whose components are arbitrary constant tensors.

Then a standard result of polynomial approximation theory [1] ensures that

$$\inf_{\overline{\mathbf{u}} \in \mathcal{L}_h} \|\, \mathbf{u} - \overline{\mathbf{u}}_h \,\|_1 \leq c_{\mathbf{u}}\, h \,|\, \mathbf{u} \,|_2\,, \quad \inf_{\overline{\boldsymbol{\sigma}} \in \mathcal{S}_h} \|\, \boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h \,\|_0 \leq c_{\boldsymbol{\sigma}}\, h \,|\, \boldsymbol{\sigma} \,|_1\,,$$

where $\|\cdot\|_m$ is the norm in the SOBOLEV space $H^m(\Omega)$ and $|\cdot|_m$ is the corresponding seminorm involving only derivatives of total order m. The error bounds provide the following linear estimates for the rate of convergence of the approximate solution to the exact one in terms of energy norms

$$\begin{cases} \|\, \mathbf{u} - \mathbf{u}_h \,\|_1 + \|\, \mathbf{g}_h \,\|_0 \leq \alpha_{\mathbf{u}}\, h \,(\,|\, \mathbf{u} \,|_2 + |\, \boldsymbol{\sigma} \,|_1\,)\,, \\ \|\, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \,\|_0 \;\leq\; \alpha_{\boldsymbol{\sigma}}\, h \,(\,|\, \mathbf{u} \,|_2 + |\, \boldsymbol{\sigma} \,|_1\,)\,. \end{cases}$$

It is worth noting that, as was to be expected, no role is played by the shape functions of the strain gap in determining the asymptotic rate of convergence. In fact the exact straingapis zero and hence every interpolating subspace does the job.

Computational Issues

Let us analyse the uniform well-posedness conditions

$i)$ $\quad \mathbf{B}\mathcal{L}_h \cap \tilde{\mathcal{D}}_h = \{\mathbf{o}\}\,, \quad \mathbf{B}\mathcal{L}_h + \tilde{\mathcal{D}}_h$ uniformly closed in $\mathcal{H}$,

$ii)$ $\quad \mathcal{S}_h \cap \mathcal{D}_h^{\perp} = \{\mathbf{o}\}\,, \quad \mathcal{S}_h + \mathcal{D}_h^{\perp}$ uniformly closed in $\mathcal{H}$.

The former involves the conforming subspace $\mathcal{L}_h$ which depends on the *a priori* unknown element assembly operations and hence cannot be checked in a finite element analysis. This shortcoming can be circumvented by considering the larger non-conforming kinematic space $\mathcal{V}_h \supseteq \mathcal{L}_h$ formed by the cartesian product of the local kinematic spaces generated by the displacement shape functions over the single elements, to get the local sufficient conditions $\mathbf{B}\mathcal{V}_h \cap \tilde{\mathcal{D}}_h = \{\mathbf{o}\}_{\mathcal{H}}$ and $\mathbf{B}\mathcal{V}_h + \tilde{\mathcal{D}}_h$ uniformly closed in $\mathcal{H}$. What we really need are conditions susceptible tobe verified on the reference element K of an isoparametric finite element mesh. Let us append the subscript $_K$ tofields defined overthereference element K. The condition $ii)$ canbeeasilysatisfied by choosing $\mathcal{D}_K$ so that $\mathcal{S}_K \subset \mathcal{D}_K$. IF the isoparametric maps are homothetic it turns out that $\mathbf{B}_K = \mathbf{B}$ and the condition $\mathbf{B}_K\mathcal{V}_K \cap \tilde{\mathcal{D}}_K = \{\mathbf{o}\}$ can be imposed on the reference element by computing the GRAM determinant of the set of shape functions which generates the space $\mathbf{B}\mathcal{V}_K \times \tilde{\mathcal{D}}_K$ and implies the sufficient condition $\mathbf{B}\mathcal{V}_h \cap \tilde{\mathcal{D}}_h = \{\mathbf{o}\}_{\mathcal{H}}$. Furtherinthecaseof homothetic mapstheuniform closednessof $\mathbf{B}\mathcal{V}_h + \tilde{\mathcal{D}}_h$ in $\mathcal{H}$ is trivially satisfied [10]. Theeffectiveness of the **SGM** requires that $\tilde{\mathcal{D}}_h = \mathcal{D}_h \cap \mathcal{S}_h^{\perp} \neq \{\mathbf{o}\}$. This condition must be checked by the evaluation of an $\mathcal{L}^2(\mathcal{P})$ inner product on each actual element $\mathcal{P}$. In performing the transformation back to the reference element K the integration will involve the unknown jacobian determinant of the isoparametric map. No problem arises if we consider affine equivalent finite element meshes since the constant jacobian determinant field is irrilevant in imposing the orthogonality conditions. In the case of general isoparametric maps the jacobian determinant is no more constant. As a consequence the integral of the product of any two fields on an actual element is no more proportional to the integral of the product of the corresponding two fields in the reference element. A skilful trick was proposed in [2] in order to overcome this shortcoming in verifying their assumption that $\mathcal{D}_h \subset \mathcal{S}_h^{\perp}$. The authors of [2] proposed in fact to define the shape functions of the enhanced strains in the reference element as the quotient of simple polynomial expressions divided by the jacobian determinant. It follows that, in performing the integral transformation, the jacobian deteminant disappears from the integral over the reference element and the orthogonality condition can be simply verified once and for all in terms of simple polynomial expressions on the reference element. This procedure was also adopted in the convergence analysis of enhanced strain methods developed in [6] and can be carried out since the interpolation properties of the enhanced strains shape functions do not play any role due to the fact the the exact field to be interpolated is the null one.

Comparison with Previous Results

The strain gap method discussed here is based on an idea first contributed by J.C. SIMO and M.S. RIFAI in [2]. In their original analysis the authors of [2] recognized that the condition $\mathbf{B}\mathcal{L}_h \cap \tilde{\mathcal{D}}_h = \{\mathbf{o}\}_{\mathcal{H}}$ was necessary to get uniqueness of the displacement solution and that the subspace $\tilde{\mathcal{D}}_h$ of effective strain gaps should be non trivial to get an enhanced flexibility for coarse meshes with respect to the standard displacement method. Thebasic difference with our approach liesin thefact that in[2] theorthogonality condition $\mathcal{D}_h \subset \mathcal{S}_h^{\perp}$ was imposed as an essential requirement of the method. The elements of the subspace $\mathcal{D}_h \subset \mathcal{S}_h^{\perp}$ were named enhanced strains. According to our scheme the subspace $\mathcal{D}_h$ is rather the direct sum of two complementary subspaces. One of them $\tilde{\mathcal{D}}_h$ plays the same role as the enhanced strains, while the other one effects the necessary control on the interpolating stress field. This approach leads to a consistent method af approximation and permits to get a well-defined variational stress recovery and a full convergence result. The convergence analysis performed in[5] and[6]lead toclaimthatforsimplicial finiteelements (suchastrianglesand tetrahedra) the convergence requirements imply that the enhanced assumed strain method collapses into the displacement method. The underlying reason for this limitation phenomenon is that in proving convergence, due to the assumption $\mathcal{D}_h \subset \mathcal{S}_h^{\perp}$, the authors of [5] and [6] were compelled to invoke the BRAMBLE-HILBERT lemma [1], [12] in order to get a bound for the term

$$\sup_{\overline{\mathbf{g}}_h \in \mathcal{X}_h} \{ ((\, \mathcal{E}\,\mathbf{Bu}\,,\, \overline{\mathbf{g}}_h\,)) \mid \| \,\overline{\mathbf{g}}_h\, \|_{\mathcal{X}} \leq 1 \}\,.$$

To this end they had to assume that the subspace of enhanced strains must be $\mathcal{L}^2(K)$-orthogonal in elastic energy to the polynomial spaces till the degree $k-1$ if k is the degree of polynomials included in the subspace interpolating the displacement fields. For simplicial elements this polynomial space is $P_k(K)$ [1] and it turns out that in two and three-dimensional continua meshed by undistorted elements the approximate strains $\mathbf{Bu}_K$ compatible with the displacement fields $\mathbf{u}_K \in \mathcal{V}_K$ belong to $P_{k-1}(K)$. As a consequence the term $((\, \mathcal{E}\,\mathbf{Bu}_K\,,\, \mathbf{g}_K\,))$ vanishes for any $\mathbf{u}_K \in \mathcal{V}_K$, $\mathbf{g}_K \in \mathcal{D}_K$ and the reduced problem collapses into the standard diplacement method. For n-cube elements the limitation phenomenon does not necessarily occour since the relevant polynomial space is $\mathrm{Q}_k(K)$ but the compatible strain space is not included in $\mathrm{Q}_{k-1}(K)$. It can be shown [10] that for undistorted simplicial elements the limitation is not motivated by convergence requirements. Our analysis, which does not assume the condition $\mathcal{D}_h \subset \mathcal{S}_h^{\perp}$, reveals that the convergence of the **SGM** (and hence of the **EAS** method) does not require the $\mathcal{L}^2(K)$-orthogonality between enhanced strains and polynomials of degree $\leq k-1$. In fact the error bound estimates provided in [10] and quoted here are based on a consistent formulation of the discrete problem in which the stress fields are not eliminated. This fact permits to get a direct bound for the bilinear form $\mathbf{j}\,(\boldsymbol{\sigma}_h, \mathbf{x}_h)$ which was instead assumed to be zero in [5] and [6]. It is remarkable that our error estimates depend on the approximation properties of both the spaces $\mathcal{L}_h$ and $\mathcal{S}_h$ interpolating the displacements and the stresses. On the contrary the estimate of the error in terms of diplacements and enhanced strains derived in [5] and [6] were independend of the interpolation properties of the space $\mathcal{S}_h$. This fact makes the difference.

References

1. CIARLET P. G. (1978) The Finite Element Method for Elliptic Problems, North Holland, Amsterdam.
2. SIMO J.C., RIFAI M.S. (1990) A class of mixed assumed strain methodsandthe methodof incompatible modes. *Int. J. Num. Methods Engrg.* **29**, 1595-1638
3. F. BREZZI, On the existence, uniqueness and approximation of saddle point problems arising from lagrangian multipliers, *R.A.I.R.O. Anal. Numer.*, **8**, 129-151 (1974).
4. BREZZI F., FORTIN M. (1991) *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York.
5. REDDY B.D, SIMO J.C. (1995) Stability and convergence of a class of enhanced strain methods, *SIAM, J. Numer. Anal.* **32**, 1705-1728.
6. ARUNAKIRINATHAR K., REDDY B.D. (1995) Further results for enhanced strain methods with isoparametric elements, *Comp. Meth. Appl. Mech. Engrg.* **127** 127-143.
7. ROMANO G., ROSATI L., DIACO M. (1999) Well Posedness of Mixed Formulations in Elasticity, ZAMM, **79**, 435-454 .
8. ROMANO G., ROSATI L., DIACO M. (2000) A General Analysis of Mixed Methods, sub. to *SIAM, J. Num. Anal.*.
9. ROMANO G., MAROTTI DE SCIARRA F., DIACO M. (2000) Well posedness and numerical performances of the strain gap method, sub. to *Int. Jour. Num. Meth. Math. Engrg.*.
10. ROMANO G., MAROTTI DE SCIARRA F., DIACO M. (2000) Stability and convergence of the strain gap method, sub. to *SIAM, J. Numer. Anal.*.
11. H. BREZIS, Analyse Fonctionnelle, Théorie et applications, Masson Editeur, Paris (1983).
12. ROMANO G. (1999) Theory of structural models, Part I, Elements of Functional Analysis, Univ. Napoli Fed. II.
13. ROMANO G. (1999) Theory of structural models, Part II, Structural models, Univ. Napoli Federico II.