Composites Part B 114 (2017) 184–188

Contents lists available at ScienceDirect

Composites Part B

journal homepage: www.elsevier.com/locate/compositesb

Stress-driven versus strain-driven nonlocal integral model for elastic nano-beams

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ARTICLE INFO

Article history:
Received 6 December 2016
Accepted 8 January 2017
Available online 13 January 2017

Keywords:
Nonlocal elasticity
Integral elastic law
Size effects
Nano-beams
Nano-bars
CNT

ABSTRACT

In the strain-driven model of nonlocal elasticity proposed by Eringen, the elastic strain is defined by a Fredholm integral equation in which the stress is the output of a convolution between the local response to an elastic strain and a smoothing kernel dependent on a nonlocal parameter. In the wake of this proposal, size effects in nano-beams were investigated in literature by adopting a differential formulation considered to be equivalent to the integral one. Recent improvements have however revealed that equivalence requires also the fulfilment of constitutive boundary conditions. Moreover, this strain-driven nonlocal elastic problem has been shown to be ill-posed, being conflicting with equilibrium requirements. A stress-driven integral constitutive law provides the natural way to get well-posed nonlocal elastic problems for application to nano-structures. The new integral constitutive law is formulated with explicit reference to plane and straight nano-beams according to the standard Bernoulli-Euler structural model. The solution procedure based on the stress-driven nonlocal law is described and adopted for the solution of a simple statically indeterminate scheme, thus showing effectiveness of the new model for the structural design of nano-devices.

1. Introduction

Size effects in elastic nano-beams are usually investigated by simulating complex phenomena at the nano-scale by means of a nonlocal elastic law. Starting data are the standard macroscopic field of local elastic stiffness $\mathbf{E}$ and the inverse field of local elastic compliance $\mathbf{C} = \mathbf{E}^{-1}$.

In the original proposal by Eringen [1], the 3D purely elastic nonlocal model was defined by assuming that the elastic strain field ε^{el} is solution of a Fredholm integral equation.

Accordingly, the stress σ is output as convolution between the local response to the elastic strain and a scalar kernel dependent on a nonlocal parameter $\lambda > 0$:

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_{\Omega} \phi_\lambda(\mathbf{x} - \xi) \cdot \mathbf{E}(\xi) \cdot \boldsymbol{\varepsilon}^{el}(\xi)\, d\Omega_\xi, \tag{1}$$

with $\mathbf{x}, \xi$ position vectors in the actual placement Ω of the body.

The notation $d\Omega_\xi$ indicates that integration over Ω is performed with respect to the ξ variable.

The stress fields $\boldsymbol{\sigma}$ are subject to equilibrium conditions, while the total strain fields

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{th} + \boldsymbol{\varepsilon}^{el}, \tag{2}$$

sum of non-elastic (e.g. thermal) and elastic strain fields, must fulfil kinematic compatibility.

The model in Eq. (1) is referred to as the *strain-driven* nonlocal integral law, since the source field is the elastic strain $\boldsymbol{\varepsilon}^{el}$.

We will deal with linearised, plane and straight Bernoulli-Euler beam model, with axial abscissa x, end-points $a, b \in \Re$ and length $L = b - a$.

The flexural nonlocal elastic law is then expressed, in terms of an elastic curvature field $\chi^{el} \in \mathscr{H}$, square integrable on $[a, b]$, and of the local elastic flexural stiffness $K \in \mathscr{H}$.

The bending interaction field $M \in \mathscr{H}$ is output by the convolution

$$M(x) = \int_a^b \phi_\lambda(x - \xi) \cdot K(\xi) \cdot \chi^{el}(\xi)\, d\xi. \tag{3}$$

The scalar kernel $\phi_\lambda : \Re \mapsto [0, +\infty)$ depends on a positive

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http://dx.doi.org/10.1016/j.compositesb.2017.01.008
1359-8368/

nonlocal parameter $\lambda > 0$ and fulfilling the properties of symmetry and limit impulsivity:

$$\begin{cases} \phi_\lambda(x-\xi) = \phi_\lambda(\xi - x) \geq 0, \\ \lim\limits_{\lambda\to 0} \phi_\lambda(x) = \delta(x), \end{cases} \tag{4}$$

where $\boldsymbol{\delta}$ is the Dirac unit impulse at $0 \in \Re$, the limit being intended in terms of distributions, according to the expression:

$$\lim_{\lambda\to 0} \int_{-\infty}^{+\infty} \phi_\lambda(x-\xi)\cdot f(\xi)\, d\xi = f(x). \tag{5}$$

for any continuous map $f : \Re \mapsto \Re$.

The total curvature χ is sum of the nonlocal elastic curvature field χ^{el} and of all other non-elastic curvature fields, henceforth represented by a thermal curvature field χ^{th}, so that:

$$\chi = \chi^{th} + \chi^{el}. \tag{6}$$

In the linearised Bernoulli-Euler beam model, the *geometric curvature* field is defined by

$$\chi_v := v'', \tag{7}$$

with $v : [a,b] \mapsto \Re$ transverse displacement of the beam axis, the apex $'$ denoting derivation along the x axis.

Kinematical compatibility requires that the total curvature field be coincident with the geometric curvature.

Equilibrium is expressed by the variational condition that the external virtual power of the loading $\langle \ell, \delta v \rangle$ is equal to the internal virtual power of the bending interaction field:

$$\langle \ell, \delta v \rangle = \int_a^b \langle M, \chi_{\delta v} \rangle dx, \tag{8}$$

for all virtual displacement fields $\delta v \in V$ which are square integrable, together with $\delta v', \delta v''$, on $[a,b]$ and fulfil homogeneous kinematical conditions imposed on boundary values of $\delta v, \delta v'$.

The bending interaction field is called to fulfil the equilibrium condition with an imposed admissible loading $\ell \in V'$, with V' dual linear space of force systems, such that

$$\langle \ell, \delta v \rangle = 0, \quad \forall\, \delta v \in V : \chi_{\delta v} = 0. \tag{9}$$

A basic difficulty with strain-driven nonlocal models is that, denoting by $\Sigma \subset \mathscr{H}$ the affine manifold of all bending interaction fields $M \in \mathscr{H}$ fulfilling the equilibrium condition Eq. (8), the corresponding set of solutions of the Fredholm integral equation Eq. (3) can be empty.

This is indeed the case for the totality of engineering statical schemes of simple beams with the usual end-constraints.

As a consequence, the assumption of existence of a solution for these nonlocal elastic schemes leads unavoidably to paradoxical results [2]. This important conclusion was first drawn in Ref. [3], after several resolutions of paradoxes were improperly claimed [4–6] and various proposals were advanced [7–15].

It is worth observing that, in statically determinate beam models, the bending interaction field M is uniquely fixed by equilibrium and independent of the nonlocal parameter $\lambda > 0$.

On the contrary, in statically indeterminate beam models the bending interaction field M is defined by the full set of equilibrium, kinematic compatibility and constitutive laws, so that the bending interaction solution will depend on the nonlocal parameter.

An enlightening example is provided by a statically determinate beam problem (a cantilever is usually adopted to model actuators) where the bending interaction field takes an analytical expression that cannot be reproduced by the convolution in Eq. (3), for any choice of a square integrable curvature field $\chi^{el} \in \mathscr{H}$.

To overcome these basic difficulties, detailedly addressed in Refs. [3,16], an innovative nonlocal model for nano-beams was recently introduced by the authors in Ref. [17].

In the new *stress-driven* model the roles of bending interaction and curvature fields are swapped with respect to the strain-driven model of Eq. (3) so that the resulting expression is:

$$\chi^{el}(x) = \int_a^b \phi_\lambda(x-\xi)\cdot C(\xi)\cdot M(\xi)\, d\xi. \tag{10}$$

The input bending interaction field $M \in \mathscr{H}$ must meet the equilibrium condition Eq. (8).

The total curvature $\chi = \chi^{th} + \chi^{el}$ sum of the thermal curvature $\chi^{th} \in \mathscr{H}$ and the output nonlocal elastic curvature $\chi^{el} \in \mathscr{H}$, has to meet kinematic compatibility.

For a 3D continuum, the stress-driven model is formulated by swapping, with respect to the strain-driven model of Eq. (1), the roles of stress and elastic strain fields:

$$\boldsymbol{\varepsilon}^{el}(\mathbf{x}) = \int_\Omega \phi_\lambda(\mathbf{x}-\xi)\cdot \mathbf{C}(\xi)\cdot \boldsymbol{\sigma}(\xi)\, d\Omega_\xi. \tag{11}$$

It is important to underline that the stress-driven and the strain-driven elastic laws are by no means one the converse of the other and lead to completely different structural models.

Remark 1.1. The original strain-driven nonlocal model Eq. (1) for a 3D continuum was reformulated in Ref. [18] by assuming a uniform local elastic compliance $\mathbf{C} = \mathbf{E}^{-1}$:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \int_{\boldsymbol{\Omega}} \phi_\lambda(\mathbf{x}-\xi)\cdot \boldsymbol{\varepsilon}^{el}(\xi)\, d\Omega_\xi, \tag{12}$$

with $\boldsymbol{\varepsilon}^{el}(\mathbf{x}) := \mathbf{C}\cdot\boldsymbol{\sigma}(\mathbf{x})$.

In adapting to Bernoulli-Euler *beams with uniform elastic flexural stiffness* K *, the nonlocal law was written in Ref.* [4] *as*

$$\overline{\chi}(x) = \int_a^b \phi_\lambda(x-\xi)\cdot \chi(\xi)\, d\xi, \tag{13}$$

with $\overline{\chi}(x) := C\cdot M(x)$ and $C = K^{-1}$.

The expressions in Eq. (12) and Eq. (13) are possibly misleading since distinct roles of stress and strain fields are not clearly evidenced. Inappropriate notations probably contributed to shadow the essential drawbacks of structural models based on strain-driven constitutive laws. For non-uniform elasticity, the nonlocal expression in Eq. (12) and Eq. (13) differ from the original Eringen *model Eq. (1) since average is performed on the strain field rather than on the local elastic response.*

2. Integral vs differential formulations

In application to nano-beams for simulation of scale effects in NEMS [19–21], the strain-driven integral elastic model Eq. (3) was replaced with the associated differential formulation, taken from the original treatment in [1].

By introducing the characteristic length $L_c := \lambda L$, the special kernel, depicted in Fig. 1, is defined by

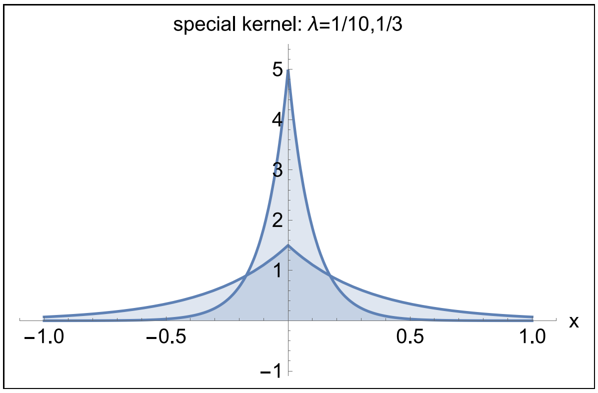


Fig. 1. Special kernel, $\lambda \in \{1/10, 1/3\}$

$$\phi_\lambda(x) := \frac{1}{2L_c} \exp\left(-\frac{|x|}{L_c}\right). \tag{14}$$

It can be shown [3] that the convolution in Eq. (3) is equivalent to the differential equation

$$M(x) - L_c^2 \cdot M''(x) = K \cdot \chi_{el}(x), \quad a \le x \le b, \tag{15}$$

with the constitutive boundary conditions

$$\begin{cases} M'(a) = \dfrac{1}{L_c} M(a), \\ M'(b) = -\dfrac{1}{L_c} M(b). \end{cases} \tag{16}$$

As a matter of fact, these boundary conditions were completely ignored until the mathematical discussion of Fredholm integral equations with the special kernel provided in Ref. [22] was put into evidence in Ref. [18].

There, the difficulties inherent to the strain-driven constitutive model, evidenced by non-existence of solutions, were by-passed by adopting a local-nonlocal mixture, as earlier suggested in Refs. [23,24] and adopted in Refs. [25,26].

The constitutive boundary conditions Eq. (16) are clearly in contrast with all natural boundary of engineering problems, since they depend explicitly on the nonlocal parameter λ through the characteristic length $L_c := \lambda L$.

This fact reveals the impossibility to fulfil the conflicting requirements imposed on the bending interaction field by equilibrium and by strain-driven constitutive nonlocal law.

This conclusion can also be reached by observing that the bending interaction field $M \in \mathscr{H}$ in the beam inherits an exponential behaviour from the adopted kernel and cannot therefore comply with the differential properties dictated by equilibrium under usual loading conditions in engineering problems.

The proof given in Ref. [3] of equivalence between the Fredholm integral equation Eq. (3) and the differential problem expressed by Eqs.(15) and (16), was decisive to conclude that Eringen strain-driven model is ill-posed and must be abandoned.

The most natural and effective strategy to get well-posedness is provided by the new stress-driven nonlocal model, recently proposed by the authors in Ref. [17].

The new integral convolution Eq. (10), with the special kernel in Eq. (14), is equivalent to the differential equation

$$\chi^{el}(x) - L_c^2 \cdot \left(\chi^{el}\right)''(x) = C \cdot M(x), \quad a \le x \le b, \tag{17}$$

with the constitutive boundary conditions

$$\begin{cases} \left(\chi^{el}\right)'(a) = \dfrac{1}{L_c} \cdot \chi^{el}(a), \\ \left(\chi^{el}\right)'(b) = -\dfrac{1}{L_c} \cdot \chi^{el}(b). \end{cases} \tag{18}$$

The signs in Eq. (16) and Eq. (18) are consequent to the fact that the first derivative of the special kernel is an odd function.

3. Example

An interesting example of application of the new theory to simple nano-beams is provided by the structural scheme of a doubly-clamped beam subject to an imposed curvature due to a zero-mean thermal gradient (i.e. butterfly shaped) along the flexure direction, and uniformly distributed along the axis.

3.1. Clamped beam with uniform thermal curvature

It is well-known that the solution of this clamped beam problem, with the standard local elastic model, leads to an identically vanishing displacement field due to a perfect compensation between uniform thermal and elastic curvature fields. The outcome of the nonlocal stress-driven problem is not so trivial.

The solution can be got by considering a basis of the linear manifold of self-equilibrated bending interaction fields (those in equilibrium under null loading, called for brevity *self-bending* fields) which by symmetry is given by a uniform non-null self-bending field, say $M_0 \in \mathbf{\Sigma}_0$.

The overall *geometric curvature* is evaluated by superposition of the uniform prescribed *thermal curvature* $\chi^{th} \in \mathscr{H}$ and of the nonlocal elastic curvature $(\chi^{el})_0 \in \mathscr{H}$ induced by the self-bending solution $M = \alpha_\lambda \cdot M_0 \in \mathbf{\Sigma}_0$ (heretofore called *self-curvature*):

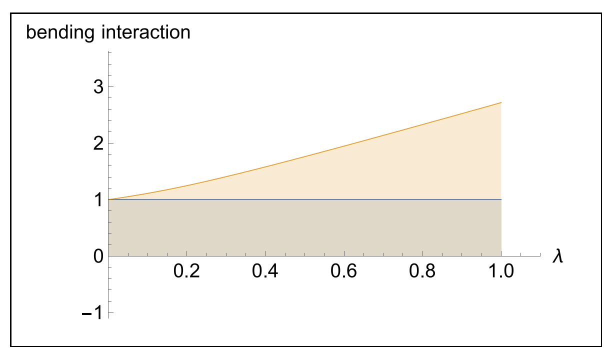


Fig. 2. Bending interaction vs nonlocal parameter.

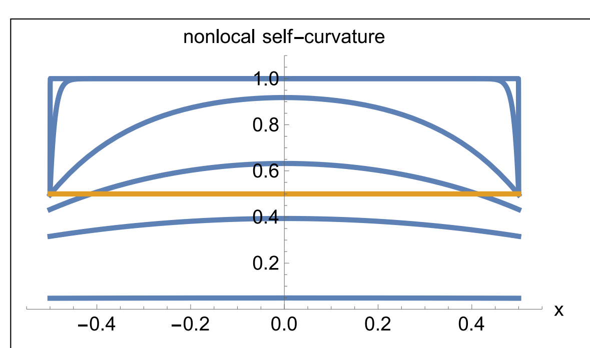


Fig. 3. Self-curvature field: $\lambda \in \{.0001, 0.01, 0.2, 0.5, 1, 10\}$

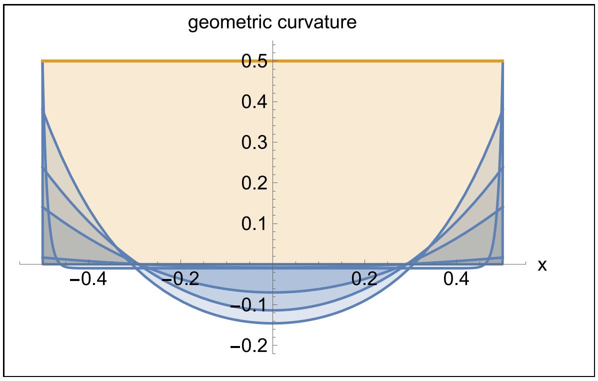


Fig. 4. Geometric curvature fields: $\lambda \in \{.0001, 0.01, 0.2, 0.5, 1, 10\}$

$$\chi = \chi^{th} + \alpha_\lambda \cdot \left(\chi^{el}\right)_0, \tag{19}$$

with

$$\left(\chi^{el}\right)_0 := \int_{-L/2}^{+L/2} \phi_\lambda(x - \xi) \cdot C(\xi) \cdot M_0(\xi)\, d\xi. \tag{20}$$

Kinematic compatibility is imposed by the virtual work equation:

$$\left\langle \delta M_0, \chi^{th} \right\rangle + \alpha_\lambda \cdot \left\langle \delta M_0, \left(\chi^{el}\right)_0 \right\rangle = 0, \tag{21}$$

where $\delta M_0 \in \mathbf{\Sigma}_0$ is any non-null self-bending field.

Computation of the convolution gives

$$\alpha_\lambda = -\frac{\chi^{th} \cdot K}{M_0} \left(\frac{1}{1 + \lambda \cdot (\exp(-1/\lambda) - 1)} \right). \tag{22}$$

The uniform self-bending solution $M_\lambda = \alpha_\lambda \cdot M_0 \in \mathbf{\Sigma}_0$ is plotted for $L = 1, \chi^{th} = -1, M_0 = 1, K = 1$, in Fig. 2. The values range form 1 at $\lambda = 0$ to e at $\lambda = 1$.

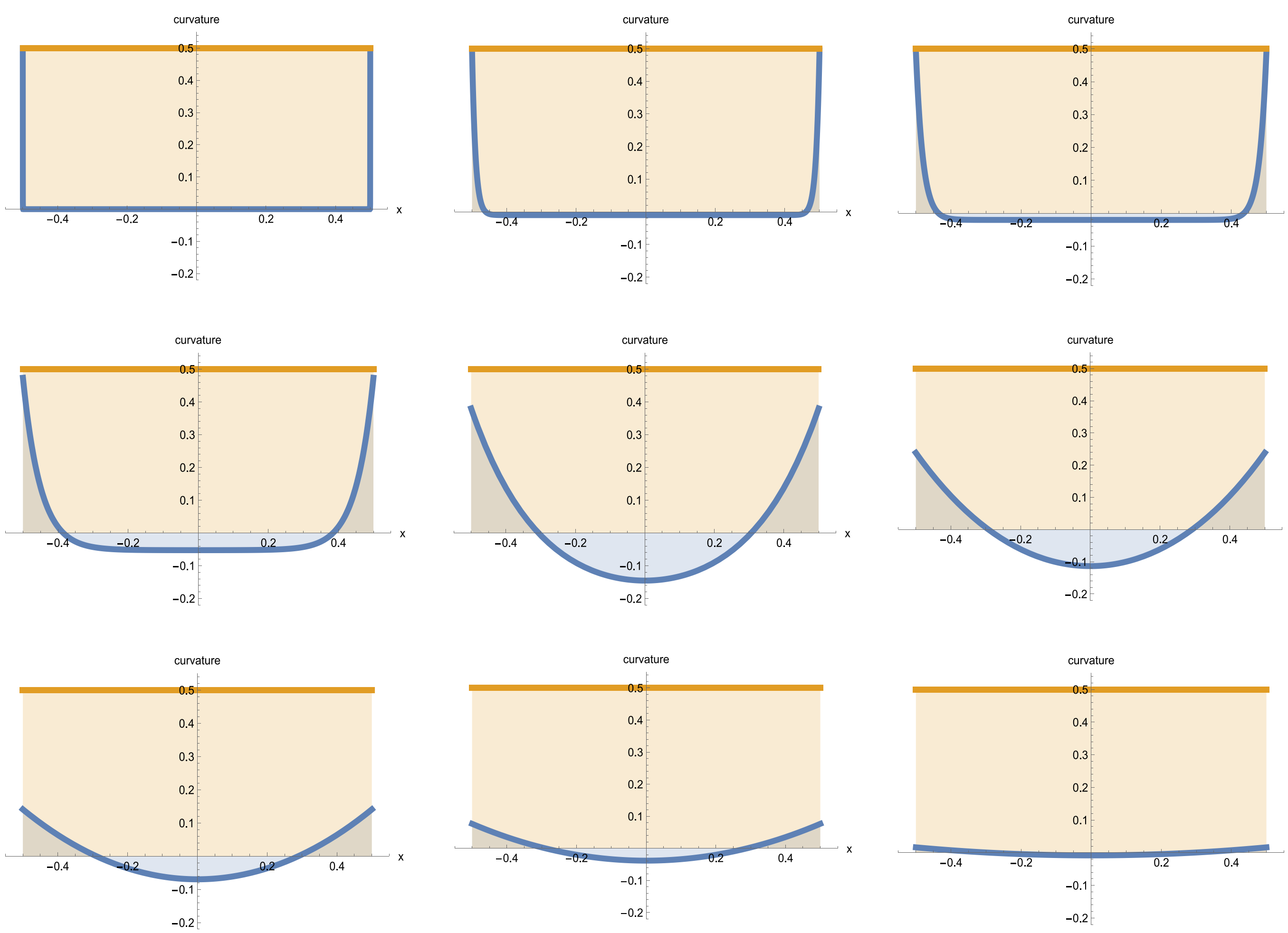
The nonlocal self-curvature, evaluated according to the convolution in Eq. (20), is given by

$$\left(\chi^{el}\right)_0 = \frac{M_0}{2 \cdot K} \left[\left(1 - \exp \frac{-1 + 2 \cdot x/L}{2 \cdot \lambda} \right) + \left(1 - \exp \frac{-1 - 2 \cdot x/L}{2 \cdot \lambda} \right) \right], \tag{23}$$

and is plotted in Fig. 3 for the nonlocal parameter ranging in the set

$$\lambda \in \{.0001, 0.01, 0.2, 0.5, 1, 10\}. \tag{24}$$

Table 1
Geometric curvature fields: $\lambda \in \{0001, 0.01, 0.02, 0.05, 0.2, 0.5, 1, 2, 10\}$



These diagrams show an elastic stiffness increasing with the nonlocal parameter λ.

The overall nonlocal curvature given by Eq. (19) is plotted in Fig. 4. We see that for $\lambda \to 0$ the boundary values of the geometric curvature tend to the value 0.5 while the values at points internal to the beam axis tend to vanish.

This behaviour is a consequence of the convolution law and of the normalisation property, as can be explained by observing the diagrams of the self-curvature $(\chi^{el})_0$ given in Fig. 3.

For $\lambda \to 0$ the values of self-curvature at internal points tend to unity, which is the uniform value of the self-bending field, while boundary values at end-points tend to the value $1/2$.

This halved value is due to elimination of the part of the kernel which falls out of the domain of integration.

As $\lambda \to 0$ the kernel tends to the Dirac impulse and the loss at boundary points tends to one-half of it (the Dirac impulse is halved!).

To better observe the evolution of the total geometric curvature field, for increasing values of the nonlocal parameter, Table 1 displays the curvature fields for

$$\lambda \in \{0001, 0.01, 0.02, 0.05, 0.2, 0.5, 1, 2, 10\}. \tag{25}$$

The progressive reduction of the maximum value of the curvature field is due to normalisation of the kernel, expressed by property Eq. (5). In fact, since the variance increases with λ, the peak value of the kernel is lowered, as shown in Fig. 1, to keep the integral equal to unity.

4. Conclusion

Main outcomes of the present paper may be summarized as follows.

1 The elastostatic problem of an inflected nano-beam formulated according to Eringen strain-driven integral model, admits a unique solution or no solution at all, depending on whether the bending interaction field fulfils constitutive boundary conditions or not. For structural schemes of applicative interest no solution exists [3].
2 The new stress-driven nonlocal integral elastic model is put into action by means of a direct solution procedure so that small-scale effects in Bernoulli-Euler nano-beams can be efficiently simulated.
3 Exact solutions of a statically indeterminate nano-beam model of engineering interest is contributed to illustrate effectiveness of the new stress-driven nonlocal model.
4 A stiffer elastic response is got for increasing values of the nonlocal parameter for any prescribed kinematic boundary conditions. This effect is due to the normalisation condition Eq. (5) on the smoothing kernel involved in the convolution integral.
5 The boundary effect of halving the elastic nonlocal curvature fields at the boundary in the limit $\lambda \to 0$ is shown to be due to cancellation of the kernel exiting from the interval of integration.

The new stress-driven nonlocal model provides an effective methodology for the analysis of nano-structures and eliminates the essential difficulties exhibited by the strain-driven model.

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