# Tangent Stiffness of Polar Shells Undergoing Large Displacements

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**Summary.** The paper deals with the definition and the evaluation of the tangent stiffness of hyperelastic polar shells without drilling rotations. The ambient space for such bodies is a nonlinear differentiable manifold. As a consequence the incremental equilibrium must be expressed as the absolute time derivative of the nonlinear equilibrium condition expressing the balance between the elastic response and the applied forces. In the absolute time derivative the classical directional derivative is substituted by the covariant derivative according to a fixed connection on the manifold. The evaluation of the tangent stiffness requires to perform the second covariant derivative of the finite deformation measure and this in turn requires an extension of the virtual displacement field in a neighborhood of the given configuration of the shell. It is explicitly shown that different choices of this extension lead to the same tangent stiffness which is symmetric since the choosen connection is torsionless.

## 1 Introduction

The evaluation of the tangent stiffness of an hyperelastic body is of a crucial importance when dealing with finite changes of configuration. The tangent stiffness provides the linear relationship between the rate of change of configuration and the corresponding rate of change of elastic response of the body in terms of forces. The analysis of small vibrations of a finitely deformed elastic body, the instability of equilibrium configurations and the prediction of the way in which an elastic body tends to move under a loading path, are all governed by the properties of the tangent stiffness.

There are many different ways of defining a deformation measure of the body and the choice of a special measure changes the way in which the modelling of the constitutive properties of the material is performed. The basic requirements which a deformation measure has to conform with are the following: the measure must be independent of superimposed rigid changes of configuration and must be a local field in the sense that its value at a point must not be affected by a change of the placement map outside any neighborhood of that point.

The definition of a rigid change of configuration is a basic item that must be given in describing the kinematical properties of the body in its motion in the ambient space. In hyperelastic bodies the Green's potential defines the local elastic properties of the material in terms of its deformation from a given natural state. The deformation field depends in turn on the map which defines the placement of the body with respect to a reference configuration in the ambient space.

Once a deformation measure has been choosen, the local elastic potential can be expressed as the composition of the local elastic energy and the deformation measure. It is then a function of the configuration change from a reference configuration in which the material is assumed to be in a natural state. The global elastic potential is obtained by integrating the local elastic potential over the whole body in the reference configuration.

In finite deformation analysis all the state variables defined in the actual configuration are transformed into the corresponding ones in the reference configuration. Accordingly, in an evolution process, the equilibrium condition at the actual configuration is written by imposing the equality between the directional derivative of the global elastic potential along a conforming virtual (tangent) displacement and the corresponding virtual work of the referential forces. The derivative of the global elastic potential is the elastic response of the body to the change of configuration. Both the elastic response and the referential forces are bounded linear forms on the linear space of conforming virtual displacements. The condition of incremental equilibrium is then obtained by taking the time derivative of the equilibrium condition.

In classical structural analysis the time derivative of the elastic response is expressed, by means of the chain rule, as the directional derivative of the elastic response along the velocity field of the body. When dealing with polar bodies this procedure must be revisited to take into account the non-affine geometrical structure of the physical space. In such a situation the time derivative must be substituted by the absolute differentiation with respect to time, defined as the covariant derivative of the elastic response along the velocity field.

To grasp the motivation of this new approach one has to consider that, when the ambient space is a nonlinear differentiable manifold, the tangent spaces of virtual displacements and their dual counterparts, the cotangent spaces of force systems, change from point to point. In general there is no way to perform a classical differentiation of a vector or of a covector field on a differentiable manifold since this would imply to perform the difference of unrelated vectors belonging to different linear spaces.

In structural mechanics the nonlinear differentiable manifold defining the ambient space is usually embedded into a larger affine space with an euclidean structure. In this cases the covariant differentiation amounts simply in taking the component of the directional derivative on the subspace tangent to the manifold.

This definition of the covariant differentiation is equivalent to consider on the manifold the LEVI-CIVITA connection associated with the riemannian metric induced by the euclidean metric of the larger affine space.

One more essential point remains to be fixed. The directional derivative of a field of linear forms on a linear space meets the LEIBNIZ rule of calculus: the directional derivative of a linear form at a vector field, is equal to the difference between the directional derivative of its value at the vector field and its value in corrispondence of the directional derivative of the vector field.

By analogy the covariant differentiation of a linear form is defined by means of a formal application of the LEIBNIZ rule: the value of the covariant derivative of a linear form at a vector field is equal to the difference between the covariant derivative of its value at the vector field and its value in corrispondence of the covariant derivative of the vector field. The definition is well posed since, although both terms in the difference depend on the values that the vector field takes in a neighborhood of the point, their difference is local and hence the covariant derivative of the linear form is tensorial.

From the discussion above it follows that the tangent stiffness must be properly defined as the covariant derivative of the elastic response. As the covariant derivative of a linear form, the tangent stiffness is then a two times covariant tensor. The evaluation of the tangent stiffness of polar elastic bodies is then a remarkable example of application of differential geometry, and specifically of calculus on manifolds, to issues of mechanics.

In previous treatments, dealing with models of polar beams and shells, the geometric tangent stiffness was simply evaluated as the inner product of the referential stress times the second directional derivative of the deformation measure. It is apparent that such an evaluation requires to perform an extension of the virtual displacement along which the first derivative is taken, to a vector field defined in a neighborhood of the given configuration.

In finite deformation analysis of polar shells without drilling rotations the ambient space is the trivial fiber bundle defined by the cartesian product of the euclidean space (the base manifold) times the unit sphere (the fiber). The corresponding tangent stiffness, computed by taking the second covariant derivative of the deformation measure, is local and symmetric when the space manifold is endowed with the LEVI-CIVITA connection induced by the larger affine space.

Two different extensions of the virtual displacement are investigated and it is shown that the former yields a symmetric second directional derivative of the deformation measure while the latter leads to a nonsymmetric second directional derivative. It is further shown, by explicit calculation, that the corresponding second covariant derivative of the deformation measure is however symmetric in both cases, as required by the theory.

## 2 Polar shells

The general theory of polar models developed in [12] has been applied in [13] to the analysis of the polar model of shear deformable beams undergoing finite configuration changes.

We shall here investigate in detail a polar model of shear deformable shells in finite deformations which is referred to in the literature as the shell without drilling rotations [7].

Let $E^3$ be the euclidean space and $V^3$ the associated linear space of translations.

The material shell $\mathcal{B}$ is a set of particles which, at each time $t \in I$, are located at points of a differentiable submanifold of the physical space $\mathbb{E} = E^3$.

The polar model of a shell without drilling rotations is a two-dimensional structural model characterized by a middle surface $\mathbb{B}$ and by vectors of prescribed length in $V^3$ attached at each of its points to simulate the constant thickness of the transversely undeformable shell. The corresponding versors, called directors, range on the unit sphere $\mathbf{S}^2$ which is a compact differentiable manifold without boundary embedded in $V^3$:

$$\mathbf{S}^2 := (\mathbf{d} \in V^3 \,:\, \|\mathbf{d}\| = 1)\,.$$

The ambient space, in which the motion of the shell takes place, is then the differentiable manifold without boundary

$$\mathbb{S} = E^3 \times \mathbf{S}^2\,,$$

a trivial fiber bundle having the euclidean space $E^3$ as base manifold and the unit sphere $\mathbf{S}^2$ as typical fiber.

The base configuration map $\boldsymbol{\chi}_t : \mathcal{B} \mapsto \mathbb{E}$ of the shell at time $t \in I$ is an injection of the material shell $\mathcal{B}$ onto the base placement $\mathbb{B}_t \subset \mathbb{E}$ which is the middle surface of the shell.

The polar structure $\mathbf{s}_t : \mathbb{B}_t \mapsto \mathbb{S}$ is a map from the middle surface at time $t$ onto the placement $\mathbb{P}_t = \mathbf{s}_t(\mathbb{B}_t)$. The map $\mathbf{s}_t : \mathbb{B}_t \mapsto \mathbb{S}$, defined by

$$\mathbf{s}_t(\mathbf{p}_t) := \{\, \mathbf{p}_t\,, \mathbf{d}_t \,\} \in \mathbb{B}_t \times \mathbf{S}^2\,,$$

is a section of the fiber bundle $\mathbb{S}$ on the submanifold $\mathbb{B}_t \subset \mathbb{E}$.

A spatial configuration of the polar shell at time $t \in I$ is an injective map $\mathbf{u}_t : \mathcal{B} \mapsto \mathbb{S}$ which assignes a placement $\mathbb{P}_t := \mathbf{u}_t(\mathcal{B}) \subset \mathbb{S}$ to the material shell $\mathcal{B}$ and is given by the composition of the base configuration map with the polar structure:

$$\mathbf{u}_t = \mathbf{s}_t \circ \boldsymbol{\chi}_t\,.$$

Let us consider the change of base configuration $\boldsymbol{\chi}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{B}_t)$ from $\boldsymbol{\chi}_s$ to $\boldsymbol{\chi}_t$ defined by

$$\boldsymbol{\chi}_{t,s} \circ \boldsymbol{\chi}_s = \boldsymbol{\chi}_t\,.$$

The configuration change from $\mathbf{u}_s$ to $\mathbf{u}_t$ is the map $\mathbf{u}_{t,s} : \mathbf{u}_s(\mathcal{B}) \mapsto \mathbf{u}_t(\mathcal{B}) \subset \mathbb{S}$ defined by

$$\mathbf{u}_{t,s} \circ \mathbf{u}_s = \mathbf{u}_t \,.$$

To extract the base point and the director from a pair $\{ \mathbf{p}_t, \mathbf{d}_t \}$ we introduce the cartesian projectors

$$\mathbf{P}_1\{ \mathbf{p}_t, \mathbf{d}_t \} := \mathbf{p}_t \,, \qquad \mathbf{P}_2\{ \mathbf{p}_t, \mathbf{d}_t \} := \mathbf{d}_t \,.$$

Accordingly we define the map $\hat{\mathbf{d}}_t : \mathbb{B}_t \mapsto S^2$ which provides the director associated to a base point on the middle surface:

$$\hat{\mathbf{d}}_t(\mathbf{p}_t) := (\mathbf{P}_2 \circ \mathbf{s}_t)(\mathbf{p}_t) \,, \qquad \mathbf{p}_t \in \mathbb{B}_t \,.$$

To simplify the notation we shall drop the $\hat{}$ and write simply $\mathbf{d}_t$ for $\hat{\mathbf{d}}_t$.

Let us consider the finite deformation measure for the polar shell model without drilling rotations that was proposed and analyzed in [7]. It consists in the triplet

$$\mathbf{A}(\mathbf{u}_{t,s}) := \begin{vmatrix} \boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s}) \\ \boldsymbol{\delta}(\mathbf{u}_{t,s}) \\ \mathbf{C}(\mathbf{u}_{t,s}) \end{vmatrix}$$

composed by

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s})(\mathbf{a},\mathbf{b}) &:= \mathbf{g}(\boldsymbol{\chi}_{t,s*}\mathbf{a},\, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\mathbf{a},\, \mathbf{b}) \,, && \text{membrane strain} \,, \\ \boldsymbol{\delta}(\mathbf{u}_{t,s})(\mathbf{a}) &:= \mathbf{g}(\mathbf{d}_t,\, \boldsymbol{\chi}_{t,s*}\mathbf{a}) - \mathbf{g}(\mathbf{d}_s,\, \mathbf{a}) \,, && \text{shear sliding} \,, \\ \mathbf{C}(\mathbf{u}_{t,s})(\mathbf{a},\mathbf{b}) &:= \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{d}_t,\, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\partial_{\mathbf{a}}\mathbf{d}_s,\, \mathbf{b}) \,, && \text{flexural curvature} \,, \end{aligned}$$

where $\mathbf{a}\,, \mathbf{b} \in \mathbb{T}_{\mathbb{B}_s}(\mathbf{p}_s)$. The push forward $\boldsymbol{\chi}_{t,s*} \in BL\,(\mathbb{T}_{\mathbb{B}_s}\,; \mathbb{T}_{\mathbb{B}_t})$ associated with the map $\boldsymbol{\chi}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s, \mathbb{B}_t)$ is defined by ( see [1], [2], [3]):

$$\boldsymbol{\chi}_{t,s*}(\mathbf{p}_s, \mathbf{a}) := \{ \boldsymbol{\chi}_{t,s}(\mathbf{p}_s) \,, \partial_{\mathbf{a}}\boldsymbol{\chi}_{t,s}(\mathbf{p}_s) \} \,.$$

The push forward maps a given tangent vector applied at a point of a manifold into the corresponding deformed tangent vector applied to the transformed point. The tangent space at $\{\mathbf{x}\,, \mathbf{d}\} \in \mathbb{S} = E^3 \times \mathbf{S}^2$ is the product manifold

$$\mathbb{T}_{\mathbb{S}}(\mathbf{x}\,, \mathbf{d}) = \mathbb{T}_{E^3}(\mathbf{x}) \times \mathbb{T}_{\mathbf{S}^2}(\mathbf{d}) = V^3 \times \mathbb{T}_{\mathbf{S}^2}(\mathbf{d}) \;.$$

The virtual displacements $\delta\mathbf{u}_{t,s} \in H^k(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})$ are defined by

$$\delta\mathbf{u}_{t,s}(\mathbf{p}_s) = \{ \mathbf{t}(\mathbf{u}_{t,s}(\mathbf{p}_s)) \,, \mathbf{X}(\mathbf{u}_{t,s}(\mathbf{p}_s)) \} \quad \text{with} \quad \begin{cases} \mathbf{t}\,(\mathbf{u}_{t,s}(\mathbf{p}_s)) & \in \mathbb{T}_{E^3}(\mathbf{p}_t) \,, \\ \mathbf{X}(\mathbf{u}_{t,s}(\mathbf{p}_s)) & \in \mathbb{T}_{\mathbf{S}^2}(\mathbf{d}_t) \,, \end{cases}$$

for any $\mathbf{p}_s \in \mathbb{B}_s$ and $\{ \mathbf{p}_t, \mathbf{d}_t \} = \mathbf{u}_{t,s}(\mathbf{p}_s)$ where $\mathbf{u}_{t,s}(\mathbf{p}_s)$ is a shortcut for $(\mathbf{u}_{t,s} \circ \mathbf{s}_s)(\mathbf{p}_s)$.

*Remark 1.* We must observe that, despite their wide acceptance (see e.g.[4], [5], [6]) the deformation measures reported above in this in section and commonly adopted in the literature for polar shells without drilling rotations, lead to physically nonplausible results in case of significant membrane strains. Indeed a simple computation reveals an unrealistic behaviour of an inflated polar spherical baloon since an increase of flexural curvature is measured when the radius increases. The effect is due to the amplification of the convected tangent vectors due to the deformation.

To get rid of this shortcoming we could redefine the deformation measures for polar shells without drilling rotations as follows:

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s})(\mathbf{a},\mathbf{b}) &:= \mathbf{g}(\boldsymbol{\chi}_{t,s*}\mathbf{a},\, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\mathbf{a},\, \mathbf{b})\,, && \text{membrane strain}\,, \\ \boldsymbol{\delta}(\mathbf{u}_{t,s})(\mathbf{a}) &:= \mathbf{g}(\mathbf{d}_t,\, \boldsymbol{\chi}_{t,s*}\mathbf{a}) - \mathbf{g}(\mathbf{d}_s,\, \mathbf{a})\,, && \text{shear sliding}\,, \\ \mathbf{C}(\mathbf{u}_{t,s})(\mathbf{a},\mathbf{b}) &:= \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{d}_t,\, \mathbf{R}_{t,s}\mathbf{b}) - \mathbf{g}(\partial_{\mathbf{a}}\mathbf{d}_s,\, \mathbf{b})\,, && \text{curvature change}\,, \end{aligned}$$

where $\mathbf{R}_{t,s}$ is the isometric transformation associated with the push forward $\boldsymbol{\chi}_{t,s*}$ according to the polar decomposition formula $\boldsymbol{\chi}_{t,s*} = \mathbf{R}_{t,s}\,\mathbf{U}_{t,s}$ where $\mathbf{U}_{t,s}$ is the right CAUCHY stretch tensor. The new expression for the curvature change correctly predicts no flexural curvature in the inflated polar spherical baloon when the radius is changed. Indded in this problem the rotation $\mathbf{R}_{t,s}$ reduces to the identity and $\mathbf{d}_t \circ \boldsymbol{\chi}_{t,s} = \mathbf{d}_s$ so that

$$(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{d}_t) \circ \boldsymbol{\chi}_{t,s} = \partial_{\mathbf{a}}\mathbf{d}_s\,.$$

The computation of the tangent stiffness for this new shell model will be dealt with in a forthcoming paper.

### 2.1 Tangent stiffness

Let $\mathbf{v}_{\mathbf{X}} := \{\mathbf{t}_{\mathbf{X}}\,, \mathbf{X}\}$ and $\mathbf{v}_{\mathbf{Y}} := \{\mathbf{t}_{\mathbf{Y}}\,, \mathbf{Y}\}$ be referential virtual displacements at the placement $\mathbb{P}_t$. For any $\mathbf{p}_s \in \mathbb{B}_s$ the position at time $t$ is given by $\{\mathbf{x}_t, \mathbf{d}_t\} = \mathbf{u}_{t,s}(\mathbf{p}_s) \in \mathbb{P}_t$, and hence the referential virtual displacements are functions of the point $\mathbf{p}_s \in \mathbb{B}_s$ and of the configuration change $\mathbf{u}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s, \mathbb{S})$. To simplify the notations we shall write $\mathbf{v}_{\mathbf{X}}$ or $\mathbf{v}_{\mathbf{X}}(\mathbf{u}_{t,s})$, dropping the explicit dependence on $\mathbf{p}_s \in \mathbb{B}_s$.

The constitutive tangent stiffness of the shell is evaluated by taking the directional derivative of the elastic potential along a virtual displacement and by subsequently taking the absolute time derivative of the directional derivative. As we have seen, by applying LEIBNIZ rule the tangent stiffness is decomposed in the sum of an elastic part and a geometric part.

The symmetric elastic tangent stiffness is the bilinear form in $\mathbf{v}_{\mathbf{X}}, \mathbf{v}_{\mathbf{Y}}$ given by the formula

$$\partial^2\varphi(\mathbf{A}(\mathbf{u}_{t,s})) \cdot (\partial\mathbf{A}(\mathbf{u}_{t,s}) \cdot \mathbf{v}_{\mathbf{Y}}) \cdot (\partial\mathbf{A}(\mathbf{u}_{t,s}) \cdot \mathbf{v}_{\mathbf{X}})\,,$$

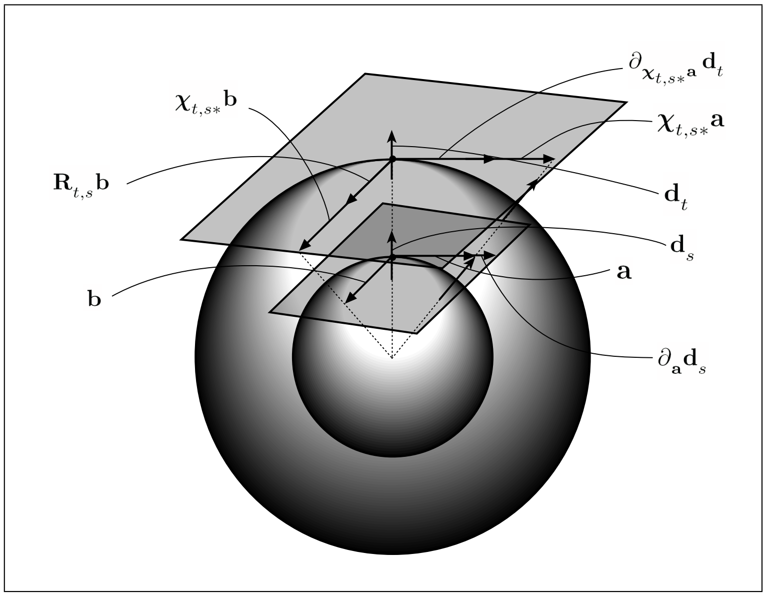


**Fig. 1.** Inflated polar spherical baloon

where the virtual displacement $\mathbf{v}_{\mathbf{X}}$ is indeed the velocity vector along the equilibrium path, which is the unknown of the incremental elastic equilibrium problem.

The geometric tangent stiffness is the bilinear form in $\mathbf{v}_{\mathbf{X}}, \mathbf{v}_{\mathbf{Y}}$ given by

$$\partial\varphi(\mathbf{A}(\mathbf{u}_{t,s}))\cdot\Big[\nabla^2_{\mathbf{v}_{\mathbf{X}}\mathbf{v}_{\mathbf{Y}}}(\mathbf{A}(\mathbf{u}_{t,s}))\Big] = \partial\varphi(\mathbf{A}(\mathbf{u}_{t,s}))\cdot\Big[(\partial_{\mathbf{v}_{\mathbf{X}}}\partial_{\hat{\mathbf{v}}_{\mathbf{Y}}} - \partial_{\nabla_{\mathbf{v}_{\mathbf{X}}}\hat{\mathbf{v}}_{\mathbf{Y}}})(\mathbf{A}(\mathbf{u}_{t,s}))\Big]$$

To compute the second covariant derivative of the deformation measure it is compelling to choose a connection on the space manifold. Such a choice determines whether symmetry of the geometric tangent stiffness is ensured or not. Indeed a torsionless connection implies the symmetry of the second covariant derivative of the deformation measure and hence the symmetry of the geometric tangent stiffness. On the other hand if the connection is not symmetric, the second covariant derivative can fail to be symmetric.

To provide a symmetric expression of the hessian of the deformation measure, let us assume that the manifold $\mathbb{S} = E^3 \times \mathbf{S}^2$ be endowed with the riemannian metric $\mathbf{g} \in BL(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}; \mathcal{R})$ induced by the usual metric in $E^3$. The Levi-Civita connection $\nabla$ on $\{\mathbb{S}, \mathbf{g}\}$ is uniquely defined by the requirements to be metric and torsionless:

*i)* $\partial_{\mathbf{c}}\,(\mathbf{g}\,(\mathbf{a},\mathbf{b})) = \mathbf{g}\,(\nabla_{\mathbf{c}}\mathbf{a},\mathbf{b}) + \mathbf{g}\,(\mathbf{a},\nabla_{\mathbf{c}}\mathbf{b})\,,$

*ii)* $\boldsymbol{T}(\mathbf{a},\mathbf{b}) := \nabla_{\mathbf{a}}\mathbf{b} - \nabla_{\mathbf{b}}\mathbf{a} - [\mathbf{a},\mathbf{b}] = \mathbf{o}\,,$

where $\mathbf{a}\,,\mathbf{b}\,,\mathbf{c} \in \mathrm{C}^1(\mathbb{S}\,;\mathbb{T}_{\mathbb{S}})$ are spatial vector fields.

The covariant derivative on $\mathbf{S}^2$ corresponding to this natural choice of the connection, can be easily computed as the projection of the directional derivative in $E^3$ on the tangent space to $\mathbf{S}^2$. Alternatively recourse can be made to the general formula due to KOSZUL [9]:

$$2\,\mathbf{g}\,(\nabla_{\mathbf{a}}\,\mathbf{b},\mathbf{c}) = d_{\mathbf{a}}\,(\mathbf{g}\,(\mathbf{b},\mathbf{c})) + d_{\mathbf{b}}\,(\mathbf{g}\,(\mathbf{c},\mathbf{a})) - d_{\mathbf{c}}\,(\mathbf{g}\,(\mathbf{a},\mathbf{b})) + \mathbf{g}\,([\mathbf{a},\mathbf{b}],\mathbf{c}) + \\ -\mathbf{g}\,([\mathbf{b},\mathbf{c}],\mathbf{a}) + \mathbf{g}\,([\mathbf{c},\mathbf{a}],\mathbf{b})\,.$$

This more involved procedure, which requires the computation of the LIE brackets appearing in the last three terms, was adopted in [8].

The evaluation of both terms at the r.h.s. in the expression of the second covariant derivative $\nabla^2_{\mathbf{v_X}\,\mathbf{v_Y}}(\mathbf{A}(\mathbf{u}_{t,s}))$ requires to perform an extension $\hat{\mathbf{v}}_{\mathbf{Y}} := \{\hat{\mathbf{t}}_{\mathbf{Y}}\,,\hat{\mathbf{Y}}\}$ of the virtual displacement $\mathbf{v}_{\mathbf{Y}} := \{\mathbf{t}_{\mathbf{Y}}\,,\mathbf{Y}\}$ along virtual trajectories in the physical space. However, as we will show, the second covariant derivative does not depend on how the extension is performed. Note that the extension of the vector $\mathbf{t}_{\mathbf{Y}}$ is trivial and consists in assuming it to be constant in the affine euclidean space $E^3$. On the other hand different extensions of the virtual displacement $\mathbf{Y}$ tangent to $\mathbf{S}^2$ at $\mathbf{d}_t$ will change the second directional derivative whilst the second covariant derivative of the deformation measure will be unchanged.

### 2.2 Extensions of the virtual displacements

We consider hereafter two extensions of the virtual displacement. The covariant derivative of the virtual displacement and the second directional derivative of the strain measure do assume different expresions in corrispondence of the two extensions. Anyway, as is to be expected on the ground of the general results, the same expression is obtained for the geometric tangent stiffness which is symmetric since the relevant connection is torsionless being induced by a riemannian metric.

#### First extension

Let us preliminarily recall that for any $\mathbf{p}_s \in \mathbb{B}_s$ it is $\{\,\mathbf{p}_t, \mathbf{d}_t\,\} = \mathbf{u}_{t,s}(\mathbf{p}_s) \in \mathbb{P}_t$. The tangent vectors $\mathbf{X}(\mathbf{u}_{t,s}), \mathbf{Y}(\mathbf{u}_{t,s}) \in \mathbb{T}_{\mathbf{S}^2}(\mathbf{d}_t)$ can be expressed as

$$\mathbf{X}(\mathbf{u}_{t,s}) = \mathbf{W}_{\mathbf{X}}\,\mathbf{P}_2\,\mathbf{u}_{t,s} = \mathbf{W}_{\mathbf{X}}\,\mathbf{d}_t = \boldsymbol{\omega}_{\mathbf{X}} \times \mathbf{d}_t\,,$$
$$\mathbf{Y}(\mathbf{u}_{t,s}) = \mathbf{W}_{\mathbf{Y}}\,\mathbf{P}_2\,\mathbf{u}_{t,s} = \mathbf{W}_{\mathbf{Y}}\,\mathbf{d}_t = \boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_t\,.$$

where $\mathbf{W}_{\mathbf{X}}$ and $\mathbf{W}_{\mathbf{Y}}$ are emisymmetric tensors in $V^3$ characterized by axial vectors $\boldsymbol{\omega}_{\mathbf{X}}$ and $\boldsymbol{\omega}_{\mathbf{Y}}$ which are assumed to be orthogonal to $\mathbf{d}_t$.

Let us now consider a virtual trajectory $\mathbf{u}_{\tau,t} \in \mathrm{C}^k(\mathbb{B}_t, \mathbb{P}_\tau)$ starting at $\mathbb{P}_t$ and having velocity $\mathbf{v}_{\mathbf{X}}(\mathbf{u}_{t,s}) \in H^k(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})$ at time $t$. We may choose the following extension for the virtual displacement $\mathbf{v}_{\mathbf{Y}}(\mathbf{u}_{t,s}) = \{\mathbf{t}_{\mathbf{Y}}(\mathbf{u}_{t,s})\,, \mathbf{Y}(\mathbf{u}_{t,s})\}$ :

$$\begin{cases} \hat{\mathbf{t}}_{\mathbf{Y}}(\mathbf{u}_{\tau,s}) := \mathbf{t}_{\mathbf{Y}}(\mathbf{u}_{t,s})\,, \\ \hat{\mathbf{Y}}(\mathbf{u}_{\tau,s}) \ := \mathbf{W}_{\mathbf{Y}}\,\mathbf{d}_{\tau} = \boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_{\tau}\,, \end{cases}$$

where $\mathbf{u}_{\tau,s} = \mathbf{u}_{\tau,t} \circ \mathbf{u}_{t,s}$. Since the vector field $\hat{\mathbf{t}}_{\mathbf{Y}}(\mathbf{u}_{\tau,s})$ has been taken constant in $V^3$ along the virtual trajectory, the evaluation of the covariant derivative of $\hat{\mathbf{v}}_{\mathbf{Y}} = \{\hat{\mathbf{t}}_{\mathbf{Y}}\,, \hat{\mathbf{Y}}\}$ at $\{\mathbf{x}_t\,, \mathbf{d}_t\}$ along $\mathbf{v}_{\mathbf{X}} = \{\mathbf{t}_{\mathbf{X}}\,, \mathbf{X}\}$ amounts in computing the covariant derivative of $\hat{\mathbf{Y}}(\mathbf{u}_{\tau,s})$ at $\mathbf{u}_{t,s}$ along $\mathbf{X}(\mathbf{u}_{t,s})$. To this end we observe that

$$\partial_{\mathbf{X}}\,\mathbf{d}_t = \frac{\partial}{\partial \tau}\bigg|_{\tau=t} \mathbf{P}_2 \circ \mathbf{u}_{\tau,s} = \mathbf{P}_2 \circ \mathbf{v}_{\mathbf{X}} = \mathbf{X}\,.$$

The directional derivative is then given by

$$\begin{aligned} (\partial_{\mathbf{X}}\hat{\mathbf{Y}})(\mathbf{u}_{t,s}) &= \mathbf{W}_{\mathbf{Y}}\mathbf{W}_{\mathbf{X}}\mathbf{d}_t = \boldsymbol{\omega}_{\mathbf{Y}} \times (\boldsymbol{\omega}_{\mathbf{X}} \times \mathbf{d}_t) = \\ &= \mathbf{g}(\boldsymbol{\omega}_{\mathbf{Y}}\,, \mathbf{d}_t)\,\boldsymbol{\omega}_{\mathbf{X}} - \mathbf{g}(\boldsymbol{\omega}_{\mathbf{Y}}\,, \boldsymbol{\omega}_{\mathbf{X}})\,\mathbf{d}_t = \\ &= -\mathbf{g}(\boldsymbol{\omega}_{\mathbf{Y}}\,, \boldsymbol{\omega}_{\mathbf{X}})\,\mathbf{d}_t = -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{d}_t\,, \end{aligned}$$

since $\mathbf{g}(\boldsymbol{\omega}_{\mathbf{Y}}\,, \mathbf{d}_t) = 0$ by assumption.

Denoting by $\boldsymbol{\Pi}$ the orthogonal projector in $E^3$ on the tangent space $\mathbb{T}_{\mathbf{S}^2}(\mathbf{d}_t)$ at the point $\mathbf{d}_t$, the formula of the covariant derivative yields

$$(\nabla_{\mathbf{X}}\hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = \boldsymbol{\Pi}\,(\partial_{\mathbf{X}}\hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = -\mathbf{g}(\boldsymbol{\omega}_{\mathbf{Y}}\,, \boldsymbol{\omega}_{\mathbf{X}})\,\boldsymbol{\Pi}\,\mathbf{d}_t = \mathbf{o}\,, \qquad \forall\, \mathbf{X} \in \mathbb{T}_{\mathbf{S}^2}(\mathbf{d}_t)\,,$$

since $\boldsymbol{\Pi}\,\mathbf{d}_t = \mathbf{o}$.
As a consequence $(\nabla\hat{\mathbf{v}}_{\mathbf{Y}})(\mathbf{u}_{t,s}) = \mathbf{o}$ and the second covariant derivative of the deformation measure at $\mathbf{u}_{t,s}$ coincides with the second directional derivative, that is

$$\nabla^2_{\mathbf{v_X}\,\mathbf{v_Y}}(\mathbf{A}(\mathbf{u}_{t,s}))(\mathbf{d}_t) = \partial_{\mathbf{v_X}}\partial_{\hat{\mathbf{v}}_{\mathbf{Y}}}(\mathbf{A}(\mathbf{u}_{t,s}))(\mathbf{d}_t)\,.$$

Let us then compute the second directional derivative of the components of the strain measure. To this end we preliminarily observe that

$$\begin{aligned} &\partial_{\mathbf{t_X}}\,\boldsymbol{\chi}_{t,s} = \mathbf{t}_{\mathbf{X}}\,, \qquad \boldsymbol{\chi}_{t,s*}\mathbf{a} = \partial_{\mathbf{a}}\,\boldsymbol{\chi}_{t,s}\,, \\ &\partial_{\mathbf{t_X}}\,\partial_{\mathbf{a}}\,\boldsymbol{\chi}_{t,s} = \partial_{\mathbf{a}}\,\partial_{\mathbf{t_X}}\,\boldsymbol{\chi}_{t,s} = \partial_{\mathbf{a}}\,\mathbf{t}_{\mathbf{X}}\,. \end{aligned}$$

The second directional derivative of the membrane strain yields for $\mathbf{a}\,, \mathbf{b} \in \mathbb{T}_{\mathbb{B}_s}$ the expression

$$\begin{aligned} \partial_{\mathbf{v_X}}\partial_{\mathbf{v_Y}}\Big[\boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s})(\mathbf{a},\mathbf{b})\Big] &= \partial_{\mathbf{v_X}}\partial_{\mathbf{v_Y}}\Big[\mathbf{g}(\boldsymbol{\chi}_{t,s*}\mathbf{a},\, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\mathbf{a},\,\mathbf{b})\Big] \\ &= \partial_{\mathbf{v_X}}\Big[\mathbf{g}(\partial_{\mathbf{a}}\mathbf{t}_{\mathbf{Y}}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\boldsymbol{\chi}_{t,s*}\mathbf{a}\,, \partial_{\mathbf{b}}\mathbf{t}_{\mathbf{Y}})\Big] \\ &= \mathbf{g}(\partial_{\mathbf{a}}\mathbf{t}_{\mathbf{Y}}\,, \partial_{\mathbf{b}}\mathbf{t}_{\mathbf{X}}) + \mathbf{g}(\partial_{\mathbf{a}}\mathbf{t}_{\mathbf{X}}\,, \partial_{\mathbf{b}}\mathbf{t}_{\mathbf{Y}})\,, \end{aligned}$$

which is apparently symmetric in $\mathbf{X}, \mathbf{Y}$.

The second directional derivatives of the shear sliding yields for $\mathbf{a} \in \mathbb{T}_{\mathbb{B}_s}$ the expression

$$\begin{aligned}\partial_{\mathbf{v}_\mathbf{X}}\partial_{\mathbf{v}_\mathbf{Y}}\Big[\boldsymbol{\delta}(\mathbf{u}_{t,s})(\mathbf{a})\Big] &= \partial_{\mathbf{v}_\mathbf{X}}\partial_{\mathbf{v}_\mathbf{Y}}\Big[\mathbf{g}(\mathbf{d}_t,\, \boldsymbol{\chi}_{t,s*}\mathbf{a}) - \mathbf{g}(\mathbf{d}_s,\, \mathbf{a})\Big]\\ &= \mathbf{g}(\partial_\mathbf{X}\hat{\mathbf{Y}}\,, \boldsymbol{\chi}_{t,s*}\mathbf{a}) + \mathbf{g}(\mathbf{Y}\,, \partial_\mathbf{a}\mathbf{t}_\mathbf{X}) + \mathbf{g}(\mathbf{X}\,, \partial_\mathbf{a}\mathbf{t}_\mathbf{Y})\\ &= -\mathbf{g}(\mathbf{X}\,,\mathbf{Y})\,\mathbf{g}(\mathbf{d}_t\,, \boldsymbol{\chi}_{t,s*}\mathbf{a}) + \mathbf{g}(\mathbf{Y}\,, \partial_\mathbf{a}\mathbf{t}_\mathbf{X}) + \mathbf{g}(\mathbf{X}\,, \partial_\mathbf{a}\mathbf{t}_\mathbf{Y})\,.\end{aligned}$$

The second directional derivative of the flexural curvature is given by

$$\begin{aligned}&\partial_{\mathbf{v}_\mathbf{X}}\partial_{\mathbf{v}_\mathbf{Y}}\big[\mathbf{C}(\mathbf{u}_{t,s})(\mathbf{a},\mathbf{b})\big] = \partial_{\mathbf{v}_\mathbf{X}}\partial_{\mathbf{v}_\mathbf{Y}}\big[\mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{d}_t,\, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\partial_\mathbf{a}\mathbf{d}_s,\, \mathbf{b})\big]\\ &= \partial_{\mathbf{v}_\mathbf{X}}\Big[\mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{Y}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{d}_t\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{Y}) + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{Y}}\mathbf{d}_t\,, \boldsymbol{\chi}_{t,s*}\mathbf{b})\Big]\\ &= \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}(\partial_\mathbf{X}\hat{\mathbf{Y}})\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{Y}\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{X}) + \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{X}\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{Y})\\ &\qquad + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{X}}\mathbf{Y}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{X}}\mathbf{d}_t\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{Y})\\ &\qquad + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{Y}}\mathbf{X}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{Y}}\mathbf{d}_t\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{X})\,.\\ &= -\mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}(\mathbf{g}(\mathbf{X}\,,\mathbf{Y})\mathbf{d}_t)\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{Y}, \partial_\mathbf{b}\mathbf{t}_\mathbf{X}) + \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{X}, \partial_\mathbf{b}\mathbf{t}_\mathbf{Y})\\ &\qquad + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{X}}\mathbf{Y}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{X}}\mathbf{d}_t\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{Y})\\ &\qquad + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{Y}}\mathbf{X}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\partial_\mathbf{a}\mathbf{t}_\mathbf{Y}}\mathbf{d}_t\,, \partial_\mathbf{b}\mathbf{t}_\mathbf{X})\,.\end{aligned}$$

From the expressions above it is apparent that the second directional derivatives of the shear sliding and of the flexural curvature are symmetric with respect to an exchange of $\mathbf{X}$ and $\mathbf{Y}$ as was to be expected. Indeed the second directional derivative coincides with the second covariant derivative for the adopted extension of the virtual displacements.

The same results are obtained by considering another, perhaps simpler, extension for the virtual displacement $\mathbf{Y} = \mathbf{Y}(\mathbf{u}_{t,s})$, defined as follows

$$\hat{\mathbf{Y}}(\mathbf{u}_{\tau,s}) := (\mathbf{I} - \mathbf{d}_\tau \otimes \mathbf{d}_\tau)\,\mathbf{Y}\,, \qquad \mathbf{d}_\tau = \mathbf{P}_2\mathbf{u}_{\tau,s}\,,$$

so that

$$\hat{\mathbf{Y}}(\mathbf{u}_{t,s}) = (\mathbf{I} - \mathbf{d}_t \otimes \mathbf{d}_t)\,\mathbf{Y} = \mathbf{Y}\,.$$

The directional derivative of $\hat{\mathbf{Y}}$ along $\mathbf{X}$ at $\mathbf{u}_{t,s}$ is given by

$$(\partial_\mathbf{X}\hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = -(\mathbf{X} \otimes \mathbf{d}_t + \mathbf{d}_t \otimes \mathbf{X})\,\mathbf{Y} = -\mathbf{g}(\mathbf{X}\,,\mathbf{Y})\,\mathbf{d}_t$$

and the covariant derivative by

$$(\nabla_\mathbf{X}\hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = \boldsymbol{\Pi}\,(\partial_\mathbf{X}\hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = -\mathbf{g}(\mathbf{X}\,,\mathbf{Y})\,\boldsymbol{\Pi}\,\mathbf{d}_t = \mathbf{o}\,.$$

**Second extension**

Let us now choose a different extension of the virtual displacement $\mathbf{v}_{\mathbf{Y}}(\mathbf{u}_{t,s})$ by setting

$$\begin{cases} \hat{\mathbf{t}}_{\mathbf{Y}}(\mathbf{u}_{\tau,s}) := \mathbf{t}_{\mathbf{Y}}(\mathbf{u}_{t,s})\,, \\ \hat{\mathbf{Y}}(\mathbf{u}_{\tau,s}) \;:= \big[1 - \mathbf{g}(\mathbf{d}_{\tau}\,, \mathbf{Y})\big]\,(\boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_{\tau})\,, \qquad \mathbf{d}_{\tau} = \mathbf{P}_2 \mathbf{u}_{\tau,s} \end{cases}$$

so that, since $\mathbf{g}(\mathbf{d}_t\,, \mathbf{Y}) = 0$ and $\boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_t = \mathbf{Y}$ it is

$$\hat{\mathbf{Y}}(\mathbf{u}_{t,s}) = \big[1 - \mathbf{g}(\mathbf{d}_t\,, \mathbf{Y})\big]\,(\boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_t) = \mathbf{Y}\,.$$

The directional derivative of $\hat{\mathbf{Y}}$ along $\mathbf{X}$ at $\mathbf{d}_t$ is given by

$$\begin{aligned} (\partial_{\mathbf{X}} \hat{\mathbf{Y}})(\mathbf{u}_{t,s}) &= -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,(\boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_t) + \big[1 - \mathbf{g}(\mathbf{d}_t\,, \mathbf{Y})\big]\,(\boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{X}) \\ &= -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,(\boldsymbol{\omega}_{\mathbf{Y}} \times \mathbf{d}_t) + \boldsymbol{\omega}_{\mathbf{Y}} \times (\boldsymbol{\omega}_{\mathbf{X}} \times \mathbf{d}_t) \\ &= -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{Y} - \mathbf{g}(\boldsymbol{\omega}_{\mathbf{Y}}\,, \boldsymbol{\omega}_{\mathbf{X}})\,\mathbf{d}_t = -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,(\mathbf{Y} + \mathbf{d}_t)\,, \end{aligned}$$

and the covariant derivative by

$$(\nabla_{\mathbf{X}} \hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = \boldsymbol{\Pi}\,(\partial_{\mathbf{X}} \hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{Y}\,.$$

The second directional derivative of the shear sliding is now given by

$$-\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{g}(\mathbf{Y} + \mathbf{d}_t\,, \boldsymbol{\chi}_{t,s*}\mathbf{a}) + \mathbf{g}(\mathbf{Y}\,, \partial_{\mathbf{a}} \mathbf{t}_{\mathbf{X}}) + \mathbf{g}(\mathbf{X}\,, \partial_{\mathbf{a}} \mathbf{t}_{\mathbf{Y}})\,,$$

and that of the flexural curvature by

$$\begin{aligned} &-\mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\,(\mathbf{g}(\mathbf{X}\,, \mathbf{Y})(\mathbf{Y} + \mathbf{d}_t))\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}} \mathbf{Y}\,, \partial_{\mathbf{b}} \mathbf{t}_{\mathbf{X}}) \\ &+\mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}} \mathbf{X}\,, \partial_{\mathbf{b}} \mathbf{t}_{\mathbf{Y}}) + \mathbf{g}(\partial_{\partial_{\mathbf{a}} \mathbf{t}_{\mathbf{X}}} \mathbf{Y}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\partial_{\mathbf{a}} \mathbf{t}_{\mathbf{X}}} \mathbf{d}_t\,, \partial_{\mathbf{b}} \mathbf{t}_{\mathbf{Y}}) \\ &+ \mathbf{g}(\partial_{\partial_{\mathbf{a}} \mathbf{t}_{\mathbf{Y}}} \mathbf{X}\,, \boldsymbol{\chi}_{t,s*}\mathbf{b}) + \mathbf{g}(\partial_{\partial_{\mathbf{a}} \mathbf{t}_{\mathbf{Y}}} \mathbf{d}_t\,, \partial_{\mathbf{b}} \mathbf{t}_{\mathbf{X}})\,. \end{aligned}$$

Both these expressions are nonsymmetric due to the lack of symmetry of the first terms.

Symmetry is however recovered by taking into account the additional term appearing in the expression of the second covariant derivative of the deformation measure which does not vanish since $(\nabla_{\mathbf{X}} \hat{\mathbf{Y}})(\mathbf{u}_{t,s}) = -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{Y}$.

In fact for the shear sliding we have

$$\partial_{\nabla_{\mathbf{v}_{\mathbf{X}}} \hat{\mathbf{v}}_{\mathbf{Y}}} \big[\boldsymbol{\delta}(\mathbf{u}_{t,s})(\mathbf{a})\big] = -\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{g}(\mathbf{Y}\,, \boldsymbol{\chi}_{t,s*}\mathbf{a})\,,$$

and for the flexural curvature

$$\partial_{\nabla_{\mathbf{v}_{\mathbf{X}}} \hat{\mathbf{v}}_{\mathbf{Y}}} \big[\mathbf{C}(\mathbf{u}_{t,s})(\mathbf{a}, \mathbf{b})\big] = -\mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}(\mathbf{g}(\mathbf{X}\,, \mathbf{Y})\,\mathbf{Y})\,, \boldsymbol{\chi}_{t,s*}\mathbf{b})\,.$$

By subtracting the last two terms to the second directional derivatives we get the symmetric expressions of the second covariant derivatives of the shear sliding and of the flexural curvature, coincident with the ones found with the first extension.

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