Tangent stiffness of elastic continua on manifolds

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Summary. Nonlinear models of beams, shells and polar continua are addressed from a general point of view with the aim of providing a clear motivation of the fact that the tangent stiffness of these structural models may be nonsymmetric. Classical and polar models of continua are investigated and a critical analysis of the commonly adopted strain measures is performed. It is emphasized that the kinematic space of a polar continuum is a nonlinear differentiable manifold. Accordingly, by choosing a connection on the manifold, the hessian operator of the elastic potential is defined as the second covariant derivative of the elastic potential. The hessian operator can be expressed as the difference between the second directional derivative along the trial and test fields and the first directional derivative in the direction of the covariant derivative of the test field along the trial field. It follows that the evaluation of the hessian operator requires the extention of the local virtual displacement to a vector field over the nonlinear kinematic manifold. Anyway the tensoriality of the hessian operator ensures that the result is independent of the choice of the extention. and its symmetry depends on whether the assumed connection is torsionless or not. Conservative and nonconservative loads are considered and it is shown that at equilibrium points, the tangent stiffness in independent of the choosen connection on the fiber manifold and symmetry holds for conservative loads.

1 Introduction

Polar models of beams and shells have been investigated by an ever increasing number of scholars since the pioneering contributions of J.C. Simo and its co-workers who in the years 1985-1989 faced the problem of providing a geometrically exact theory of polar beams and shells undergoing large deformations and a numerical implementation scheme for the related elastostatic and elastodynamic problems (see [4], [5], [6], [8], [9], [12], [13], [15], [17], [18]).

Until quite recently, a number of papers have been devoted to the formulation of a suitable interpolation of the kinematic variables in finite element approximations of polar continua (see e.g. [36], [39], [42], [43], [53]).

A list of recent contributions to the theoretical and computational analysis of polar beams and shells is provided in the references at the end of the paper.

Polar models of continua include one-dimensional polar beams (also called Timoshenko beams or shear deformable beams), two-dimensional polar shells (Reissner-Mindlin shells or shear deformable shells) and three-dimensional polar continua (Cosserat continua).

The peculiar features of polar models is that the evolution processes of the body take place in an ambient space which is no more the usual three-dimensional euclidean space but instead a more general geometrical object, a nonlinear manifold. This is due to the fact that the polar structure of the continuum is represented by means of an additional set of kinematic variables which, at each point of the parent classical continuum, vary over a nonlinear manifold, the fiber manifold. In polar beams the fiber manifold is the special orthogonal group of rotations which allows to monitorize the orientation of the cross sections of the beam, assumed to be rigid bodies, hinged to the beam axis, which can rotate independently **indenpendently** of the position of the beam axis. In polar shells the fiber manifold is the unit sphere, i.e. the locus which the tickness-directors belong to. Indeed the shell is described by a field of *neddles* (or *rigid hairs*) attached at each point of the middle surface. The common length of the needles is equal to the constant tickness of the shell but they can be *combed* indenpendently of the position of the middle surface. This model is referred to in the literature as a shell without drilling rotations since rotations of the **thec** needles around their axis are not taken into account. To accomodate for the interaction betweeen shell and beam models assembled toghether to design a stiffened shell, another model of shell has also been introduced, in which the polar structure is described by the rotations of a triad hinged at each point of the middle surface. This model is referred to in the literature as a shell with drilling rotations.

In Cosserat continua the fiber manifold is the special orthogonal group of rotations depicting the orientation of the rigid balls centered at each particle of the three-dimensional body which can rotate indenpendently of the position of the parent particle.

The ambient spaces in which the evolution processes of these polar continua take place are trivial fiber bundles formed by the cartesian product of the euclidean three-space times a nonlinear fiber manifold.

The analysis of such polar models requires to deal with nonlinear geometrical objects and hence to rely on concepts and results of differential geometry. This circumstance was underestimated **underextimated** in the first investigations on polar beams, [4], [5], [6], and in authors' opinion has not yet been fully digested in spite of the contribution [14] provided by Simo to explain why the tangent stiffness of the polar beams evaluated in [5], [6] was apparently nonsymmetric. Indeed the discussion performed in [14] takes no concern of the way in which the directional derivatives of the virtual displacement are defined, makes reference only to riemannian connections and hence cannot explains why a nonsymmetric but tensorial tangent stiffness may occur. Further in [14] it is claimed that the right symmetric tangent stiffness can be obtained by simply taking the symmetric part of the nonsymmetric one, at least for conservative loadings. It can be shown [50] that this is special property holds true only for the polar beam model and that its validity is strictly connected to the special extension of virtual displacements considered in [14].

As we shall see, in general, the expression of the tensorial tangent stiffness at nonequilibrium points depends on the choice of the connection over the fiber manifold which describes the polar behaviour of the continuum. At equilibrium point however the tangent stiffness is independet of the choosen connection and symmetry holds for conservative referential loads. At nonequilibrium points a nonsymmetric but tensorial stiffness may occur if the torsion of the connection does not vanish and the covariant derivative of the choosen extension of the virtual displacement vanishes identically [49], [50].

The aim of the present paper is to provide an outline a self-consistent treatment of nonlinear equilibrium problems of an elastic continuum endowed with a polar structure. Special emphasis is put on the problem concerning the evaluation of the tangent stiffness of polar continua.

The basic notions of configuration maps and tangent (virtual) displacements are reformulated in an way suitable to deal with polar models.

The appropriate ambient space for polar continua is a nonlinear manifold which has the geometric structure of a fiber bundle. In structural models of engineering interest this fiber bundle is simply the cartesian product of the physical space (the three-dimensional euclidean space) times a nonlinear manifold which characterizes the local structure of the polar continuum.

The space of configurations is a nonlinear manifold of continuously differentiable mappings which map the base manifold of a reference placement into the actual placement in the ambient manifold. Virtual displacements are defined as tangent vectors to the manifold of admissible configurations.

A general discussion of finite strain measures is provided and the equilibrium condition of the polar continuum in a reference placement is formulated by invoking a consistency property of finite strain measures.

It is shown that the notion of a connection over the fiber manifold allows one to define, on the manifold of configuration maps, the covariant derivative of one-forms which have the physical meaning of force systems acting on the body. The covariant differentiation leads to the notion of absolute (or covariant) time derivative which, applied to the equilibrium condition, provides the incremental equilibrium condition governed by the tangent stiffness operator.

It is emphasized that the evaluation of the covariant derivative of one-forms requires that the virtual displacements tangent at a given configuration be extended to vector fields in a neighborhood of the configuration.

The roles played, in evaluating of the tensorial tangent stiffness, by the connection assumed on the fiber manifold and by the choosen extension of the virtual displacements, are discussed in detail. It is shown that, at equilibrium points, the tangent stiffness is independent of the assumed connection and its symmetry depends on whether the referential loads are conservative or not.

2 Differentiable manifolds

Let us provide here for sake of completeness and clarity some basic facts and definitions about differentiable manifolds (see e.g. [7]).

- Let $\mathbb{M}$ be a set and E a BANACH space. A *chart* $\{ U, \varphi \}$ on $\mathbb{M}$ is a pair with $\varphi : U \mapsto E$ bijection between the subset $U \subset \mathbb{M}$ and an open set in E. A C^k-*atlas* $\mathcal{A}$ on $\mathbb{M}$ is a family of charts $\{ \{ U_i, \varphi_i \} \,|\, i \in I \}$ such that $\{ \cup U_i \,|\, i \in I \}$ is a covering of $\mathbb{M}$ and that the overlap maps are C^k-diffeomorfisms.

- Two atlases are equivalent if their union is still a C^k-atlas and the union of all the atlases equivalent to a given one $\mathcal{A}$ is called the *differentiable structure* generated by $\mathcal{A}$.

- A C^k *differentiable manifold* modeled on the BANACH space E is a pair $\{ \mathbb{M}, \mathcal{D} \}$ where $\mathcal{D}$ is an equivalence class of C^k-atlases on $\mathbb{M}$. The space E is called the model space.

- A subset $\mathcal{O}$ of a differentiable manifold $\mathbb{M}$ is said to be open if for each $\mathbf{x} \in \mathcal{O}$ there is a chart $\{ U, \varphi \}$ such that $\mathbf{x} \in U$ and $U \subset \mathcal{O}$.

- A morphism between two differentiable manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$ is a differentiable map $\phi : \mathbb{M}_1 \mapsto \mathbb{M}_2$.

- A C^k-diffeomorphism $\phi \in \mathrm{C}^k(\mathbb{M}_1 \,; \mathbb{M}_2)$ is a morphism which is invertible and C^k with its inverse.

- The *tangent space* $\mathbb{T}_{\mathbb{M}}(\mathbf{x})$ at a point $\mathbf{x} \in \mathbb{M}$ is the linear space of *tangent vectors* $\{\mathbf{x}\,, \mathbf{v}\} : \mathrm{C}^r(\mathbf{x}, U) \mapsto \mathrm{C}^{r-1}(\mathbf{x}, U)$ where $\mathrm{C}^r(\mathbf{x}, U)$ is the germ of scalar functions which are r-times continuously differentiable in a neighborhood U of $\mathbf{x} \in \mathbb{M}$. Tangent vectors at a point are uniquely defined by requiring that they fulfil the formal properties of a *point derivation*:

$$\begin{cases} (\mathbf{v}_1 + \mathbf{v}_2)(f) = \mathbf{v}_1(f) + \mathbf{v}_2(f)\,, & \textit{additivity}, \\ \mathbf{v}(a\,f) = a\,\mathbf{v}(f)\,, \quad a \in \mathcal{R}\,, & \textit{homogenity}, \end{cases} \Bigg\} \ \mathcal{R}\text{-linearity,}$$
$$\quad \mathbf{v}(fg) = \mathbf{v}(f)\,g + f(\mathbf{v}(g))\,, \qquad \text{LEIBNIZ rule,}$$

  where $f \in \mathrm{C}^r(\mathbf{x}, U)$. This point of view, that identifies the tangent vectors at a point with the directional derivatives of smooth scalar functions at that point, results to be the most convenient to get basic results of differential geometry.

- The *tangent bundle* $\mathbb{T}_{\mathbb{M}}$ to the manifold $\mathbb{M}$ is the disjoint union of the pairs $\{\mathbf{x}\,, \mathbb{T}_{\mathbb{M}}(\mathbf{x})\}$ with $\mathbf{x} \in \mathbb{M}$. An element $\{\mathbf{x}\,, \mathbf{v}\} \in \{\mathbf{x}\,, \mathbb{T}_{\mathbb{M}}(\mathbf{x})\}$ is said to be a tangent vector applied at the base point $\mathbf{x} \in \mathbb{M}$. We shall denote by $\boldsymbol{\tau}_{\mathbb{M}} : \mathbb{T}_{\mathbb{M}} \mapsto \mathbb{M}$ the projection on the base point: $\boldsymbol{\tau}_{\mathbb{M}}(\{\mathbf{x}\,, \mathbf{v}\}) = \mathbf{x}$.

- The *cotangent bundle* $\mathbb{T}^*_{\mathbb{S}}$ to the manifold $\mathbb{M}$ is the disjoint union of the pairs $\{\mathbf{x}\,,\mathbb{T}^*_{\mathbb{M}}(\mathbf{x})\}$ with $\mathbb{T}^*_{\mathbb{M}}(\mathbf{x})$ topological dual space of $\mathbb{T}_{\mathbb{M}}(\mathbf{x})$. The elements of the cotangent bundle are called *covectors*. We shall denote by $\mathbb{T}_{\mathbb{M}}(\mathcal{P}) \subseteq \mathbb{T}_{\mathbb{M}}$ the disjoint union of the pairs $\{\mathbf{x}\,,\mathbb{T}_{\mathbb{M}}(\mathbf{x})\}$ with $\mathbf{x} \in \mathcal{P} \subseteq \mathbb{M}$.

- A *finite dimensional* differentiable manifold is a manifold modeled on a finite dimensional normed linear space. All the tangent spaces to a finite dimensional differentiable manifold are finite dimensional linear spaces of the same dimension.

- A C^k-*fiber bundle* with typical fiber C^k-manifold $\mathbb{F}$ and base C^k-manifold $\mathbb{B}$ is a C^k-surjective map $\boldsymbol{\pi}_{\mathbb{S}} : \mathbb{S} \mapsto \mathbb{B}$ which is locally a cartesian product. This means that the C^k-manifold $\mathbb{B}$ has an open atlas $\{\,\{\,U_i, \varphi_i\,\}\,|\, i \in I\,\}$ such that for each $i \in I$ there is a C^k-diffeomorphism $\phi_i : \boldsymbol{\pi}_{\mathbb{S}}^{-1}(U_i) \mapsto U_i \times \mathbb{F}$ such that $\boldsymbol{\tau}_i \circ \phi_i = \boldsymbol{\pi}_{\mathbb{S}}$ where $\boldsymbol{\tau}_i : U_i \times \mathbb{F} \mapsto U_i$ is the canonical projection. If $\mathbb{S} = \mathbb{B} \times \mathbb{F}$ the fiber bundle is said to be trivial. If the fiber $\mathbb{F}$ is a vector space the bundle is said to be a *vector bundle*. The *tangent bundle* $\mathbb{T}_{\mathbb{M}}$ to a manifold $\mathbb{M}$ is a vector bundle whose fibers are the tangent spaces to $\mathbb{M}$.

- A *fiber bundle morphism* $\boldsymbol{\chi} : \mathbb{S} \mapsto \mathbb{S}'$ between two differentiable manifolds $\mathbb{S}, \mathbb{S}'$ is a morphism fulfilling the *fiber preserving property*:

  $$\boldsymbol{\pi}_{\mathbb{S}}(\mathbf{x}) = \boldsymbol{\pi}_{\mathbb{S}}(\mathbf{y}) \quad \Longrightarrow \quad (\boldsymbol{\pi}_{\mathbb{S}'} \circ \boldsymbol{\chi})(\mathbf{x}) = (\boldsymbol{\pi}_{\mathbb{S}'} \circ \boldsymbol{\chi})(\mathbf{y}) \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{S}\,.$$

  A fiber bundle morphism induces a base morphism $\boldsymbol{\chi}_{\mathbb{B}} : \mathbb{B} \mapsto \mathbb{B}'$ according to the relation

  $$\boldsymbol{\chi}_{\mathbb{B}} \circ \boldsymbol{\pi}_{\mathbb{S}} = \boldsymbol{\pi}_{\mathbb{S}'} \circ \boldsymbol{\chi}\,.$$

- A *section* of the fiber bundle $\boldsymbol{\pi}_{\mathbb{S}} : \mathbb{S} \mapsto \mathbb{B}$ is a smooth map $\mathbf{s} : \mathbb{B} \mapsto \mathbb{S}$ such that

  $$(\boldsymbol{\pi}_{\mathbb{S}} \circ \mathbf{s})(\mathbf{x}) = \mathbf{x}\,, \quad \forall\, \mathbf{x} \in \mathbb{B}\,.$$

  Vector fields $\hat{\mathbf{v}} : \mathbb{M} \mapsto \mathbb{T}_{\mathbb{M}}$ on a manifold $\mathbb{M}$ are sections of the tangent vector bundle $\boldsymbol{\tau}_{\mathbb{M}} : \mathbb{T}_{\mathbb{M}} \mapsto \mathbb{M}$, indeed they meet the property

  $$(\boldsymbol{\tau}_{\mathbb{M}} \circ \hat{\mathbf{v}})(\mathbf{x}) = \mathbf{x}\,, \quad \forall\, \mathbf{x} \in \mathbb{M}\,.$$

  This means that the applied vector $\hat{\mathbf{v}}(\mathbf{x}) \in \mathbb{T}_{\mathbb{M}}$ has $\mathbf{x} \in \mathbb{M}$ as base point or equivalently that $\hat{\mathbf{v}}(\mathbf{x}) \in \mathbb{T}_{\mathbb{M}}(\mathbf{x})$.

- A *submanifold* $\mathbb{P} \subset \mathbb{M}$ is a subset of the manifold $\mathbb{M}$ such that for each $\mathbf{x} \in \mathbb{P}$ there is a chart $\{\,U, \varphi\,\}$ in $\mathbb{M}$, with $\mathbf{x} \in U$, fulfilling the *submanifold property*:

  $$\varphi : U \mapsto E = E_1 \times E_2\,, \quad \varphi(U \cap \mathbb{P}) = \varphi(U) \cap (E_1 \times \{\,0\,\})\,.$$

  Every open subset of the manifold $\mathbb{M}$ is a submanifold.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be BANACH spaces and let us denote by $BL(\mathcal{A}, \mathcal{B}\,; \mathcal{C})$ the space of bounded maps taking values in $\mathcal{C}$ and separately linear in the arguments ranging in $\mathcal{A}$ and $\mathcal{B}$. In the sequel square brackets will denote linear dependence on the enclosed arguments.

- A *riemannian metric* on the manifold $\mathbb{S}$ is a field of twice covariant, symmetric and positive definite tensors $\mathbf{g}_{\mathbb{S}} : \mathbb{S} \mapsto BL(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,; \mathcal{R})$.

Any tensor field, say $\mathbf{T}_{\mathbb{S}} : \mathbb{S} \mapsto BL(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,; \mathcal{R})$, *lives at points* in the sense that at each $\mathbf{x} \in \mathbb{S}$ there exists a tensor $\mathbf{T}_{\mathbf{x}} \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}), \mathbb{T}_{\mathbb{S}}(\mathbf{x})\,; \mathcal{R})$ such that

$$\mathbf{T}_{\mathbb{S}}(\mathbf{x})\,[\mathbf{X}, \mathbf{Y}] = \mathbf{T}_{\mathbf{x}}\,[\mathbf{X}(\mathbf{x}), \mathbf{Y}(\mathbf{x})]\,, \quad \forall\, \mathbf{X}, \mathbf{Y} \in \mathbb{T}_{\mathbb{S}}\,.$$

A riemannian metric is naturally induced in each submanifold $\mathbb{M} \subset \mathbb{S}$ of a riemannian manifold $\{\mathbb{S}\,, \mathbf{g}_{\mathbb{S}}\}$ by the canonical injection of the tangent space $\mathbb{T}_{\mathbb{M}}(\mathbf{x})$ at any $\mathbf{x} \in \mathbb{M}$ into the tangent space $\mathbb{T}_{\mathbb{S}}(\mathbf{x})$ at the same point $\mathbf{x} \in \mathbb{S}$.

3 Polar continua

The mechanical description of a *polar continuum* is based on the following items.

- The *ambient space* $\mathbb{S}$ is a finite dimensional differentiable manifold without boundary in which the body undergoes evolution processes. The ambient space of a polar continuum is a fiber bundle with projection $\pi_{\mathbb{S}} : \mathbb{S} \mapsto \mathbb{E}$ and tipical fiber $\mathbb{F}$. Then locally the manifold $\mathbb{S}$ can be diffeomorphically related to the cartesian product $\mathbb{E} \times \mathbb{F}$ of the *base manifold* $\mathbb{E}$ times the *fiber manifold* $\mathbb{F}$. Both are finite dimensional differentiable manifolds without boundary. The fiber manifold $\mathbb{F}$ provides the geometric description of the local kinematics of the polar continuum. The base manifold $\mathbb{E}$ is called the *physical space* and its points are called *positions.*

- The *material body* $\mathcal{B}$ is a set of *particles* which, at each time $t \in I$, are located at points of a differentiable submanifold of the physical space $\mathbb{E}$.

- The *base configuration* map $\boldsymbol{\chi}_t : \mathcal{B} \mapsto \mathbb{E}$ is an injection of the material body $\mathcal{B}$ onto the *base placement* $\mathbb{B}_t = \boldsymbol{\chi}_t(\mathcal{B}) \subseteq \mathbb{E}$ which is a submanifold of the physical space $\mathbb{E}$.

- The *polar structure* $\mathbf{s}_t : \mathbb{B}_t \mapsto \mathbb{S}$ is a map from the base placement at time t onto the placement $\mathbb{P}_t = \mathbf{s}_t(\mathbb{B}_t)$. The map $\mathbf{s}_t : \mathbb{B}_t \mapsto \mathbb{S}$ meets the property

  $$(\boldsymbol{\pi}_{\mathbb{S}} \circ \mathbf{s}_t)(\mathbf{p}) = \mathbf{p}\,, \qquad \forall\, \mathbf{p} \in \mathbb{B}_t \subset \mathbb{E}\,,$$

  and is then a *section* of the fiber bundle $\mathbb{S}$ defined on the submanifold $\mathbb{B}_t \subset \mathbb{E}$

- A *spatial configuration* of the polar body at time $t \in I$ is an injective map $\mathbf{u}_t : \mathcal{B} \mapsto \mathbb{S}$ which assignes a *placement* $\mathbb{P}_t := \mathbf{u}_t(\mathcal{B}) \subset \mathbb{S}$ to the material body $\mathcal{B}$ and is given by the composition of the base configuration map with the polar structure:
$$\mathbf{u}_t = \mathbf{s}_t \circ \boldsymbol{\chi}_t \,.$$

In nonpolar continua the section $\mathbf{s}_t : \mathbb{B}_t \mapsto \mathbb{S}$ reduces to the identity on $\mathbb{B}_t$.

Remark 1. A very important property of the polar models of interest in structural mechanics is that the base manifold $\mathbb{E}$ and the fiber manifold $\mathbb{F}$ are both embedded in finite dimensional affine spaces, respectively denoted by $\{E\,, \mathbf{g}_E\}$ and $\{F\,, \mathbf{g}_F\}$, which are endowed with the euclidean metrics $\mathbf{g}_E \in BL\,(\mathbb{T}_E, \mathbb{T}_E\,; \mathcal{R})$ and $\mathbf{g}_F \in BL\,(\mathbb{T}_F, \mathbb{T}_F\,; \mathcal{R})$.

The ambient space $\mathbb{S}$ is a then a Riemann's manifold with the metric $\mathbf{g}_{\mathbb{S}} \in BL\,(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,; \mathcal{R})$ induced by the euclidean metrics in E and F via the inclusions $\mathbb{T}_{\mathbb{E}} \subseteq \mathbb{T}_E$ and $\mathbb{T}_{\mathbb{F}} \subseteq \mathbb{T}_F$.

Remark 2. In polar models of beams and shells and in Cosserat continua, the fiber bundle $\mathbb{S}$ is a trivial bundle, that is a cartesian product $\mathbb{S} = \mathbb{E} \times \mathbb{F}$. The physical space $\mathbb{E}$ is the euclidean space $E(3)$.

The fiber manifold $\mathbb{F}$ is $\mathrm{SO}(3)$ (the special orthogonal group of rotations) for beams and Cosserat continua, and is S^2 (the unit sphere) for shells without drilling rotations and $\mathrm{SO}(3)$ for shells with drilling rotations.

Other examples of polar continua are provided by the mathematical models of *liquid cristals* (see e.g. [3] p. 139) which are modeled by assuming that $\mathbb{E} = E(3)$, the euclidean space, and

- $\mathbb{F} = S^2$ for *cholesteric* liquid cristals (inextensible directed rodlike molecules) and
- $\mathbb{F} = P^2$ for *nematic* liquid cristals (inextensible undirected rodlike molecules) where P^2 is the real projective two-space obtained by identifying the antipodal points on S^2.

Let $\mathbf{u}_s : \mathcal{B} \mapsto \mathbb{S}$ and $\mathbf{u}_t : \mathcal{B} \mapsto \mathbb{S}$ be the reference and the current configuration of the body in the ambient space $\mathbb{S}$ and let $\boldsymbol{\chi}_s : \mathcal{B} \mapsto \mathbb{E}$ be the base map of the reference configuration.

- The *change of base configuration* from $\boldsymbol{\chi}_s$ to $\boldsymbol{\chi}_t$ is the diffeomorphism $\boldsymbol{\chi}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{B}_t)$ defined by
$$\boldsymbol{\chi}_{t,s} \circ \boldsymbol{\chi}_s = \boldsymbol{\chi}_t \,,$$
where the index k denotes a suitable integer.
- The *change of configuration* from $\mathbf{u}_s$ to $\mathbf{u}_t$ is the map $\mathbf{u}_{t,s} : \mathbf{u}_s(\mathcal{B}) \mapsto \mathbf{u}_t(\mathcal{B}) \subset \mathbb{S}$ defined by
$$\mathbf{u}_{t,s} \circ \mathbf{u}_s = \mathbf{u}_t \,.$$

The composition rules are given by

$$\mathbf{u}_{\tau,t} \circ \mathbf{u}_{t,s} = \mathbf{u}_{\tau,s} \,, \qquad \boldsymbol{\chi}_{\tau,t} \circ \boldsymbol{\chi}_{t,s} = \boldsymbol{\chi}_{\tau,s} \,.$$

Since $\boldsymbol{\chi}_{s,s} \in \mathrm{C}^k(\mathbb{B}_s\,;\mathbb{B}_s)$ and $\mathbf{u}_{s,s} : \mathbb{P}_s \mapsto \mathbb{P}_s$ are identity maps, the maps $\boldsymbol{\chi}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,;\mathbb{B}_t)$ and $\mathbf{u}_{t,s} : \mathbf{u}_s(\mathcal{B}) \mapsto \mathbf{u}_t(\mathcal{B})$ are invertible and the inverses are given by

$$(\boldsymbol{\chi}_{t,s})^{-1} = \boldsymbol{\chi}_{s,t} \,, \qquad (\mathbf{u}_{t,s})^{-1} = \mathbf{u}_{s,t} \,.$$

Remark 3. The requirement of regularity of the configuration changes must be expressed in terms of maps between manifolds. Now, while the base placements $\mathbb{B}_s = \boldsymbol{\chi}_s(\mathcal{B})$ and $\mathbb{B}_t = \boldsymbol{\chi}_t(\mathcal{B})$ are manifolds, the placements $\mathbb{P}_s = \mathbf{u}_s(\mathcal{B})$ and $\mathbb{P}_t = \mathbf{u}_t(\mathcal{B})$ are not manifolds but instead images of sections of the fiber bundle $\mathbb{S}$ defined on submanifolds of the physical space. Accordingly we require that

$$\mathbf{u}_{t,s} \circ \mathbf{s}_s \in \mathrm{C}^k(\mathbb{B}_s\,;\mathbb{S}) \,,$$

but shall write simply $\mathbf{u}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,;\mathbb{S})$.

- The base configuration changes can be depicted as a two-parameter family of diffeomorphisms $\boldsymbol{\chi}_{t,s} : \mathcal{B} \mapsto \mathrm{C}^k(\mathbb{S},\mathbb{S})$ which is called a *flow* of the material manifold $\mathcal{B}$ into the physical space $\mathbb{S}$. The flow $\boldsymbol{\chi}_{t,s}$ maps the position $\boldsymbol{\chi}_s(\mathbf{p})$ at time $s \in I$ of a particle $\mathbf{p} \in \mathcal{B}$ into its position $\boldsymbol{\chi}_t(\mathbf{p})$ at time $t \in I$ and, as seen above, meets the Chapman-Kolmogorov composition rule [7]

$$\boldsymbol{\chi}_{\tau,s} = \boldsymbol{\chi}_{\tau,t} \circ \boldsymbol{\chi}_{t,s} \,, \qquad \boldsymbol{\chi}_{t,t}(\mathbf{x}) = \mathbf{x} \,, \quad \forall\, \mathbf{x} \in \mathbb{B}_t \,.$$

- The *space of configuration changes* from $\mathbf{u}_s$ is the differentiable manifold $\mathbb{M} := \mathrm{C}^k(\mathbb{B}_s\,;\mathbb{S})$ modeled on the Banach space $\mathrm{C}^k(\mathbb{B}_s\,;\mathcal{R}^d)$, $d = \dim \mathbb{S}$.

When a reference configuration $\mathbf{u}_s$ is fixed, we shall often call a configuration change $\mathbf{u}_{t,s}$ simply a configuration by identifying it with $\mathbf{u}_t = \mathbf{u}_{t,s} \circ \mathbf{u}_s$.

- The *push forward* of a vector field $\mathbf{v}_s \in \mathrm{C}^k(\mathbb{B}_s, \mathbb{T}_{\mathbb{S}})$ along the flow $\boldsymbol{\chi}_{t,s}$ is the vector field $\boldsymbol{\chi}_{t,s\,*}\mathbf{v}_s \in \mathrm{C}^k(\mathbb{B}_t, \mathbb{T}_{\mathbb{S}})$ locally defined by

$$((\boldsymbol{\chi}_{t,s\,*}\mathbf{v}_s)\, f)(\boldsymbol{\chi}_{t,s}\,\mathbf{p}) = (\mathbf{v}_s(f \circ \boldsymbol{\chi}_{t,s}))(\mathbf{p}) \,, \quad \forall\, f \in \mathrm{C}^1(\boldsymbol{\chi}_{t,s}\,\mathbf{p}, U) \,, \mathbf{p} \in \mathbb{B}_s \,.$$

  The set $\mathrm{C}^1(\mathbf{x}, U)$ is the *germ* of continuously differentiable functions in the neighborhood U of $\mathbf{x} \in \mathbb{B}_t$. The push forward maps tangent vectors applied at points of a manifold into the corresponding deformed tangent vectors applied at the transformed points.

- The *pull back* $\boldsymbol{\chi}^*_{t,s} = \boldsymbol{\chi}^{-1}_{t,s\,*}$ is the push induced by the inverse diffeomorphism.

- The *push forward* of a tensor field $\mathbf{a}_s \in \mathrm{C}^k(\mathbb{S}, BL(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,; \mathcal{R}))$ is the tensor field $\boldsymbol{\chi}_{t,s\,*}\mathbf{a}_s \in \mathrm{C}^k(\mathbb{S}, BL(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,; \mathcal{R}))$ locally defined by the relation

$$(\boldsymbol{\chi}_{t,s\,*}\mathbf{a}_s)\,(\boldsymbol{\chi}_{t,s\,*}\mathbf{v}_s, \boldsymbol{\chi}_{t,s\,*}\mathbf{w}_s) := \boldsymbol{\chi}_{t,s\,*}\,(\mathbf{a}_s\,(\mathbf{v}_s, \mathbf{w}_s))\,,$$

  and for any $\mathbf{v}_s, \mathbf{w}_s \in \mathrm{C}^k(\mathbb{S}, \mathbb{T}_{\mathbb{S}})$.

- The Lie derivative of a tensor field $\mathbf{a}_t \in \mathrm{C}^k(\mathbb{S}, BL(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,; \mathcal{R}))$ along a flow $\boldsymbol{\chi}_{\tau,t} : \mathcal{B} \mapsto \mathrm{C}^k(\mathbb{S}, \mathbb{S})$, evaluated at the configuration at time $t \in I$ is the time derivative of the tensor field pulled back to the configuration at time $t \in I$.

$$\mathcal{L}_{\mathbf{X}_t}\mathbf{a}_t := \frac{d}{d\tau}\bigg|_{\tau=t} \boldsymbol{\chi}_{t,\tau\,*}\mathbf{a}_\tau\,,$$

  where $\mathbf{X}_t$ is the velocity field of the flow $\boldsymbol{\chi}_{\tau,t}$ at time $t \in I$:

$$\frac{d}{d\tau}\bigg|_{\tau=t} \boldsymbol{\chi}_{\tau,t} = \mathbf{X}_t\,.$$

3.1 Finite deformation measures

A *finite deformation measure* is a nonlinear operator

$$\mathbf{A} \in \mathrm{C}^2(\mathbb{M}\,; \mathrm{C}^0(\mathbb{B}_s\,; D))\,,$$

that maps the configuration changes $\mathbf{u}_{t,s} \in \mathbb{M} = \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{S})$ into the corresponding finite deformation fields $\mathbf{A}_{\mathbf{u}} = \mathbf{A}(\mathbf{u}_{t,s}) \in \mathrm{C}^0(\mathbb{B}_s\,; D)$. The space D is the finite dimensional linear space of local strain values.
Deformation measures are differential operators and hence the value $\mathbf{A}_{\mathbf{u}}(\mathbf{p})$ at a point $\mathbf{p} \in \mathbb{B}_s$ is independent of the values of the map $\mathbf{u}_{t,s}$ outside any given neighborhood of $\mathbf{p} \in \mathbb{B}_s$. This locality property is in fact characteristic of (linear or nonlinear) differential operators (see [3] p. 189).

The definition of the subset $\mathcal{R} \subset \mathbb{M}$ of *rigid configuration changes* is a cornerstone in the formulation of a continuous structural model. It is natural to assume that the identity map is a rigid configuration change.

The basic property to be enjoyed by a deformation measure is that it vanishes if and only if the configuration change is rigid:

$$\mathbf{u}_{t,s} \in \mathcal{R} \quad \Longleftrightarrow \quad \mathbf{A}(\mathbf{u}_{t,s}) = 0 \in \mathrm{C}^0(\mathbb{B}_s\,; D)\,.$$

- Two deformation measures $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{C}^2(\mathbb{M}\,; \mathrm{C}^0(\mathbb{B}_s\,; D))$ are said to be equivalent if

$$\mathbf{A}_1(\mathbf{u}_{t,s}) = 0 \quad \Longleftrightarrow \quad \mathbf{A}_2(\mathbf{u}_{t,s}) = 0\,.$$

  Let $D = D_1 \oplus D_2$ be a decomposition of the linear space D into the direct sum of two complementary subspaces and let the associated projectors be denoted by $\boldsymbol{\Pi}_1 \in BL(D\,; D_1)$, $\boldsymbol{\Pi}_2 \in BL(D\,; D_2)$.

- A deformation measure $\mathbf{A} \in \mathrm{C}^2(\mathbb{M}\,; \mathrm{C}^0(\mathbb{B}_s\,; D))$ is said to be *redundant* if there exists a nontrivial decomposition $D = D_1 \oplus D_2$ such that

$$(\boldsymbol{\Pi}_1 \circ \mathbf{A})(\mathbf{u}_{t,s}) = 0 \quad \Longrightarrow \quad \mathbf{A}(\mathbf{u}_{t,s}) = 0\,.$$

  A non *nonredundant* deformation measure is be said to be *minimal* in its class of equivalence.

- In a referential description of kinematics it is also essential to require that the deformation measure meets the following *consistency property*:

$$\mathbf{A}(\mathbf{u}_{\tau,s}) = \mathbf{A}(\mathbf{u}_{t,s}) + \mathbf{S}(\mathbf{A}(\mathbf{u}_{\tau,t}), \mathbf{u}_{t,s})\,,$$

  where $\mathbf{S}$ is a nonlinear differentiable operator such that

$$\mathbf{A}(\mathbf{u}_{\tau,t}) = 0 \quad \Longrightarrow \quad \mathbf{S}(\mathbf{A}(\mathbf{u}_{\tau,t}), \mathbf{u}_{t,s}) = 0\,, \quad \forall\, \mathbf{u}_{t,s} \in \mathbb{M}\,.$$

  The latter requirement ensures that the deformation measure is indifferent to superimposed rigid changes of configuration and hence ensures also the invariance of the deformation measure under a change of observer.

The relevance of the consistency property will be clearly illustrated in section 5.

Finite deformation fields are evaluated pointwise according to the following scheme. At any point $\mathbf{x} = (\mathbf{u} \circ \mathbf{s})(\mathbf{p})$, $\mathbf{p} \in \mathbb{B}$ we consider a local operator $\mathbf{A}_{\mathbf{x}} \in \mathrm{C}^2(\mathbb{M}\,; D)$ defined by

$$\mathbf{A}_{\mathbf{x}}(\mathbf{u}) := \mathbf{N}(\mathbf{D}\mathbf{u})_{\mathbf{x}}\,,$$

where $\mathbf{D}$ is a linear differential operator of order k acting on the space variable $\mathbf{p} \in \mathbb{B}$ and $\mathbf{N}$ is a smooth local nonlinear operator mapping the local values of the field $\mathbf{D}\mathbf{u}$ into the linear space D. The operator $\mathbf{A}$ is then defined pointwise by setting

$$\mathbf{A}_{\mathbf{u}}(\mathbf{x}) := \mathbf{A}_{\mathbf{x}}(\mathbf{u})\,, \quad \forall\, \mathbf{u} \in \mathbb{M}\,, \quad \forall\, \mathbf{x} \in (\mathbf{u} \circ \mathbf{s})(\mathbb{B})\,.$$

3.2 Virtual displacements

A *referential virtual displacement* at the configuration $\mathbf{u}_{t,s} \in \mathbb{M} = \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{S})$ is a vector field tangent to $\mathbb{M}$ at $\mathbf{u}_{t,s} \in \mathbb{M}$, that is a map $\mathbf{X} \in \mathrm{C}^k(\mathbb{M}\,; \mathbb{T}_{\mathbb{M}})$ such that

$$\mathbf{X}(\mathbf{u}_{t,s}) \in \mathbb{T}_{\mathbb{M}}(\mathbf{u}_{t,s})\,.$$

Virtual displacements are then vector fields which are defined on the space $\mathbb{M}$ of admissible configurations and take values in its tangent bundle $\mathbb{T}_{\mathbb{M}}$.

Since the reference placement $\mathbb{B}_s$ is fixed, virtual displacements can be represented as vector fields $\delta\mathbf{u}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})$ defined on the base reference placement $\mathbb{B}_s$ and taking values in the tangent bundle $\mathbb{T}_{\mathbb{S}}$ to the ambient space $\mathbb{S}$ ([3] p.170).

Accordingly the linear space of virtual displacements can be defined by

$$\mathbb{T}_{\mathbb{M}}(\mathbf{u}_{t,s}) = \{\, \delta\mathbf{u}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}}) \mid \boldsymbol{\tau}_{\mathbb{S}} \circ \delta\mathbf{u}_{t,s} = \mathbf{u}_{t,s} \,\}\,,$$

so that

$$\delta\mathbf{u}_{t,s}(\mathbf{p}) \in \mathbb{T}_{\mathbb{S}}(\mathbf{u}_{t,s}(\mathbf{p}))\,, \quad \forall\, \mathbf{p} \in \mathbb{B}_s\,.$$

The fields $\delta\mathbf{u}_{t,s}$ will also be called *referential virtual displacements*.

A *virtual displacement* at the configuration $\mathbf{u}_{t,s} \in \mathbb{M}$ is a field of vectors on $\mathbb{B}_t$ tangent to the ambient space $\mathbb{S}$, that is a map $\mathbf{v}_t \in \mathrm{C}^k(\mathbb{B}_t\,; \mathbb{T}_{\mathbb{S}})$ such that

$$\mathbf{v}_t(\mathbf{x}) \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})\,, \quad \forall\, \mathbf{x} \in \mathbb{P}_t = \mathbf{u}_{t,s}(\mathbb{P}_s)\,.$$

Hence a virtual displacement $\mathbf{v}_t \in \mathrm{C}^k(\mathbb{B}_t\,; \mathbb{T}_{\mathbb{S}})$ and the corresponding referential virtual displacement $\delta\mathbf{u}_{t,s} \in \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})$ are related by the composition rule

$$\delta\mathbf{u}_{t,s} = \mathbf{v}_t \circ \boldsymbol{\chi}_{t,s}\,.$$

4 Classical and polar models of continua

We shall present here some basic models of classical and polar continua to illustrate the general topics analyzed in the previous sections.

4.1 Cauchy continuum

A placement $\mathbb{B}$ of a Cauchy continuum is a regular region of the three-dimensional euclidean space E^3.

The tangent space at each point $\mathbf{p} \in \mathbb{B}$ is the three-dimensional linear space V^3 of translations in E^3 endowed with the usual inner product.

The tangent bundle $\mathbb{T}_{\mathbb{B}}$ is made of the disjoint union of copies of the linear translation space V^3 attached at each point of the affine space E^3.

The local structure of Cauchy continuum reduces to the one of its tangent bundle. Therefore Cauchy's model is lacking of a polar structure.

The ambient space $\mathbb{S}$ is the affine space E^3.

A placement at time $t \in I$ of the material body $\mathcal{B}$ is a diffeomorphism

$$\boldsymbol{\chi}_t \in \mathrm{C}^k(\mathcal{B}\,; E^3)\,.$$

A flow is given by a two-parameter family of diffeomorphisms $\boldsymbol{\chi}_{t,s} : \mathcal{B} \mapsto \mathrm{C}(E^3\,; E^3)$ defined by

$$\boldsymbol{\chi}_{t,s} \circ \boldsymbol{\chi}_s = \boldsymbol{\chi}_t\,.$$

Rigid configuration changes are isometric transformations in E^3 described by a translation vector and a rotation. The set of configuration changes from a given placement is then a six-dimensional manifold.

GREEN's finite deformation measure $\mathfrak{D}(\boldsymbol{\chi}_{t,s})$ associated with the flow $\boldsymbol{\chi}_{t,s}$ is the twice covariant tensor field defined by (see e.g. [40]):

$$\mathfrak{D}(\boldsymbol{\chi}_{t,s})\,(\mathbf{X},\mathbf{Y}) = \frac{1}{2}(\boldsymbol{\chi}_{s,t*}\mathbf{g}_{\mathbb{S}} - \mathbf{g}_{\mathbb{S}})\,(\mathbf{X},\mathbf{Y})\,,$$

where $\mathbf{X},\mathbf{Y} \in \mathbb{T}_{\mathbb{S}}$ are tangent vector fields on $\boldsymbol{\chi}_s(\mathcal{B})$ and $\mathbf{g}_{\mathbb{S}}$ is the metric tensor of the euclidean space $\mathbb{S}$.

GREEN's strain measure satisfies the *consistency property* since

$$\boldsymbol{\chi}_{s,\tau*}\mathbf{g}_{\mathbb{S}} - \mathbf{g}_{\mathbb{S}} = (\boldsymbol{\chi}_{s,t*} \circ \boldsymbol{\chi}_{t,\tau*})\,\mathbf{g}_{\mathbb{S}} - \mathbf{g}_{\mathbb{S}} = \boldsymbol{\chi}_{s,t*}(\boldsymbol{\chi}_{t,\tau*}\mathbf{g}_{\mathbb{S}} - \mathbf{g}_{\mathbb{S}}) + (\boldsymbol{\chi}_{s,t*}\mathbf{g}_{\mathbb{S}} - \mathbf{g}_{\mathbb{S}})\,.$$

The tangent deformation at time $t \in I$ associated with the GREEN's strain measure at the configuration $\boldsymbol{\chi}_t$ is given by [40]:

$$\frac{1}{2}\,(\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\mathbb{S}})_t(\mathbf{X},\mathbf{Y}) = \left.\frac{d}{ds}\right|_{s=t} (\boldsymbol{\chi}_{t,s*}\,\mathbf{g}_{\mathbb{S}})(\mathbf{X},\mathbf{Y}) = \mathbf{g}_{\mathbb{S}}\,((\mathrm{sym}\,\partial\mathbf{v}_t)\mathbf{X},\mathbf{Y})$$

where $\partial\mathbf{v}_t$ is the spatial derivative of the velocity $\mathbf{v}_t$ of the flow $\boldsymbol{\chi}_{t,s}$ and $\mathbf{X},\mathbf{Y} \in \mathbb{T}_{\mathbb{S}}$ are tangent vector fields on $\boldsymbol{\chi}_t(\mathcal{B})$.

4.2 Cables and Membranes

A placement $\mathbb{B}$ of a cable is a one-dimensional manifold (a curve) embedded in the euclidean space $\mathbb{S} = E^3$. The tangent bundle $\mathbb{T}_{\mathcal{B}}$ is made of the disjoint union of the one-dimensional tangent spaces to $\mathbb{B}$.

A placement $\mathbb{B}$ of a membrane is a two-dimensional manifold embedded in the euclidean space $\mathbb{S} = E^3$ The tangent bundle $\mathbb{T}_{\mathcal{B}}$ is the disjoint union of the two-dimensional tangent spaces to $\mathbb{B}$.

The models of cables and membranes are lacking of a polar structure.

Rigid configuration changes are isometric transformations of the one or two-dimensional manifold and hence the set of configuration changes from a given placement is then a non-finite dimensional manifold.

Let us consider the metric tensor field on $\mathbb{B}$:

$$\mathbf{g}_{\mathbb{B}}\,(\mathbf{X},\mathbf{Y}) := \mathbf{g}_{\mathbb{S}}\,(\boldsymbol{\Pi}^T\mathbf{X}, \boldsymbol{\Pi}^T\mathbf{Y})\,,$$

GREEN's deformation measure for the cable (or for the membrane) is given by [40]

$$\mathfrak{D}(\boldsymbol{\chi}_{t,s})\,(\mathbf{X},\mathbf{Y}) = \tfrac{1}{2}\left[\,(\boldsymbol{\chi}_{s,t*}\mathbf{g}_{\mathbb{S}} - \mathbf{g}_{\mathbb{S}})\,(\boldsymbol{\Pi}^T\mathbf{X}, \boldsymbol{\Pi}^T\mathbf{Y})\right],$$

where $\mathbf{X},\mathbf{Y} \in \mathbb{T}_{\mathbb{B}}$ are tangent vectors fields on $\boldsymbol{\chi}_s(\mathcal{B})$ and $\boldsymbol{\Pi} \in BL\,(\mathbb{T}_{\mathbb{S}}\,;\mathbb{T}_{\mathbb{B}})$ is the orthogonal projector from $\mathbb{T}_{\mathbb{S}}$ on $\mathbb{T}_{\mathbb{B}}$. Its transpose $\boldsymbol{\Pi}^T \in BL\,(\mathbb{T}_{\mathbb{B}}\,;\mathbb{T}_{\mathbb{S}})$ is the canonical injection of $\mathbb{T}_{\mathbb{B}}$ into $\mathbb{T}_{\mathbb{S}}$.

The tangent deformation at time $t \in I$ associated with the GREEN's strain measure is given by [40]:

$$\tfrac{1}{2}\,(\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\mathbb{B}})_t(\mathbf{X},\mathbf{Y}) := \frac{d}{ds}\bigg|_{s=t} (\boldsymbol{\chi}_{t,s_*}\,\mathbf{g}_{\mathbb{S}})(\boldsymbol{\Pi}^T\mathbf{X},\boldsymbol{\Pi}^T\mathbf{Y})$$

$$= \mathbf{g}_{\mathbb{B}}\,((\mathrm{sym}\,(\boldsymbol{\Pi}\,\partial\mathbf{v}_t\,\boldsymbol{\Pi}^T))\mathbf{X},\mathbf{Y})\,,$$

where $\mathbf{v}_t \in \mathrm{C}^k(\mathbb{B},\mathbb{T}_{\mathbb{S}})$ is a virtual displacement and $\mathbf{X},\mathbf{Y} \in \mathbb{T}_{\mathbb{S}}$ are tangent vector fields on $\mathbb{B} = \boldsymbol{\chi}_t(\mathcal{B})$.

4.3 Cosserat continuum

In the COSSERAT continuum the ambient space is the trivial fiber bundle defined by the projection $\boldsymbol{\pi}_{\mathbb{S}} : \mathbb{S} = E^3 \times \mathrm{SO}(3) \mapsto E^3$ onto the three-dimensional euclidean space E^3. The fiber manifold $\mathrm{SO}(3)$ is the three-dimensional special orthogonal compact group of rotations. The tangent bundle $\mathbb{T}_{\mathbb{S}}$ is the disjoint union of the linear spaces $V^3 \times (\mathrm{so}(3)\,\mathbf{Q})$ with $\mathbf{Q} \in \mathrm{SO}(3)$, Here $\mathrm{so}(3) \subset BL\,(V^3\,;V^3)$ is the linear subspace of skew symmetric mixed tensors and the linear space $\mathrm{so}(3)\,\mathbf{Q}$ is defined by [40]:

$$\mathrm{so}(3)\,\mathbf{Q} = \{\,\mathbf{T} \in BL\,(V^3\,;V^3) \;:\; \mathbf{T} = \mathbf{W}\,\mathbf{Q}\,, \quad \mathbf{W} \in \mathrm{so}(3)\,, \mathbf{Q} \in \mathrm{SO}(3)\,\}\,.$$

A base configuration at time $t \in I$ of a COSSERAT continuum is an injective map $\boldsymbol{\chi}_t : \mathcal{B} \mapsto E^3$ whose image is a compact domain in E^3. A configuration at time $t \in I$ is an injective map $\mathbf{u}_t : \mathcal{B} \mapsto E^3 \times \mathrm{SO}(3)$ defined at each particle $\mathbf{p} \in \mathcal{B}$ by

$$\mathbf{u}_t(\mathbf{p}) = \{\boldsymbol{\chi}_t(\mathbf{p})\,, \mathbf{Q}_t(\mathbf{p})\} \in E^3 \times \mathrm{SO}(3)\,,$$

where $\mathbf{Q}_t \in \mathcal{B} \mapsto \mathrm{SO}(3)$ is a rotation field with respect to a given reference triad. A flow is represented by a pair $\{\boldsymbol{\chi}_{t,s}\,, \mathbf{Q}_{t,s}\}$ with

$$\boldsymbol{\chi}_{t,s} \circ \boldsymbol{\chi}_s = \boldsymbol{\chi}_t\,, \qquad \mathbf{Q}_{t,s} \circ \mathbf{Q}_s = \mathbf{Q}_t\,.$$

A finite deformation measure for the COSSERAT continuum is given by [?], [?], [40]

$$\mathfrak{D}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s}) = \{\mathbf{C}(\mathbf{Q}_{t,s})\,, \boldsymbol{\Delta}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s})\}$$

where

$$\begin{cases} \mathbf{C}(\mathbf{Q}_{t,s}) := \boldsymbol{\Omega}_{t,s}\,, & \text{curvature change,} \\ \boldsymbol{\Delta}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s}) := \mathbf{Q}^T_{t,s}\,\partial\boldsymbol{\chi}_{t,s} - \mathbf{I}_s\,, & \text{strain gap,} \end{cases}$$

with $\mathbf{I}_s \in \mathrm{L}(V^3\,;V^3)$ identity at time $s \in I$ and

$$\boldsymbol{\Omega}_{t,s}[\mathbf{h}] := \mathrm{axial}\,(\mathbf{Q}^T_{t,s}\,\partial\mathbf{Q}_{t,s}\,[\mathbf{h}])\,, \quad \forall\,\mathbf{h} \in V^3\,.$$

Then $\boldsymbol{\Omega}_{t,s} \in \mathrm{L}(V^3\,;V^3)$ and $D = \mathrm{L}(V^3\,;V^3) \times \mathrm{L}(V^3\,;V^3)$.

4.4 Timoshenko beam

A placement of a TIMOSHENKO beam is described by a regular curve in E^3 named the axis of the beam, and by a field of rotations $\mathbf{Q} \in \mathrm{SO}(3)$, attached at each point of the beam axis, which simulate the rigid body kinematics of the cross sections of the beam. The ambient space $\mathbb{S}$ is the trivial fiber bundle $\boldsymbol{\pi}_{\mathbb{S}} : E^3 \times \mathrm{SO}(3) \mapsto E^3$.

A base configuration at time $t \in I$ is an injective map $\mathbf{r}_t : \mathcal{B} \mapsto E^3$ whose image is a regular curve in E^3. A configuration at time $t \in I$ is an injective map $\mathbf{u}_t : \mathcal{B} \mapsto E^3 \times \mathrm{SO}(3)$ defined at each particle $\mathbf{p} \in \mathcal{B}$ by

$$\mathbf{u}_t(\mathbf{p}) = \{\mathbf{r}_t(\mathbf{p}) \,, \mathbf{Q}_t(\mathbf{p})\} \in E^3 \times \mathrm{SO}(3) \,,$$

where $\mathbf{Q}_t \in \mathcal{B} \mapsto \mathrm{SO}(3)$ is a rotation field with respect to a given reference triad. A flow is represented by a pair $\{\mathbf{r}_{t,s} \,, \mathbf{Q}_{t,s}\}$ where

$$\mathbf{r}_{t,s} \circ \mathbf{r}_s = \mathbf{r}_t \,, \qquad \mathbf{Q}_{t,s} \circ \mathbf{Q}_s = \mathbf{Q}_t \,.$$

A finite deformation measure is provided by the pair [5], [40]

$$\mathfrak{D}(\mathbf{r}_{t,s}, \mathbf{Q}_{t,s}) = \{\mathbf{c}(\mathbf{Q}_{t,s}) \,, \boldsymbol{\delta}(\mathbf{r}_{t,s}, \mathbf{Q}_{t,s})\}$$

where

$$\begin{cases} \mathbf{c}(\mathbf{Q}_{t,s}) := \mathrm{axial}\,(\mathbf{Q}^T_{t,s}\mathbf{Q}'_{t,s}) & \text{flexural-torsional curvature change,} \\ \boldsymbol{\delta}(\mathbf{r}_{t,s}, \mathbf{Q}_{t,s}) := \mathbf{Q}^T_{t,s}\,\mathbf{r}'_{t,s} - \mathbf{t}_s & \text{axial-shear sliding,} \end{cases}$$

with $\mathbf{t}_s \in V^3$ unit tangent to the beam axis at time $s \in I$ and $\mathbf{c}(\mathbf{Q}_{t,s}) \in V^3$. Then $D = V^3 \times V^3$.

The apex $(\cdot)'$ denotes the derivative with respect to the curvilinear abscissa along the beam axis at the initial configuration of the time step, so that $\mathbf{Q}'_{t,s}$ is derived with respect to ξ_s and $\mathbf{Q}'_{\tau,t}$ is derived with respect to ξ_t.

4.5 Polar shells

Two basic models of polar shells have been proposed in the literature. In the former the local polar structure is simulated by means of oriented rigid hairs attached at the points of the middle surface. No drilling rotations of the hairs (i.e. rotations around their axis) are considered. The latter is the two dimensional analogue of the three-dimensional COSSERAT continuum an will be referred to as the COSSERAT shell model or shell with drilling rotations: a rigid triedron is attached at the points of the middle surface and arbitrary rotations are allowed for. Both models are briefly illustrated in the sequel.

Polar shells with drilling rotations

A placement of a Cosserat shell is described by a regular surface in E^3, the middle surface of the shell, and by a field of rotations $\mathbf{Q} \in \mathrm{SO}(3)$ defined at each point of the middle surface which simulate a rigid body kinematics along the thickness of the shell. The ambient space $\mathbb{S}$ is the trivial fiber bundle $\boldsymbol{\pi}_{\mathbb{S}} : E^3 \times \mathrm{SO}(3) \mapsto E^3$. A base configuration at time $t \in I$ is an injective map $\boldsymbol{\chi}_t : \mathcal{B} \mapsto E^3$ whose image is a regular surface in E^3. A configuration at time $t \in I$ is an injective map $\mathbf{u}_t : \mathcal{B} \mapsto E^3 \times \mathrm{SO}(3)$ defined at each particle $\mathbf{p} \in \mathcal{B}$ by

$$\mathbf{u}_t(\mathbf{p}) = \{\boldsymbol{\chi}_t(\mathbf{p}) \,, \mathbf{Q}_t(\mathbf{p})\} \in E^3 \times \mathrm{SO}(3) \,,$$

where $\mathbf{Q}_t \in \mathcal{B} \mapsto \mathrm{SO}(3)$ is a rotation field with respect to a given reference triad. A finite deformation measure is provided by the pair [19], [20]

$$\mathfrak{D}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s}) = \{\mathbf{C}(\mathbf{Q}_{t,s}) \,, \boldsymbol{\Delta}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s})\}$$

where

$$\begin{cases} \mathbf{C}(\mathbf{Q}_{t,s}) := \boldsymbol{\Omega}_{t,s} \,, & \text{curvature change,} \\ \boldsymbol{\Delta}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s}) := \mathbf{Q}^T_{t,s} \, \partial \boldsymbol{\chi}_{t,s} - \mathbf{I}_s \,, & \text{strain gap,} \end{cases}$$

with $\mathbf{I}_s \in \mathrm{L}(V^3 \,; V^3)$ identity at time $s \in I$ and

$$\boldsymbol{\Omega}_{t,s}[\mathbf{h}] := \mathrm{axial}\,(\mathbf{Q}^T_{t,s} \, \partial \mathbf{Q}_{t,s} \, [\mathbf{h}]) \,, \quad \forall \, \mathbf{h} \in \mathbb{T}_{\mathbb{B}_t}(\boldsymbol{\chi}_t(\mathbf{p})) \,.$$

Then $\boldsymbol{\Omega}_{t,s} \in \mathrm{L}(V^2 \,; V^3)$ and $D = \mathrm{L}(V^2 \,; V^3) \times \mathrm{L}(V^3 \,; V^3)$.

Polar shells without drilling rotations

A placement of a polar shell without drilling rotations is described by a middle surface in E^3 and by a field of unit vectors attached at each of its points which simulate the kinematics of the shell in the transverse direction. The ambient space is the trivial fiber bundle $\boldsymbol{\pi}_{\mathbb{S}} : \mathbb{S} = E^3 \times S^2 \mapsto E^3$. The fiber manifold is the two-dimensional unit sphere S^2 in E^3. The finite deformation measure proposed and analyzed in [8], [9] consists in the triplet

$$\mathbf{A}(\mathbf{u}_{t,s}) := \begin{vmatrix} \boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s}) \\ \boldsymbol{\delta}(\mathbf{u}_{t,s}) \\ \boldsymbol{\chi}(\mathbf{u}_{t,s}) \end{vmatrix}$$

composed by

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s})(\mathbf{a}, \mathbf{b}) &:= \mathbf{g}(\boldsymbol{\chi}_{t,s*}\mathbf{a}, \, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\mathbf{a}, \, \mathbf{b}) \,, && \text{membrane strain} \,, \\ \boldsymbol{\delta}(\mathbf{u}_{t,s})(\mathbf{a}) &:= \mathbf{g}(\mathbf{d}_t, \, \boldsymbol{\chi}_{t,s*}\mathbf{a}) - \mathbf{g}(\mathbf{d}_s, \, \mathbf{a}) \,, && \text{shear sliding} \,, \\ \boldsymbol{\chi}(\mathbf{u}_{t,s})(\mathbf{a}, \mathbf{b}) &:= \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}} \, \mathbf{d}_t, \, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\partial_{\mathbf{a}} \mathbf{d}_s, \, \mathbf{b}) \,, && \text{curvature change} \,, \end{aligned}$$

where $\mathbf{a} \,, \mathbf{b} \in V^3 = \mathbb{T}_{E^3}$ and $\mathbf{g}$ is the metric tensor on E^3.

These measures vanish if and only if the shell undergoes a rigid body transformation i.e. when the membrane deformation vanishes and the directors and tangent planes to the middle surface are rotated according to a constant rotation field. Indeed the vanishing of the membrane strain imposes that the middle surface transformation be isometric and the vanishing of the shear sliding imposes that the directors be invariant when seen by observers co-rotating with the tangent planes.

From the vanishing of the flexural curvature, which is an estrinsic quantity, one infers that the second fundamental form of the surface does not change and hence that the surface must undergo a rigid body transformation with a rotation equal to the rotation of the directors.

This finite strain measure for shells is not redundant since the vanishing of any proper subset of the strain measures does not ensure that the transformation be rigid.

4.6 Consistency, redundancy an physical plausibility

It is interesting to underline the formal analogy existing between the deformation measures pertaining to TIMOSHENKO beams, to polar shells with drilling rotations and to COSSERAT continua. Such measures satisfy the consistency condition. Indeed for the TIMOSHENKO beam we have that

$$\begin{aligned}\mathbf{Q}^T_{\tau,s}\,\mathbf{Q}'_{\tau,s} &= (\mathbf{Q}_{\tau,t}\mathbf{Q}_{t,s})^T(\mathbf{Q}_{\tau,t}\mathbf{Q}_{t,s})' = \mathbf{Q}^T_{t,s}\,\mathbf{Q}^T_{\tau,t}\,(\mathbf{Q}_{\tau,t}\mathbf{Q}_{t,s})' \\ &= \mathbf{Q}^T_{t,s}\,\mathbf{Q}^T_{\tau,t}\left(\mathbf{Q}'_{\tau,t}\,\frac{d\xi_t}{d\xi_s}\,\mathbf{Q}_{t,s} + \mathbf{Q}_{\tau,t}\mathbf{Q}'_{t,s}\right) \\ &= \mathbf{Q}^T_{t,s}\left(\mathbf{Q}^T_{\tau,t}\,\mathbf{Q}'_{\tau,t}\,\frac{d\xi_t}{d\xi_s}\right)\mathbf{Q}_{t,s} + \mathbf{Q}^T_{t,s}\mathbf{Q}'_{t,s}\,.\end{aligned}$$

Then the emisymmetric curvature tensor $\mathbf{C}(\mathbf{Q}_{t,s}) = \mathbf{Q}^T_{t,s}\mathbf{Q}'_{t,s}$ satisfies the relation

$$\mathbf{C}(\mathbf{Q}_{\tau,s}) = \mathbf{Q}^T_{t,s}\,\mathbf{C}(\mathbf{Q}_{\tau,t})\mathbf{Q}_{t,s}\frac{d\xi_t}{d\xi_s} + \mathbf{C}(\mathbf{Q}_{t,s})\,.$$

In the same way

$$\begin{aligned}\mathbf{Q}^T_{\tau,s}\,\boldsymbol{\chi}'_{\tau,s} - \mathbf{r}'_s &= \mathbf{Q}^T_{t,s}\,\mathbf{Q}^T_{\tau,t}\,\mathbf{r}'_{\tau,s} - \mathbf{r}'_s \\ &= \mathbf{Q}^T_{t,s}\,(\mathbf{Q}^T_{\tau,t}\,\mathbf{r}'_{\tau,t} - \mathbf{r}'_t)\,\frac{d\xi_t}{d\xi_s} + (\mathbf{Q}^T_{t,s}\mathbf{r}'_{t,s} - \mathbf{r}'_s)\,.\end{aligned}$$

Then the axial-shear sliding satisfies the relation

$$\boldsymbol{\delta}(\mathbf{r}_{\tau,s},\mathbf{Q}_{\tau,s}) = \mathbf{Q}^T_{t,s}\,\boldsymbol{\delta}(\mathbf{r}_{\tau,t},\mathbf{Q}_{\tau,t})\,\frac{d\xi_t}{d\xi_s} + \boldsymbol{\delta}(\mathbf{r}_{t,s},\mathbf{Q}_{t,s})\,,$$

and the consistency property is proved.

A similar proof can be carried out for the deformation measure pertaining to polar shells and to COSSERAT continua.

The formal analogy between the deformation measures of TIMOSHENKO beams, of polar shells with drilling rotations and of COSSERAT continua, has led some authors to consider the former two as special, respectively one and two-dimensional, cases of the latter [21].

Anyway, despite the increasing popularity of COSSERAT continua, and their application to model special phenomena in various field of structural mechanics (see e.g.), a simple analysis shows that the finite deformation measure of the three-dimensional COSSERAT continuum is redundant (see section 3.1). Indeed it can be proved [51] that the vanishing of the field of strain gaps implies the vanishing of the field of curvature changes:

$$\boldsymbol{\Delta}(\boldsymbol{\chi}_{t,s}, \mathbf{Q}_{t,s}) = 0 \quad \Longrightarrow \quad \mathbf{C}(\mathbf{Q}_{t,s}) = 0 \,.$$

The redundancy is due to the integrability conditions fulfilled by the field $\partial\boldsymbol{\chi}_{t,s}$. A more difficult to be proved redundancy argument should then also apply to the two-dimensional model of polar shells with drilling rotations while the one-dimensional beam model is certainly non redundant due to the absence of integrability **intergability** conditions.

But for the three-dimensional COSSERAT continua worse things are to came: if the redundant field of curvature changes is removed, in the attempt to get a nonredundant deformation measure, the three-dimensional COSSERAT continuum collapses into a CAUCHY continuum [51]. This shortcoming should lead to the conclusion that the three-dimensional COSSERAT continuum is based on an ill-posed kinematical model.

On the other hand we must observe that, despite their wide acceptance (see e.g.[8], [9], [12], [43]) the deformation measures reported in section 4.5 and commonly adopted in the literature for polar shells without drilling rotations, lead to physically nonplausible results in case of significant membrane strains. Indeed a simple computation reveals an unrealistic behaviour of an inflated polar spherical baloon since an increase of flexural curvature is measured when the radius increases. The effect is due to the amplification of the convected tangent vectors due to the deformation.

To get rid of this shortcoming we may redefine the deformation measures for polar shells without drilling rotations as follows:

$$\begin{aligned}
\boldsymbol{\varepsilon}(\boldsymbol{\chi}_{t,s})(\mathbf{a},\mathbf{b}) &:= \mathbf{g}(\boldsymbol{\chi}_{t,s*}\mathbf{a},\, \boldsymbol{\chi}_{t,s*}\mathbf{b}) - \mathbf{g}(\mathbf{a},\, \mathbf{b})\,, && \text{membrane strain}\,,\\
\boldsymbol{\delta}(\mathbf{u}_{t,s})(\mathbf{a}) &:= \mathbf{g}(\mathbf{d}_t,\, \boldsymbol{\chi}_{t,s*}\mathbf{a}) - \mathbf{g}(\mathbf{d}_s,\, \mathbf{a})\,, && \text{shear sliding}\,,\\
\boldsymbol{\chi}(\mathbf{u}_{t,s})(\mathbf{a},\mathbf{b}) &:= \mathbf{g}(\partial_{\boldsymbol{\chi}_{t,s*}\mathbf{a}}\mathbf{d}_t,\, \mathbf{R}_{t,s}\mathbf{b}) - \mathbf{g}(\partial_{\mathbf{a}}\mathbf{d}_s,\, \mathbf{b})\,, && \text{curvature change}\,,
\end{aligned}$$

where $\mathbf{R}_{t,s}$ is the isometric transformation associated with the push forward $\boldsymbol{\chi}_{t,s*}$ according to the polar decomposition formula. The new expression for the curvature change correctly predicts no flexural curvature in the inflated polar spherical baloon when the radius is changed. A detailed discussion on these topics is provided in [57].

5 Equilibrium

The proof of the *virtual work principle*, which is the basic theoretical result in continuum mechanics, requires that virtual displacements be considered as vector fields belonging to a larger space. More precisely virtual displacements at any $\mathbf{u} \in \mathbb{M}$ are assumed to belong to the SOBOLEV space $H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}}) \supset \mathrm{C}^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})$ and the differential of the strain measure from $\mathbf{u} \in \mathbb{M}$ is assumed to be a bounded linear differential operator of KORN's type [34], [40]:

$$\partial\mathbf{A}(\mathbf{i}_t) \in BL\,(H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})\,;\mathcal{L}^2(\mathbb{B}_t\,;D))\,,$$

where $\mathbf{i}_t = \mathbf{u}_{t,t}$ is the identity on $\mathbb{P}_t$.

Virtual displacements in the kernel of the tangent deformation operator $\partial\mathbf{A}(\mathbf{i}_t)$ are said to be *rigid* at $\mathbf{u} \in \mathbb{M}$.

Since virtual displacements belong to the HILBERT space $H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})$, the force systems $\mathbf{f}_t \in BL\,(H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})\,;\mathcal{R})$ belong to the dual HILBERT space.

Equilibrium of a force system is expressed by the condition of orthogonality to any admissible rigid virtual displacement:

$$\langle \mathbf{f}_t, \mathbf{v}_t \rangle = 0\,, \qquad \forall\, \mathbf{v}_t \in \mathrm{Ker}\partial\mathbf{A}(\mathbf{i}_t)^{\perp} \subset H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})\,.$$

In presence of kinematic constraints, admissible virtual displacements belong to a closed linear subspace $\mathcal{V}(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}}) \subseteq H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})$ and referential admissible virtual displacements to the closed linear subspace $\mathcal{V}(\mathbb{B}_s\,;\mathbb{T}_{\mathbb{S}}) \subseteq H^k(\mathbb{B}_s\,;\mathbb{T}_{\mathbb{S}})$.

- The *virtual work theorem* [40] ensures that if a force system

  $$\mathbf{f}_t \in BL\,(H^k(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})\,;\mathcal{R})\,,$$

  is in equilibrium there exists a stress field $\boldsymbol{\sigma} \in \mathcal{L}^2(\mathbb{B}_t\,;S)$ fulfilling the variational condition:

  $$\int_{\mathbb{B}_t} \boldsymbol{\sigma}_{\mathbf{x}_t} : (\partial\mathbf{A}(\mathbf{i}_t)\cdot\mathbf{v})_{\mathbf{x}_t}\, d\mu_t = \langle \mathbf{f}_t, \mathbf{v}_t \rangle\,, \qquad \forall\, \mathbf{v}_t \in \mathcal{V}(\mathbb{B}_t\,;\mathbb{T}_{\mathbb{S}})\,.$$

The local values $\boldsymbol{\sigma}_{\mathbf{x}_t}$ of the stress field belong to the finite dimensional space S dual of D.

When transformed to the reference configuration, the virtual work condition reads:

$$\int_{\mathbb{B}} \mathfrak{S}_{\mathbf{x}_s} : \big(\partial\mathbf{A}(\mathbf{u}_{t,s})\cdot\delta\mathbf{u}_{t,s}\big)_{\mathbf{x}_s} d\mu_s = \langle \mathbf{G}(\mathbf{u}_{t,s})\cdot\mathbf{f}_t, \delta\mathbf{u}_{t,s} \rangle\,, \qquad \forall\, \delta\mathbf{u}_{t,s} \in \mathcal{V}(\mathbb{B}_s\,;\mathbb{T}_{\mathbb{S}})\,,$$

where $\mathbf{G}(\mathbf{u}_{t,s})\cdot\mathbf{f}_t$ is the equivalent force in the reference configuration, defined by the identity

$$\langle \mathbf{G}(\mathbf{u}_{t,s})\cdot\mathbf{f}_t, \delta\mathbf{u}_{t,s} \rangle := \langle \mathbf{f}_t, \delta\mathbf{u}_{t,s} \circ \boldsymbol{\chi}_{t,s}^{-1} \rangle\,, \qquad \forall\, \delta\mathbf{u}_{t,s} \in \mathcal{V}(\mathbb{B}_s\,;\mathbb{T}_{\mathbb{S}})\,,$$

and $\mathfrak{S} \in \mathcal{L}^2(\mathbb{B}\,; S)$ is the referential stress measure conjugate to the finite deformation $\mathbf{A}(\mathbf{u}_{t,s}) \in \mathrm{C}^0(\mathbb{B}_s\,; D)$, locally defined as

$$\mathfrak{S}_{\mathbf{x}_s} = \mathbf{L}_{\mathbf{x}_s}(\mathbf{u}_{t,s})^{-T} \cdot \boldsymbol{\sigma}_{\mathbf{x}_t}\,,$$

where

$$\mathbf{L}_{\mathbf{x}_s}(\mathbf{u}_{t,s}) := \partial_1 \mathbf{S}_{\mathbf{x}_s}(0, \mathbf{u}_{t,s}) \in BL\,(S\,; S)\,,$$

is assumed to be invertible. The nonlinear operator $\mathbf{S}$ was introduced in section 3.1 in stating the consistency property and ∂_1 denotes the partial derivative with respect to the first argument.

The directional derivative $\partial\mathbf{A}(\mathbf{u}_{t,s}) \cdot \delta\mathbf{u}_{t,s}$ is defined pointwise by considering a virtual trajectory thru $\mathbf{u}_{t,s}$ with tangent $\delta\mathbf{u}_{t,s}$ and setting:

$$\begin{aligned}(\partial\mathbf{A}(\mathbf{u}_{t,s}) \cdot \delta\mathbf{u}_{t,s})_{\mathbf{x}_s} &:= \partial_{\delta\mathbf{u}_{t,s}} \mathbf{A}_{\mathbf{x}_s}(\mathbf{u}_{t,s}) = \frac{\partial}{\partial t}\,\mathbf{A}_{\mathbf{x}_s}(\mathbf{u}_{t,s}) \\ &= \partial\mathbf{N}((\mathbf{D}\mathbf{u}_{t,s})_{\mathbf{x}_s}) \cdot \frac{\partial}{\partial t}(\mathbf{D}\mathbf{u}_{t,s})_{\mathbf{x}_s} = \partial\mathbf{N}((\mathbf{D}\mathbf{u}_{t,s})_{\mathbf{x}_s}) \cdot (\mathbf{D}\delta\mathbf{u}_{t,s})_{\mathbf{x}_s}\,.\end{aligned}$$

The dot denotes linear dependence on the subsequent term.

6 Elastic equilibrium

Green's *elastic energy* is a scalar function $\varphi_{\mathbf{x}_s} \in \mathrm{C}^2(D\,; \mathcal{R})$ that maps the local values of the finite elastic deformation $\mathfrak{D}_{\mathbf{x}_s} \in D$ into the corresponding elastic energy $\varphi_{\mathbf{x}_s}(\mathfrak{D}_{\mathbf{x}_s})$ per unit volume in the reference placement $\mathbb{P}_s$.

- The *elastic law* relates the *local deformation measure* $\mathfrak{D}_{\mathbf{x}_s} \in D$ to the conjugate *local stress state* $\mathfrak{S}_{\mathbf{x}_s} \in S$:

$$\mathfrak{S}_{\mathbf{x}_s} = \partial\varphi_{\mathbf{x}}(\mathfrak{D}_{\mathbf{x}_s})\,.$$

The reference placement $\mathbb{P}_s$ is assumed to be a *natural state* for the material. This means that $\mathfrak{S}_{\mathbf{x}_s} = (\partial\varphi_{\mathbf{x}_s})(\mathfrak{D}_{\mathbf{x}_s})$ vanishes if $\mathfrak{D}_{\mathbf{x}_s} = 0$. The *global elastic energy* $\varphi \in \mathrm{C}^2(\mathcal{L}^2(\mathbb{B}_s\,; D)\,; \mathcal{R})$ of the body is the integral of the specific elastic energy over the base manifold:

$$\varphi(\mathfrak{D}) = \int_{\mathbb{B}} \varphi_{\mathbf{x}_s}(\mathfrak{D}_{\mathbf{x}_s})\, d\mu_s\,.$$

Hereafter the suffices t, s will be dropped whenever not strictly necessary.

The *global elastic potential* $\phi \in \mathrm{C}^2(\mathrm{C}^k(\mathbb{B}_s, \mathbb{P}_t)\,; \mathcal{R})$ provides the elastic energy associated with the configuration change $\mathbf{u} \in \mathrm{C}^k(\mathbb{B}_s, \mathbb{P}_t)$ and is given by

$$\phi(\mathbf{u}) := (\varphi \circ \mathbf{A})(\mathbf{u}) = \int_{\mathbb{B}} (\varphi_{\mathbf{x}} \circ \mathbf{A}_{\mathbf{x}})(\mathbf{u})\, d\mu\,.$$

Enforcing the constitutive law in terms of the elastic potential, the referential equilibrium of the body at time $t \in I$ is expressed by

$$\langle \partial\phi(\mathbf{u}), \delta\mathbf{u} \rangle \;=\; \langle \mathbf{G}(\mathbf{u}) \cdot \mathbf{f}, \delta\mathbf{u} \rangle \qquad \forall\, \delta\mathbf{u} \in \mathcal{V}(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})\,.$$

The bounded linear functionals $\mathbf{G}(\mathbf{u}) \cdot \mathbf{f} \in BL\,(\mathcal{V}(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})\,; \mathcal{R})$ and $\partial\phi(\mathbf{u}) \in BL\,(\mathcal{V}(\mathbb{B}_s\,; \mathbb{T}_{\mathbb{S}})\,; \mathcal{R})$ provide respectively the referential applied load and the referential elastic response of the body. In the sequel the terms *form*, *covector* and *bounded linear functional* should be considered as synonyms.

Setting $\mathbf{G}_{\mathbf{f}}(\mathbf{u}) := \mathbf{G}(\mathbf{u}) \cdot \mathbf{f}$, the equilibrium condition may equivalently be written by imposing the vanishing of the resultant force system on the body:

$$(\partial\phi - \mathbf{G}_{\mathbf{f}})(\mathbf{u}) \;=\; 0\,.$$

6.1 Incremental equilibrium

The incremental equilibrium is imposed by taking the total time derivative of the nonlinear condition along the equilibrium path:

$$\frac{d}{dt}\Big[(\partial\phi - \mathbf{G}_{\mathbf{f}})(\mathbf{u})\Big] \;=\; 0\,.$$

Since both the configuration change $\mathbf{u}$ and the force map $\mathbf{f}$ depend on $t \in I$ the incremental equilibrium condition is given by

$$\partial_{\dot{\mathbf{u}}}(\partial\phi - \mathbf{G}_{\mathbf{f}})(\mathbf{u}) \;=\; \mathbf{G}(\mathbf{u}) \cdot \dot{\mathbf{f}}\,,$$

where as usual a superimposed dot denotes the time derivative.

The *total tangent stiffness* of the body is the directional derivative

$$\mathbf{K}(\mathbf{u}) := \partial(\partial\phi - \mathbf{G}_{\mathbf{f}})(\mathbf{u})\,,$$

and the incremental equilibrium is accordingly written as

$$\mathbf{K}(\mathbf{u}) \cdot \dot{\mathbf{u}} = \mathbf{G}(\mathbf{u}) \cdot \dot{\mathbf{f}}\,.$$

However when dealing with polar continua the directional derivative of the form valued map $(\partial\phi - \mathbf{G}_{\mathbf{f}})$ at a configuration $\mathbf{u} \in \mathbb{M}$ cannot be taken in the classical way since the ambient space $\mathbb{S}$ is a nonlinear manifold and hence also the configuration space $\mathbb{M} = \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{S})$ is a nonlinear manifold of maps. Indeed in this case the evaluation of the directional derivative would require to perform the limit of differences between covectors defined on distinct tangent spaces and these differences would have no meaning until a further geometric structure is given to the space manifold. The issue will be illustrated in the next sections.

7 Affine connections and covariant differentiation

An affine connection on a differentiable manifold $\mathbb{M}$ is a map $\mathbf{X} \mapsto \nabla\mathbf{X}$ which associates to any vector field $\mathbf{X} : \mathbb{M} \mapsto \mathbb{T}_{\mathbb{M}}$ a tensor field

$$\nabla\mathbf{X} : \mathbb{M} \mapsto BL\,(\mathbb{T}_{\mathbb{M}}\,;\mathbb{T}_{\mathbb{M}})$$

of type $(1,1)$ such that for any pair of tangent vectors $\mathbf{Y}_{\mathbf{u}}, \mathbf{Z}_{\mathbf{u}} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$ the following characteristic properties of a derivation are met:

$$\begin{aligned}
&i) \quad && \nabla f = \partial f\,,\\
&ii) \quad && \nabla\mathbf{X}\,[\alpha\,\mathbf{Y}_{\mathbf{u}} + \beta\,\mathbf{Z}_{\mathbf{u}}] = \alpha\,\nabla\mathbf{X}\,[\mathbf{Y}_{\mathbf{u}}] + \beta\,\nabla\mathbf{X}\,[\mathbf{Z}_{\mathbf{u}}]\,,\\
&iii) \quad && \begin{cases} \nabla(\mathbf{X}_1 + \mathbf{X}_2) = \nabla\mathbf{X}_1 + \nabla\mathbf{X}_2\,,\\ \nabla(f\,\mathbf{X})\,[\mathbf{Y}_{\mathbf{u}}] = (\partial f\,[\mathbf{Y}_{\mathbf{u}}])\,\mathbf{X} + f\,(\nabla\mathbf{X}\,[\mathbf{Y}_{\mathbf{u}}])\,,\end{cases}
\end{aligned}$$

where $\alpha, \beta \in \mathcal{R}$, $f \in \mathrm{C}^1(\mathbf{u}, U)$ where $U(\mathbf{u}) \subseteq \mathbb{M}$ is a neighborhood of $\mathbf{u} \in \mathbb{M}$ and ∂ denotes the directional differentiation.

- Property $i)$ assesses that directional and covariant derivative are the same for scalar fields.
- Property $ii)$ expresses the $(1,1)$ tensoriality of $\nabla\mathbf{X}$.
- Properties $iii_1, iii_2)$ are characteristic of a derivation.

The local value at $\mathbf{u} \in \mathbb{M}$ of the tangent vector field $\nabla_{\mathbf{Y}_{\mathbf{u}}}\mathbf{X} : \mathbb{M} \mapsto \mathbb{T}_{\mathbb{M}}$ is the covariant derivative of the tangent vector field $\mathbf{X} : \mathbb{M} \mapsto \mathbb{T}_{\mathbb{M}}$ along the tangent vector $\mathbf{Y}_{\mathbf{u}} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$.

The covariant derivative of a tensor field $\mathbf{a} \in BL\,(\mathbb{T}_{\mathbb{M}}, \mathbb{T}_{\mathbb{M}}\,;\mathcal{R})$ is defined so that Leibniz rule holds

$$(\nabla_{\mathbf{Z}}\,\mathbf{a})(\mathbf{X},\mathbf{Y}) := \partial_{\mathbf{Z}}(\mathbf{a}(\mathbf{X},\mathbf{Y})) - \mathbf{a}(\nabla_{\mathbf{Z}}\mathbf{X},\mathbf{Y}) + \mathbf{a}(\mathbf{X},\nabla_{\mathbf{Z}}\mathbf{Y})\,.$$

The definition is well posed because the l.h.s. does not depend on the extension of the vectors $\mathbf{X}_{\mathbf{u}}, \mathbf{Y}_{\mathbf{u}} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$ to vector fields $\mathbf{X}, \mathbf{Y} : U(\mathbf{u}) \mapsto \mathbb{T}_{\mathbb{M}}$ even if each one of the addends at the r.h.s. depends on such an extension.

This property ensures that the expression above defines a three-times covariant tensor field on $\mathbb{M}$ and can be easily assessed by applying the following tensoriality criterion [2], [50].

Theorem 1. *A multilinear application*

$$A : \overbrace{\mathbb{T}_{\mathbb{M}} \times \ldots \times \mathbb{T}_{\mathbb{M}}}^{k\ times} \mapsto \mathcal{R}\,,$$

which is linear on the space $\mathrm{C}^\infty(\mathbb{M})$, *in the sense that*

$$A(\mathbf{v}_1, \ldots, f\,\mathbf{v}_i, \ldots, \mathbf{v}_k) = f\,A(\mathbf{v}_1, \ldots, \mathbf{v}_k)\,, \quad \forall\, i = 1, \ldots, k\,, \quad \forall\, f \in \mathrm{C}^\infty(\mathbb{M})\,,$$

can be pointwise represented by a unique tensor field T *on* $\mathbb{M}$. *In other words we have that* $A = A_T$ *where*

$$A_T(\mathbf{v}_1 \ldots \mathbf{v}_k)(\mathbf{p}) := T(\mathbf{p})(\mathbf{v}_1(\mathbf{p}), \ldots, \mathbf{v}_k(\mathbf{p}))\,, \quad \forall\, \mathbf{p} \in \mathbb{M}\,,$$

is the multilinear application pointwise defined by the tensor field T *on* $\mathbb{M}$.

7.1 Parallel transport and connection

It is known from differential geometry (see e.g. [3]) that the parallel transport $\mathcal{T}^{\mathbb{S}}_{\lambda,\xi} : \mathbb{T}_{\mathbb{S}}(\mathbf{c}(\xi)) \mapsto \mathbb{T}_{\mathbb{S}}(\mathbf{c}(\lambda))$ along a regular curve $\mathbf{c}$ in the ambient space manifold $\mathbb{S}$ is solution of of the ordinary differential equation

$$\nabla_{\dot{\mathbf{c}}(\lambda)}(\mathcal{T}^{\mathbb{S}}_{\lambda,\xi}\mathbf{v}_\xi) = 0 \quad \forall\, \lambda, \xi \in I\,.$$

By the uniqueness of the solution of an ODE we infer the validity of the composition rule

$$\mathcal{T}^{\mathbb{S}}_{\lambda,\mu} = \mathcal{T}^{\mathbb{S}}_{\lambda,\xi} \circ \mathcal{T}^{\mathbb{S}}_{\xi,\mu}\,,$$

The parallel transport induces a connection ∇ on the manifold according to the following formula for covariant differentiation:

$$\nabla_{\dot{\mathbf{c}}(\lambda)}\mathbf{v}_\lambda := \frac{\partial}{\partial\xi}\bigg|_{\xi=\lambda} (\mathcal{T}^{\mathbb{S}}_{\lambda,\xi}\mathbf{v}_\xi)\,,$$

where $\mathbf{v}_\lambda := \mathbf{v}(\mathbf{c}(\lambda)) \in \mathbb{T}_{\mathbb{S}}(\mathbf{c}(\lambda))$ is a vector field tangent to $\mathbb{S}$. Note that the time derivative makes sense since

$$\mathcal{T}^{\mathbb{S}}_{\lambda,\xi}\mathbf{v}_\xi \in \mathbb{T}_{\mathbb{S}}(\mathbf{c}(\lambda)) \quad \forall\, \xi \in I\,.$$

It is easy to check that the field $\mathcal{T}^{\mathbb{S}}_{\lambda,\xi}\mathbf{v}_\xi$ is parallel trasported along $\mathbf{c}$ according to the connection since

$$\nabla_{\dot{\mathbf{c}}(\lambda)}(\mathcal{T}^{\mathbb{S}}_{\lambda,\xi}\mathbf{v}_\xi) = \frac{\partial}{\partial\mu}\bigg|_{\mu=\lambda} (\mathcal{T}^{\mathbb{S}}_{\lambda,\mu}\,\mathcal{T}^{\mathbb{S}}_{\mu,\xi}\mathbf{v}_\xi) = \frac{\partial}{\partial\mu}\bigg|_{\mu=\lambda} (\mathcal{T}^{\mathbb{S}}_{\lambda,\xi}\mathbf{v}_\xi) = 0\,.$$

A connection on the finite dimensional space manifold $\mathbb{S}$, which is modeled on the linear space $\mathcal{R}^d$, induces a corresponding connection on the infinite dimensional manifold $\mathbb{M} = \mathrm{C}^k(\mathbb{B}_s\,;\mathbb{S})$ of admissible configuration changes which is modeled on the Banach space $\mathrm{C}^k(\mathbb{B}_s\,;\mathcal{R}^d)$.

Indeed the notion of parallel transport $\mathcal{T}^{\mathbb{M}}_{\tau,t}$ of a vector feld $\delta\mathbf{u}_{t,s} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u}_{t,s})$ along curves $\{\mathbf{u}_{t,s}, t \in I\}$ on the manifold $\mathbb{M}$ is defined pointwise by setting

$$(\mathcal{T}^{\mathbb{M}}_{\tau,t}\delta\mathbf{u}_{t,s})(\mathbf{p}) := \mathcal{T}^{\mathbb{S}}_{\tau,t}\,(\delta\mathbf{u}_{t,s}(\mathbf{p}))\,, \quad \forall\, \mathbf{p} \in \mathbb{B}_s\,.$$

Accordingly the covariant derivative on $\mathbb{M}$ is also defined pointwise by

$$(\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}_{t,s}}\,\delta\mathbf{u}_{t,s})(\mathbf{p}) := \nabla^{\mathbb{S}}_{\dot{\mathbf{u}}_{t,s}(\mathbf{p})}\,\delta\mathbf{u}_{t,s}(\mathbf{p})\,, \quad \forall\, \mathbf{p} \in \mathbb{B}_s\,,$$

and is related to the parallel transport by the relation

$$\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}_{t,s}}\,\delta\mathbf{u}_{t,s} = \frac{\partial}{\partial\tau}\bigg|_{\tau=t}\,(\mathcal{T}^{\mathbb{M}}_{t,\tau}\delta\mathbf{u}_{\tau,s})\,.$$

8 Tangent stiffness

Once a connenction has been defined on the manifold $\mathbb{M}$ of admissible configuration changes, the total tangent stiffness may be computed by performing covariant derivatives instead of directional derivatives to get the expression

$$\mathbf{K}(\mathbf{u}) := \nabla^{\mathbb{M}}(\partial\phi - \mathbf{G}_{\mathbf{f}})(\mathbf{u}) = \nabla^{\mathbb{M}}\boldsymbol{\alpha}(\mathbf{u})\,.$$

where

$$\boldsymbol{\alpha} = \partial\phi - \mathbf{G}_{\mathbf{f}}\,.$$

is the equilibrium gap resulting from the difference between the covector fields representing the elastic response $\partial\phi : \mathbb{M} \mapsto \mathbb{T}^*_{\mathbb{M}}$ and the referential load $\mathbf{G}_{\mathbf{f}} : \mathbb{M} \mapsto \mathbb{T}^*_{\mathbb{M}}$.

Accordingly the bounded linear functional $\boldsymbol{\alpha}(\mathbf{u}) \in \mathbb{T}^*_{\mathbb{M}}(\mathbf{u}) = BL\,(\mathbb{T}_{\mathbb{M}}(\mathbf{u})\,;\mathcal{R})$ provides the resultant referential force corresponding to the configuration change $\mathbf{u} \in \mathbb{M}$.

As shown in section 7, the covariant derivative $\nabla_{\dot{\mathbf{u}}}\,\boldsymbol{\alpha}(\mathbf{u})$ is defined by means of a formal application of Leibniz rule of calculus:

$$(\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}}\,\boldsymbol{\alpha}(\mathbf{u}))\,[\delta\mathbf{u}] := \partial_{\dot{\mathbf{u}}}\,(\boldsymbol{\alpha}(\mathbf{u})\,[\hat{\delta\mathbf{u}}]) - \boldsymbol{\alpha}(\mathbf{u})[\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}}\,\hat{\delta\mathbf{u}}]\,.$$

The vector field $\hat{\delta\mathbf{u}} \in \mathbb{T}_{\mathbb{M}}(U(\mathbf{u}))$ is an extension of the vector $\delta\mathbf{u} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$ to a neighborhood $U(\mathbf{u}) \subseteq \mathbb{M}$ of $\mathbf{u} \in \mathbb{M}$. Recall that $\mathbf{u} \in \mathbb{M}$ is an admissible configuration and that $\delta\mathbf{u} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$ is a virtual displacement from that configuration.

Although both derivatives at the r.h.s. of Leibniz formula depend on the assumed extension of the virtual displacement $\delta\mathbf{u}$, the l.h.s. is independent of such an extension and hence is tensorial in $\delta\mathbf{u}$ by the tensoriality criterion provided in Theorem 1.

Hereafter the suffix $^{\mathbb{M}}$ will be dropped unless necessary.

The hessian of the elastic potential $\phi = \varphi \circ \mathbf{A}$ provides the constitutive stiffness and is the twice covariant tensor field on the manifold $\mathbb{M}$ defined by

$$\nabla^2_{\dot{\mathbf{u}}\,\delta\mathbf{u}}\phi(\mathbf{u}) := (\nabla_{\dot{\mathbf{u}}}\partial\,\phi(\mathbf{u}))\,[\delta\mathbf{u}] = \partial_{\dot{\mathbf{u}}}\,\partial_{\hat{\delta\mathbf{u}}}\,\phi(\mathbf{u}) - \partial\phi(\mathbf{u})\,[\nabla_{\dot{\mathbf{u}}}\,\hat{\delta\mathbf{u}}]\,.$$

Applying the chain rule to $\phi(\mathbf{u}) = (\varphi \circ \mathbf{A})(\mathbf{u})$ and the Leibniz rule, the evaluation of the first term at the r.h.s. yields

$$\begin{aligned}\partial_{\dot{\mathbf{u}}}(\partial\varphi(\mathbf{A}(\mathbf{u}))\cdot\partial\mathbf{A}(\mathbf{u})\cdot\hat{\delta\mathbf{u}}) = \partial^2\varphi(\mathbf{A}(\mathbf{u}))\cdot(\partial\mathbf{A}(\mathbf{u})\cdot\hat{\delta\mathbf{u}})\cdot(\partial\mathbf{A}(\mathbf{u})\cdot\dot{\mathbf{u}}) +\\ + \partial\varphi(\mathbf{A}(\mathbf{u}))\cdot(\partial_{\dot{\mathbf{u}}}\,\partial_{\hat{\delta\mathbf{u}}}\mathbf{A})(\mathbf{u}).\end{aligned}$$

The first term at the r.h.s. is the *elastic tangent stiffness* which is a symmetric bilinear form in $\dot{\mathbf{u}}\,,\delta\mathbf{u} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$. The symmetry of the second directional derivative of the functional $\varphi \in \mathrm{C}^2(\mathcal{L}^2(\mathbb{B}_s\,;D)\,;\mathcal{R})$ is a classical result since $\mathcal{L}^2(\mathbb{B}_s\,;D)$ is a linear space.

The remainder provides the *geometric tangent stiffness*, a bilinear form in $\dot{\mathbf{u}}\,,\delta\mathbf{u} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$ given by

$$\partial\varphi(\mathbf{A}(\mathbf{u}))\cdot\Big[(\partial_{\dot{\mathbf{u}}}\partial_{\hat{\delta\mathbf{u}}} - \partial_{\nabla_{\dot{\mathbf{u}}}\hat{\delta\mathbf{u}}})\mathbf{A}\Big](\mathbf{u}) = \partial\varphi(\mathbf{A}(\mathbf{u}))\cdot\Big(\nabla^2_{\dot{\mathbf{u}}\,\delta\mathbf{u}}\,\mathbf{A}\Big)(\mathbf{u})\,.$$

We remark that the directional derivative of $\mathbf{A}$ at $\mathbf{u}$ is well defined since $\mathbf{A}(\mathbf{u})$ belongs to the linear space $\mathcal{L}^2(\mathbb{B}_s\,;D)$.

8.1 Torsion and symmetry

The *torsion* of the connection $\nabla^{\mathbb{S}}$ is the mixed tensor field $\boldsymbol{T}^{\mathbb{S}} \in \mathrm{L}(\mathbb{T}_{\mathbb{S}}, \mathbb{T}_{\mathbb{S}}\,;\mathbb{T}_{\mathbb{S}})$, twice covariant and one time contravariant, defined by

$$\boldsymbol{T}^{\mathbb{S}}(\mathbf{v},\mathbf{w}) = \nabla^2_{\mathbf{v},\mathbf{w}} - \nabla^2_{\mathbf{w},\mathbf{v}} = [\mathbf{v},\mathbf{w}] - \nabla_{\mathbf{v}}\mathbf{w} + \nabla_{\mathbf{w}}\mathbf{v}\,.$$

The second equality follows from the formula for the second covariant derivative of a scalar field $f \in \mathrm{C}^2(\mathbb{S}\,;\mathcal{R})$:

$$\nabla^2_{\mathbf{v},\mathbf{w}}\,f = \partial_{\mathbf{v}}\partial_{\mathbf{w}}\,f - (\nabla_{\mathbf{v}}\mathbf{w})\,f\,,$$

where $\mathbf{h}\,f := \partial f\cdot\mathbf{h}$ denotes the directional derivative of $f \in \mathrm{C}^2(\mathbb{S}\,;\mathcal{R})$ along $\mathbf{h} \in \mathbb{T}_{\mathbb{S}}$. Hence

$$\boldsymbol{T}^{\mathbb{S}}(\mathbf{v},\mathbf{w})\,f = (\nabla^2_{\mathbf{v},\mathbf{w}} - \nabla^2_{\mathbf{w},\mathbf{v}})\,f = (\partial_{\mathbf{v}}\partial_{\mathbf{w}} - \partial_{\mathbf{w}}\partial_{\mathbf{v}} - \nabla_{\mathbf{v}}\mathbf{w} + \nabla_{\mathbf{w}}\mathbf{v})\,f\,.$$

The formula then follows by recalling the definition of the Lie bracket:

$$[\mathbf{v},\mathbf{w}]\,f = (\partial_{\mathbf{v}}\partial_{\mathbf{w}} - \partial_{\mathbf{w}}\partial_{\mathbf{v}})\,f\,.$$

A well-known result of differential geometry states that the Lie bracket is equal to the Lie derivative, according to the formula

$$[\mathbf{X},\mathbf{Y}]_s = (\mathcal{L}_{\mathbf{X}}\mathbf{Y})_s := \frac{d}{dt}\bigg|_{t=s} \boldsymbol{\chi}_{s,t\,*}\mathbf{Y}_t$$

The *torsion* $\boldsymbol{T}^{\mathbb{M}} \in \mathrm{L}(\mathbb{T}_{\mathbb{M}}, \mathbb{T}_{\mathbb{M}}\,; \mathbb{T}_{\mathbb{M}})$ of the connection $\nabla^{\mathbb{M}}$ on the infinite dimensional manifold $\mathbb{M} = \mathrm{C}^k(\mathbb{B}_s\,; \mathbb{S})$ is defined pointwise in terms of the parent torsion $\boldsymbol{T}^{\mathbb{S}}$ of $\nabla^{\mathbb{S}}$ by the identity

$$\Big(\boldsymbol{T}^{\mathbb{M}}(\mathbf{X}_{\mathbf{u}}, \mathbf{Y}_{\mathbf{u}})\Big)_{\mathbf{p}} = \boldsymbol{T}^{\mathbb{S}}((\mathbf{X}_{\mathbf{u}})_{\mathbf{p}}, (\mathbf{Y}_{\mathbf{u}})_{\mathbf{p}}) \in \mathbb{T}_{\mathbb{S}}(\mathbf{u}_{\mathbf{p}})\,, \qquad \forall\, \mathbf{p} \in \mathbb{B}_s\,.$$

Hence we have that

$$\boldsymbol{T}^{\mathbb{M}}(\mathbf{X}_{\mathbf{u}}, \mathbf{Y}_{\mathbf{u}}) = (\nabla^2_{\mathbf{X}_{\mathbf{u}},\mathbf{Y}_{\mathbf{u}}} - \nabla^2_{\mathbf{Y}_{\mathbf{u}},\mathbf{X}_{\mathbf{u}}}) = [\mathbf{X}_{\mathbf{u}}\,, \mathbf{Y}_{\mathbf{u}}] - \nabla_{\mathbf{X}_{\mathbf{u}}}\mathbf{Y}_{\mathbf{u}} + \nabla_{\mathbf{Y}_{\mathbf{u}}}\mathbf{X}_{\mathbf{u}}\,.$$

A vanishing torsion $\boldsymbol{T}^{\mathbb{S}}$ implies that the hessian of any $f \in \mathrm{C}^2(\mathbb{S}\,; \mathcal{R})$ is symmetric. The finite dimensionality of D ensures that also the hessian

$$(\nabla^2_{\mathbf{X}_{\mathbf{u}},\mathbf{Y}_{\mathbf{u}}}\mathbf{A}_{\mathbf{x}})(\mathbf{u}) \in D\,,$$

of the local deformation map $\mathbf{A}_{\mathbf{x}} \in \mathrm{C}^2(\mathbb{M}\,; D)$ is symmetric. If follows that the geometric tangent stiffness is symmetric too.

9 Conservative vs nonconservative loads

Let the referential force system acting on the body be positional and conservative in the sense that there exists a scalar potential $F_{\mathbf{f}} \in \mathrm{C}^1(\mathbb{M}\,; \mathcal{R})$ linearly dependent on $\mathbf{f}$ and such that

$$\mathbf{G}_{\mathbf{f}}(\mathbf{u}) = \mathbf{G}(\mathbf{u}) \cdot \mathbf{f} = -\partial F_{\mathbf{f}}(\mathbf{u})\,.$$

Then, in terms of the total potential $P = \phi + F_{\mathbf{f}}$, sum of the elastic potential $\phi = \varphi \circ \mathbf{A}$ and of the referential load potential $F_{\mathbf{f}}$, the condition of elastic equilibrium becomes

$$\partial P(\mathbf{u}) = \mathbf{o}\,.$$

A solution $\mathbf{u} \in \mathbb{M}$ is then a *critical point* of P. Accordingly the incremental equilibrium condition writes

$$\nabla_{\dot{\mathbf{u}}}\,\partial P(\mathbf{u}) = -\partial F_{\dot{\mathbf{f}}}(\mathbf{u})\,,$$

and in variational form

$$\nabla^2_{\dot{\mathbf{u}}\,\delta\mathbf{u}}P(\mathbf{u}) = \partial_{\dot{\mathbf{u}}}\,\partial_{\hat{\delta\mathbf{u}}}\,P(\mathbf{u}) - \partial P(\mathbf{u})\,[\nabla_{\dot{\mathbf{u}}}\,\hat{\delta\mathbf{u}}] = \partial_{\dot{\mathbf{u}}}\,\partial_{\hat{\delta\mathbf{u}}}\,P(\mathbf{u}) = -\langle \partial F_{\dot{\mathbf{f}}}(\mathbf{u}), \delta\mathbf{u}\rangle\,,$$

$\forall\, \delta\mathbf{u} \in \mathbb{T}_{\mathbb{M}}(\mathbf{u})$ and $\forall\, \hat{\delta\mathbf{u}} \in \mathbb{T}_{\mathbb{M}}(U(\mathbf{u}))$ which is an extension of $\delta\mathbf{u}$ to a neighborhood $U(\mathbf{u}) \subseteq \mathbb{M}$. The second equality in the formula above holds since the derivative $\partial P(\mathbf{u})$ vanishes at equilibrium points $\mathbf{u} \in \mathbb{M}$.

From the previous results we get that the hessian of the total potential at a critical point can be computed as the second directional derivative of the potential (the classical formula) by performing an arbitrary extension of the virtual displacement. Remarkably the result is tensorial and symmetric since it depends neither on the extension nor on the choosen connection. Since a torsionless connection can be considered, we infer that the hessian has to be symmetric. It follows that the tangent stiffness $\mathbf{K}(\mathbf{u})$ at equilibrium points $\mathbf{u} \in \mathbb{M}$ is symmetric and defined by

$$\langle \mathbf{K}(\mathbf{u})\, \dot{\mathbf{u}}, \delta\mathbf{u} \rangle := \partial_{\dot{\mathbf{u}}}\, \partial_{\hat{\delta\mathbf{u}}}\, P(\mathbf{u})\,.$$

This observation was underlined with some contradictions in [5] and in a more clear but still incomplete form in [14]. Indeed the discussion performed in [14] takes no concern of the way the directional derivatives of the virtual displacement are defined and makes reference only to riemannian connections.

Numerical evidence of the symmetry of the tangent stiffness at equilibrium points in the case of positional and conservative loads was provided in [5].

It is worth noting that the authors of [5] found a nonsymmetric but tensorial expression of the tangent stiffness for polar beams by adopting the expression above at nonequilibrium points. Indeed at noncritical points the hessian should be evaluated by the tensorial formula

$$\langle \mathbf{K}(\mathbf{u})\, \dot{\mathbf{u}}, \delta\mathbf{u} \rangle := \nabla^2_{\dot{\mathbf{u}}\,\delta\mathbf{u}} P(\mathbf{u}) = \partial_{\dot{\mathbf{u}}}\, \partial_{\hat{\delta\mathbf{u}}}\, P(\mathbf{u}) - \partial P(\mathbf{u})\, [\nabla_{\dot{\mathbf{u}}}\, \hat{\delta\mathbf{u}}]\,,$$

which requires the definition of a connection and the choice of an extension of the virtual displacements.

The relevance of the role played by the torsion of the connection and by the extension choosen for the virtual dispacement, in explaining why a nonsymmetric but tensorial stiffness may occur, was enlighted in [46] when the author was not yet aware of the paper [14].

More generally, if the referential load is nonconservative, the tangent stiffness has to be defined by the formula

$$\langle \mathbf{K}(\mathbf{u})\, \dot{\mathbf{u}}, \delta\mathbf{u} \rangle := (\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}}\, \boldsymbol{\alpha}(\mathbf{u}))\, [\delta\mathbf{u}] = \partial_{\dot{\mathbf{u}}}\, (\boldsymbol{\alpha}(\mathbf{u})\, [\hat{\delta\mathbf{u}}]) - \boldsymbol{\alpha}(\mathbf{u})[\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}}\, \hat{\delta\mathbf{u}}]\,,$$

where the resultant referential force, given by

$$\boldsymbol{\alpha} = \partial\phi - \mathbf{G}_{\mathbf{f}} : \mathbb{M} \mapsto \mathbb{T}^*_{\mathbb{M}}\,,$$

vanishes at equilibrium points. In the general case the tangent stiffness is then tensorial but possibly nonsymmetric also at equilibrium points. Anyway at these points the expression of the tangent stiffness is independent of the choosen connection and is given by the formula

$$\langle \mathbf{K}(\mathbf{u})\, \dot{\mathbf{u}}, \delta\mathbf{u} \rangle = (\nabla^{\mathbb{M}}_{\dot{\mathbf{u}}}\, \boldsymbol{\alpha}(\mathbf{u}))\, [\delta\mathbf{u}] = \partial_{\dot{\mathbf{u}}}\, (\boldsymbol{\alpha}(\mathbf{u})\, [\hat{\delta\mathbf{u}}])\,.$$

In [14] it was claimed that the correct symmetric stiffness for polar beams is obtained by taking the symmetric part of the nonsymmetric one. We remark that this statement is correct only for the special extension of the virtual dispacement choosen there. A comprehensive analysis of the evaluation of the tangent stiffness for polar beams can be found in [49], [50].

10 Conclusions

On a nonlinear manifold there is no preferential way of defining a connection among tangent spaces at different points.

The choice of a connection determines the covariant differentiation of vector fields belonging to the tangent bundle and of related covector and tensor fields. On the contrary, in the special case of an affine manifold, there is a standard connection, the euclidean one.

If the nonlinear manifold is embedded in an affine space endowed with an euclidean metric, there is a canonical way to define a riemannian metric through the Levi-Civita connection. This connection is uniquely detected as the one that mimics some basic properties of euclidean geometry, that is invariance of the local metric and symmetry of the second covariant derivative of scalar fields.

This connection is also the most natural to be considered due to the simple computation of the related covariant derivative in terms of the directional derivative in the container euclidean space.

In fact, in the polar models that we have considered, the fiber manifold is always embedded in a linear space with inner product and, according to the Levi-Civita conection, the covariant derivative on the manifold is given by the orthogonal projection of the directional derivative in the parent linear space onto the tangent bundle.

Our analysis reveals that, in a general model of polar elastic continua, the tangent stiffness must be defined as the covariant derivative of the resultant referential force which is a covector field on the manifold of configuration changes.

At equilibrium points the resultant referential force vanishes and the tangent stiffness is indepenedet of the assumed connection on the fiber manifold but in general may fail to be symmetric.

The circumstance that at nonequilibrium points the expression of the tangent stiffness of polar continua and its symmetry property depend directly on the connection choosen on the fiber manifold, should not disturb any *physical sense*. In fact it is known that also in the euclidean space nonconventional connections may be defined to get special geometric models capable e.g. to provide a mathematical model of continuous distributions of dislocations [1].

In the special case of conservative referential loads, the tangent stiffness is provided by the hessian of the total potential and is then tensorial and symmetric at equilibrium points indipendently of the choice of the connection and of the estension of virtual displacements required for its evaluation

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