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Well-Posedness of Mixed Formulations in Elasticity

Mixed formulations in elasticity are analysed and existence and uniqueness of the solution are discussed in the context of Hilbert space theory. New results, referred to in the analysis of elasticity problems, are proved. They are concerned with the closedness of the product of two linear operators and a projection property equivalent to the closedness of the sum of two closed subspaces. A set of two necessary and sufficient conditions for the well-posedness of an elastic problem with a singular elastic compliance provides the most general result of this kind in linear elasticity. Sufficient criteria for the well-posedness of elastic problems in structural mechanics including the presence of supporting elastic beds are contributed and applications are exemplified.

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1. Introduction

Mixed formulations in elasticity, in which both the stress and the kinematic fields are taken as basic unknowns of the problem, are motivated either by singularities of the constitutive operators or by computational requirements.

The pioneering contributions by I. Babuška [1] and F. Brezzi [2] have provided mixed formulations leading to saddle-point problems with a sound mathematical foundation. A comprehensive presentation of the state of art can be found in chapter II of [3] where existence and uniqueness results and *a priori* error estimates are contributed.

The present paper is devoted to the abstract analysis of elasticity problems in which the elastic compliance is allowed to be singular so that the elastic strains are subject to a linear constraint. Problems involving such constraints have been recently analysed in [4, 5] and critically reviewed in [6].

Our aim is to provide criteria for the assessment of the well-posedness property for these problems. Well-posedness corresponds to the engineering expectation that a (possibly non-unique) solution of a problem must exist under suitable variational conditions of admissibility on the data.

An elastic model capable to encompass all the usual engineering applications must then include a possibly singular elastic compliance and external elastic constraints characterized by a non-coercive stiffness operator.

The treatment of such general kind of models is out of the range of applicability of the results that can be found in treatises on the foundation of elasticity [see e.g. 7, 8]. New necessary and sufficient conditions for the existence of a solution and applicable criteria for their fulfilment are thus needed.

Banach's fundamental results in Functional Analysis and classical properties of Hilbert spaces are the essential background for the investigation [9, 10]. A review of the essential notions and propositions can be found in [11] and [12].

To provide a self-consistent presentation we devote an appendix to a brief exposition of classical results referred to in the subsequent analysis. We further give simpler proofs in Hilbert spaces of some basic results of Functional Analysis usually dealt with in the more troublesome context of Banach spaces.

The proof of some original results is also contributed in a preliminary section. They are concerned with a variant of an inequality which characterizes the closedness of the sum of two closed subspaces and with a criterion for the closedness of the image of the product of two operators.

An abstract treatment of linear problems governed by symmetric bilinear forms yields a reference framework for the subsequent analysis. The characteristic properties of structural models are then illustrated and the problems of equilibrium and of kinematic compatibility are discussed.

The mixed formulation of an elastic structural problem with a singular behaviour of the constitutive operator and of the external elastic constraints is then discussed.

The analysis is based on the split of the stress field into its elastically effective and ineffective parts. By expressing the effective part in terms of the strain field an equivalent problem in terms of the kinematic field and of the ineffective stress field is obtained. The discussion of this problem is illuminating and reveals which condition must be fulfilled for its equivalence to a reduced problem whose sole unknown is the kinematic field. This is a classical symmetric one-field problem in which trial and test fields belong to the same space. The necessary and sufficient conditions for well-posedness of the reduced problem are discussed in detail and applicable criteria for their fulfilment are contributed.

The well-posedness of the more challenging situation in which the external elastic energy is not semielliptic is then discussed. This extension is motivated by the analysis of elastic structures resting on elastic beds. The treatment starts with the observation that, in the applications, the external elastic energy can be assumed to be semielliptic with respect to rigid kinematisms and is based on an original result named the elastic bed inequality.

It is shown that the condition ensuring the equivalence of the mixed problem to a reduced one and the well-posedness criteria of the reduced problem are always met for simple structural models, defined to be those in which the subspaces of rigid displacements and of self-stresses are finite dimensional. This result provides a theoretical basis to engineers' confidence to get a solution of structural assemblies composed by one dimensional elements such as bars and beams with possibly singular elastic compliances and resting on elastic beds.

The discussion of two- or three-dimensional structural models with singular elastic compliance is by far more difficult and the answer to well-posedness is generally negative due to the infinite dimensionality of the subspace of self-stresses. The condition which fails to be met is the one ensuring the equivalence between the mixed problem and the corresponding reduced one. Actually, a singularity of the elastic compliance imposes a constraint on the strain fields. The compatibility requirement induces a corresponding constraint on the kinematic fields and hence reactive forces are originated.

The equivalence above requires the existence of elastically ineffective stresses in equilibrium with the reactive forces. The trouble arises from the fact that only very special singularities of the elastic compliance ensure the existence of such stress fields. This difficulty explains why the discussion of mixed problems is by far more challenging than the discussion of one-field problems.

2. Preliminary results

To provide a comprehensive presentation of the subject we report in the Appendix some basic definitions and results of Functional Analysis which all subsequent developments will make reference to.

Further we present here some new results which have been discovered in the development of the investigation on mixed problems.

First we quote a variant of proposition A.5 providing an inequality which plays a basic role in the analysis carried out in section 7. The result is due to the first author.

Proposition 2.1. A projection property: *Let $\mathscr{X}$ be a Hilbert space and $\mathscr{A} \subseteq \mathscr{X}$ and $\mathscr{B} \subseteq \mathscr{X}$ closed subspaces such that their sum $\mathscr{A} + \mathscr{B}$ is closed. Let us further denote by $\boldsymbol{\Pi}_{\mathscr{A}}$ and $\boldsymbol{\Pi}_{\mathscr{B}}$ the orthogonal projectors on $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{X}$. Then there exists a constant $k > 0$ such that*

$$\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}} + k\|\boldsymbol{\Pi}_{\mathscr{A}}\mathbf{x}\|_{\mathscr{X}/\mathscr{B}} \quad \forall \mathbf{x} \in \mathscr{X},$$

$$\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \|\mathbf{x}\|_{\mathscr{X}/\mathscr{B}} + k\|\boldsymbol{\Pi}_{\mathscr{B}}\mathbf{x}\|_{\mathscr{X}/\mathscr{A}} \quad \forall \mathbf{x} \in \mathscr{X}.$$

Proof: The proof of proposition A.5 shows that

$$\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \|\mathbf{x} - \mathbf{a}\|_{\mathscr{X}} + c\|\mathbf{a} + \mathbf{b}\|_{\mathscr{X}} \quad \forall \mathbf{a} \in \mathscr{A}, \quad \mathbf{b} \in \mathscr{B}.$$

Setting $\mathbf{a} = \boldsymbol{\Pi}_{\mathscr{A}}\mathbf{x}$ and taking the infimum with respect to $\mathbf{b} \in \mathscr{B}$ we get the first inequality. Setting $\mathbf{b} = \boldsymbol{\Pi}_{\mathscr{B}}\mathbf{x}$ and taking the infimum with respect to $\mathbf{a} \in \mathscr{A}$ we get the second one. □

A simple geometrical sketch of the previous result is given in Fig. 2.1.

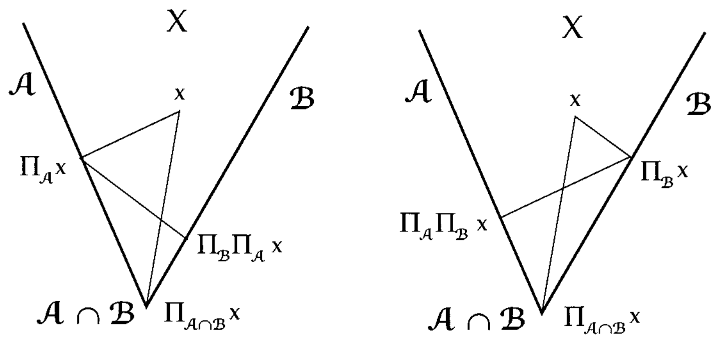


Fig. 2.1. Geometrical interpretation of the projection property

Remark 2.1: For any pair $\{\mathbf{x}, \mathbf{y}\} \in \mathscr{X} \times \mathscr{X}$ we have

$$(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2} \le \|\mathbf{x}\| + \|\mathbf{y}\| \le \sqrt{2}\,(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2},$$

and hence the inequalities in propositions A.5 and 2.1 can be rewritten as

$$\|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \bar{c}(\|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{A}} + \|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{B}}) \qquad \forall \mathbf{x} \in \mathscr{X},$$

$$\|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \bar{k}(\|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{A}} + \|\boldsymbol{\Pi}_{\mathscr{A}}\mathbf{x}\|^2_{\mathscr{X}/\mathscr{B}}) \quad \forall \mathbf{x} \in \mathscr{X},$$

$$\|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \bar{k}(\|\mathbf{x}\|^2_{\mathscr{X}/\mathscr{B}} + \|\boldsymbol{\Pi}_{\mathscr{B}}\mathbf{x}\|^2_{\mathscr{X}/\mathscr{A}}) \quad \forall \mathbf{x} \in \mathscr{X},$$

with obvious definitions of the constants. These inequalities are the ones directly invoked in our analysis.

We derive hereafter a useful criterion for the closedness of the image of a product operator.

To this end we premise the following lemma.

Proposition 2.2. An equivalence between closedness properties: *Let $\mathscr{X}$ be a Hilbert space and $\mathscr{A}$, $\mathscr{B}$ subspaces of $\mathscr{X}$ with $\mathscr{B}$ closed. Then $\mathscr{A}+\mathscr{B}$ is closed in $\mathscr{X}$ if and only if the subspace $(\mathscr{A}+\mathscr{B})/\mathscr{B}$ is closed in the factor space $\mathscr{X}/\mathscr{B}$.*

Proof: Let $\mathscr{A}+\mathscr{B}$ be closed in $\mathscr{X}$. Then $\mathscr{A}+\mathscr{B}$ is a Hilbert space for the topology of $\mathscr{X}$ and hence the subspace $(\mathscr{A}+\mathscr{B})/\mathscr{B}$ is closed for the topology of $\mathscr{X}/\mathscr{B}$.

Now let $(\mathscr{A}+\mathscr{B})/\mathscr{B}$ be closed in $\mathscr{X}/\mathscr{B}$. A Cauchy sequence $\{\mathbf{a}_n+\mathbf{b}_n\}$ with $\mathbf{a}_n \in \mathscr{A}$ and $\mathbf{b}_n \in \mathscr{B}$ will converge to an element $\mathbf{x} \in \mathscr{X}$ and we have to show that $\mathbf{x} \in \mathscr{A}+\mathscr{B}$. First we observe that

$$\|\mathbf{a}_n+\mathbf{b}_n-\mathbf{x}\|_{\mathscr{X}} \geq \inf_{\mathbf{b}\in\mathscr{B}} \|\mathbf{a}_n-\mathbf{x}+\mathbf{b}\|_{\mathscr{X}} = \|\mathbf{a}_n-\mathbf{x}\|_{\mathscr{X}/\mathscr{B}}\,.$$

Hence by the closedness of $(\mathscr{A}+\mathscr{B})/\mathscr{B}$ the sequence $\{\mathbf{a}_n+\mathscr{B}\} \subset (\mathscr{A}+\mathscr{B})/\mathscr{B}$ will converge to the element $\mathbf{x}+\mathscr{B} \in (\mathscr{A}+\mathscr{B})/\mathscr{B}$. It follows that $\mathbf{x} \in \mathscr{A}+\mathscr{B}$ which was to be proved. □

We can now state the result concerning the range of a product operator.

Proposition 2.3. Product operators: *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be Hilbert spaces and $\mathbf{F} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{Y}\}$ and $\mathbf{G} \in \mathrm{Lin}\{\mathscr{Y}, \mathscr{Z}\}$ be continuous linear operators and $\mathbf{F}' \in \mathrm{Lin}\{\mathscr{Y}', \mathscr{X}'\}$ and $\mathbf{G}' \in \mathrm{Lin}\{\mathscr{Z}', \mathscr{Y}'\}$ their duals. Let $\mathrm{Im}\,\mathbf{F}$ be closed in $\mathscr{Y}$. Then the following equivalence holds:*

$$\mathrm{Im}\,\mathbf{GF} \text{ closed in } \mathscr{Z} \iff \mathrm{Im}\,\mathbf{G}' + \mathrm{Ker}\,\mathbf{F}' \text{ closed in } \mathscr{Y}'\,,$$

that is, the image $\mathrm{Im}\,\mathbf{GF}$ of the product operator $\mathbf{GF} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{Z}\}$ is closed in $\mathscr{Z}$ if and only if the subspace $\mathrm{Im}\,\mathbf{G}' + \mathrm{Ker}\,\mathbf{F}'$ is closed in $\mathscr{Y}'$.

Proof: Let us consider the operator $\mathbf{G}_o \in \mathrm{Lin}\{\mathrm{Im}\,\mathbf{F}, \mathscr{Z}\}$ and its dual $\mathbf{G}_o' \in \mathrm{Lin}\{\mathscr{Z}', \mathscr{Y}'/\mathrm{Ker}\,\mathbf{F}'\}$ which are defined by

$$\mathbf{G}_o\mathbf{y} := \mathbf{G}\mathbf{y} \quad \forall \mathbf{y} \in \mathrm{Im}\,\mathbf{F}\,; \qquad \mathbf{G}_o'\mathbf{z}' := \mathbf{G}'\mathbf{z}' + \mathrm{Ker}\,\mathbf{F}' \quad \forall \mathbf{z}' \in \mathscr{Z}\,.$$

Proposition A.3 shows that $\mathrm{Im}\,\mathbf{G}_o = \mathrm{Im}\,\mathbf{GF}$ is closed if and only if $\mathrm{Im}\,\mathbf{G}_o' = (\mathrm{Im}\,\mathbf{G}' + \mathrm{Ker}\,\mathbf{F}')/\mathrm{Ker}\,\mathbf{F}'$ is closed in $\mathscr{Y}'/\mathrm{Ker}\,\mathbf{F}'$. By Proposition 2.2 this property is equivalent to the closedness of $\mathrm{Im}\,\mathbf{G}' + \mathrm{Ker}\,\mathbf{F}'$ in $\mathscr{Y}'$. □

3. Symmetric linear problems

In view of its application to the theory of linear elastic problems we discuss here an abstract symmetric linear problem in a Hilbert space.

Let $\mathbf{a}$ be a continuous symmetric bilinear form on the product space $\mathscr{X} \times \mathscr{X}$ and $\mathbf{A} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{X}'\}$ the associated symmetric continuous operator, so that

$$\mathbf{a}(\mathbf{x}, \mathbf{y}) = \mathbf{a}(\mathbf{y}, \mathbf{x}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle \quad \forall \mathbf{x}, \mathbf{y} \in \mathscr{X}\,.$$

Given a closed subspace $\mathscr{L}$ of $\mathscr{X}$ and a functional $\ell \in \mathscr{X}'$, we consider the linear problem

$$\mathbb{P}) \quad \mathbf{a}(\mathbf{x}, \mathbf{y}) = l(\mathbf{y}) \quad \mathbf{x} \in \mathscr{L} \quad \forall \mathbf{y} \in \mathscr{L}\,.$$

The duality between $\mathscr{X}$ and $\mathscr{X}'$ induces a duality between $\mathscr{L} \subseteq \mathscr{X}$ and the quotient space $\mathscr{X}'/\mathscr{L}^\perp$ by setting for any $\bar{\mathbf{x}} \in \mathscr{X}'/\mathscr{L}^\perp$

$$\langle \bar{\mathbf{x}}, \mathbf{y}\rangle := \langle \mathbf{x}, \mathbf{y}\rangle \quad \forall \mathbf{y} \in \mathscr{L} \quad \forall \mathbf{x} \in \bar{\mathbf{x}}\,.$$

It is then convenient to provide an alternative formulation of the problem in terms of a reduced operator $\mathbf{A}_o \in \mathrm{Lin}\{\mathscr{L}, \mathscr{X}'/\mathscr{L}^\perp\}$ and of a reduced functional $l_o \in \mathscr{X}'/\mathscr{L}^\perp$ defined by

$$\mathbf{A}_o\mathbf{x} := \mathbf{A}\mathbf{x} + \mathscr{L}^\perp \quad \forall \mathbf{x} \in \mathscr{L}; \quad l_o := l + \mathscr{L}^\perp\,.$$

We have

$$\mathbf{a}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{A}_o\mathbf{x}, \mathbf{y}\rangle \quad \forall \mathbf{x}, \mathbf{y} \in \mathscr{L}$$

and problem $\mathbb{P}$ can now be rewritten as

$$\mathbb{P}) \quad \mathbf{A}_o\mathbf{x} = l_o \quad \mathbf{x} \in \mathscr{L}\,.$$

Definition 3.1. Well-posedness: The symmetric problem $\mathbb{P}$ is said to be *well-posed* if it admits a solution for any data $l_o \in (\mathrm{Ker}\,\mathbf{A}_o)^\perp$.

Banach's closed range theorem, Proposition A.3, shows that the well-posedness of problem $\mathbb{P}$ is equivalent to the closedness of $\mathrm{Im}\,\mathbf{A}_o$ in $\mathscr{X}'/\mathscr{L}^\perp$. The basic properties of well-posed symmetric linear problems are reported hereafter.

Proposition 3.2. Existence and uniqueness properties: *The solution set of a well-posed symmetric problem* $\mathbb{P}$ *enjoys the following characteristic alternative:*

i) *If* $\operatorname{Ker}\mathbf{A}_o \neq \{\mathbf{o}\}$ *the solution set is a non-empty linear variety parallel to* $\operatorname{Ker}\mathbf{A}_o$ *for any admissible data* $\ell_o \in (\operatorname{Ker}\mathbf{A}_o)^{\perp}$.

ii) *If* $\operatorname{Ker}\mathbf{A}_o = \{\mathbf{o}\}$ *the solution is unique for every data* $l_o \in \mathscr{X}'/\mathscr{L}^{\perp}$.

We notice that the range and the kernel of the reduced operator are given by

$$\operatorname{Im}\mathbf{A}_o = (\mathbf{A}\mathscr{L} + \mathscr{L}^{\perp})/\mathscr{L}^{\perp}\,,$$

$$\operatorname{Ker}\mathbf{A}_o = (\mathbf{A}^{-1}\mathscr{L}^{\perp}) \cap \mathscr{L} = (\mathrm{A}\mathscr{L})^{\perp} \cap \mathscr{L}\,.$$

The closedness of $\operatorname{Im}\mathbf{A}_o$ can be expressed by stating that the bilinear form $\mathbf{a}$ is closed on $\mathscr{L} \times \mathscr{L}$ and is equivalently expressed by the conditions

$$\text{i)}\quad \|\mathbf{A}_o\mathbf{x}\|_{\mathscr{X}'/\mathscr{L}^{\perp}} \geq c_{\mathbf{a}}\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o} \qquad \forall \mathbf{x} \in \mathscr{L}\,,$$

$$\text{ii)}\quad \sup_{\mathbf{y}\in\mathscr{L}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{y}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o}} \geq c_{\mathbf{a}}\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o} \quad \forall \mathbf{x} \in \mathscr{L}\,,$$

$$\text{iii)}\quad \inf_{\mathbf{x}\in\mathscr{L}} \sup_{\mathbf{y}\in\mathscr{L}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o}\|\mathbf{y}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o}} \geq c_{\mathbf{a}} > 0\,,$$

$$\text{iv)}\quad \mathbf{A}\mathscr{L} + \mathscr{L}^{\perp} \quad \text{is closed in } \mathscr{X}'\,.$$

Property iv) is a direct consequence of proposition 2.2.

It is important to provide an expression of the kernel of the reduced operator in terms of the kernel of the symmetric bilinear form $\mathbf{a}$ defined by

$$\operatorname{Ker}\mathbf{a} = \operatorname{Ker}\mathbf{A} := \{\mathbf{x} \in \mathscr{X} \mid \mathbf{a}(\mathbf{x},\,\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \mathscr{X}\}\,.$$

Although in general we have only that

$$\operatorname{Ker}\mathbf{A}_o = (\mathbf{A}\mathscr{L})^{\perp} \cap \mathscr{L} \supseteq \operatorname{Ker}\mathbf{a} \cap \mathscr{L}\,,$$

the next result provides a sufficient condition to get an equality in the expression above.

Proposition 3.3. A formula for the kernel: *Let the symmetric bilinear form* $\mathbf{a}$ *be positive on the whole space* $\mathscr{X}$*:*

$$\mathbf{a}(\mathbf{x},\,\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathscr{X}\,.$$

Then we have

$$\operatorname{Ker}\mathbf{A}_o = (\mathbf{A}\mathscr{L})^{\perp} \cap \mathscr{L} = \operatorname{Ker}\mathbf{a} \cap \mathscr{L}\,.$$

Proof: We first observe that

$$\begin{aligned} \mathbf{x} \in (\mathbf{A}\mathscr{L})^{\perp} \cap \mathscr{L} \quad &\Longleftrightarrow \quad \mathbf{a}(\mathbf{x},\,\mathbf{y}) = \langle \mathbf{A}\mathbf{x},\,\mathbf{y}\rangle = \langle \mathbf{A}\mathbf{y},\,\mathbf{x}\rangle = 0 \quad \mathbf{x} \in \mathscr{L} \quad \forall \mathbf{y} \in \mathscr{L} \\ &\Rightarrow \mathbf{a}(\mathbf{x},\,\mathbf{x}) = 0 \quad \mathbf{x} \in \mathscr{L}\,. \end{aligned}$$

By the positivity of $\mathbf{a}$ the zero value is an absolute minimum of $\mathbf{a}$ in $\mathscr{X}$ so that any directional derivative will vanish at a minimum point. Hence we have

$$\mathbf{a}(\mathbf{x},\,\mathbf{x}) = 0 \quad \mathbf{x} \in \mathscr{L} \Rightarrow \mathbf{a}(\mathbf{x},\,\mathbf{y}) = 0 \quad \mathbf{x} \in \mathscr{L} \quad \forall \mathbf{y} \in \mathscr{X} \quad \Longleftrightarrow \quad \mathbf{x} \in \operatorname{Ker}\mathbf{a} \cap \mathscr{L}$$

and the proposition is proved. □

The next result provides a criterion for the closedness of $\mathbf{a}$ on $\mathscr{L} \times \mathscr{L}$.

Proposition 3.4. A sufficient closedness condition: *The inequality*

$$\mathbf{a}(\mathbf{x},\,\mathbf{x}) \geq c_{\mathbf{a}}\|\mathbf{x}\|^2_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o} \quad c_{\mathbf{a}} > 0 \quad \forall \mathbf{x} \in \mathscr{L}$$

implies the closedness of $\mathbf{a}$ *on* $\mathscr{L} \times \mathscr{L}$.

Proof: It suffices to observe that the inequality

$$\inf_{\mathbf{x}\in\mathscr{L}} \sup_{\mathbf{y}\in\mathscr{L}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o}\|\mathbf{y}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o}} \geq \inf_{\mathbf{x}\in\mathscr{L}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|^2_{\mathscr{X}/\operatorname{Ker}\mathbf{A}_o}} \geq c_{\mathbf{a}} > 0$$

provides the result. □

By propositions 3.3 and 3.4 we get the result which will be directly referred to in the discussion of elastic problems.

Proposition 3.5. Semi-ellipticity: *Let the symmetric bilinear form* $\mathbf{a}$ *be positive on the whole space* $\mathscr{X}$*. Then the property of semi-ellipticity of* $\mathbf{a}$ *on* $\mathscr{L}$

$$\mathbf{a}(\mathbf{x},\,\mathbf{x}) \geq c_{\mathbf{a}} \|\mathbf{x}\|^2_{\mathscr{X}/(\operatorname{Ker}\mathbf{a}\cap\mathscr{L})} \quad \forall \mathbf{x} \in \mathscr{L}$$

is sufficient to ensure the closedness of $\mathbf{a}$ *on* $\mathscr{L} \times \mathscr{L}$*.*

4. Linear structural problems

The formal framework for the analysis of linear structural models is provided by two pairs of dual Hilbert spaces:

- the kinematic space $\mathscr{V}$ and the force space $\mathscr{F}$,
- the strain space $\mathscr{D}$ and the stress space $\mathscr{S}$,

and a pair of dual operators:

- the kinematic operator $\mathbf{B} \in \operatorname{Lin}\{\mathscr{V},\,\mathscr{D}\}$,
- the equilibrium operator $\mathbf{B}' \in \operatorname{Lin}\{\mathscr{S},\,\mathscr{F}\}$.

Remark 4.1: In applications stresses and strains are defined to be square integrable fields. Accordingly we shall identify the stress space $\mathscr{S}$ and the strain space $\mathscr{D}$ with a pivot Hilbert space. The inner product in $\mathscr{D} = \mathscr{S}$ will be denoted by $((\cdot,\,\cdot))$ and the duality pairing between $\mathscr{V}$ and $\mathscr{F}$ by $\langle\cdot,\,\cdot\rangle$.

The kinematic and the equilibrium operators are the dual counterparts of a fundamental bilinear form $\mathbf{b}$ which describes the geometry of the model:

$$\mathbf{b}(\mathbf{v},\boldsymbol{\sigma}) := ((\boldsymbol{\sigma},\,\mathbf{B}\mathbf{v})) = \langle \mathbf{B}'\boldsymbol{\sigma},\,\mathbf{v}\rangle \quad \forall \boldsymbol{\sigma} \in \mathscr{S},\ \mathbf{v} \in \mathscr{V}\,.$$

As we shall see, the well-posedness of the structural model requires the closedness of the fundamental form $\mathbf{b}$ on $\mathscr{S} \times \mathscr{V}$ which is expressed by the inf-sup condition [13]

$$\inf_{\boldsymbol{\sigma}\in\mathscr{S}} \sup_{\mathbf{v}\in\mathscr{V}} \frac{\mathbf{b}(\mathbf{v},\,\boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/\operatorname{Ker}\mathbf{B}'}\,\|\mathbf{v}\|_{\mathscr{V}/\operatorname{Ker}\mathbf{B}}} = \inf_{\mathbf{v}\in\mathscr{V}} \sup_{\boldsymbol{\sigma}\in\mathscr{S}} \frac{\mathbf{b}(\mathbf{v},\,\boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/\operatorname{Ker}\mathbf{B}'}\,\|\mathbf{v}\|_{\mathscr{V}/\operatorname{Ker}\mathbf{B}}} > 0\,.$$

This means that the kinematic and the equilibrium operator have closed ranges and can be expressed by stating any one of the equivalent inequalities

$$\|\mathbf{B}\mathbf{v}\|_{\mathscr{D}} \geq c_{\mathbf{b}} \|\mathbf{v}\|_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \quad \forall \mathbf{v} \in \mathscr{V} \quad \Longleftrightarrow \quad \|\mathbf{B}'\boldsymbol{\sigma}\|_{\mathscr{F}} \geq c_{\mathbf{b}} \|\boldsymbol{\sigma}\|_{\mathscr{S}/Ker\,\mathbf{B}'} \quad \forall \boldsymbol{\sigma} \in \mathscr{S}\,,$$

where $c_{\mathbf{b}}$ is a positive constant.

4.1 Linear constraints

Rigid bilateral constraints acting on the structure are modeled by considering a closed subspace $\mathscr{L} \subseteq \mathscr{V}$ of conforming kinematisms.

The duality between $\mathscr{V}$ and $\mathscr{F}$ induces a duality pairing between the closed subspace $\mathscr{L}$ and the quotient space $\mathscr{F}/\mathscr{L}^\perp$ by setting

$$\langle \bar{\mathbf{f}},\,\mathbf{v}\rangle := \langle \mathbf{f},\,\mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathscr{L} \quad \forall \mathbf{f} \in \bar{\mathbf{f}} \in \mathscr{F}/\mathscr{L}^\perp\,.$$

It is convenient to introduce the following pair of reduced dual operators:

- the reduced kinematic operator $\mathbf{B}_{\mathscr{L}} \in \operatorname{Lin}\{\mathscr{L},\,\mathscr{D}\}$, defined as the restriction of $\mathbf{B}$ to $\mathscr{L}$,
- the reduced equilibrium operator $\mathbf{B}'_{\mathscr{L}} \in \operatorname{Lin}\{\mathscr{S},\,\mathscr{F}/\mathscr{L}^\perp\}$, defined by the position $\mathbf{B}'_{\mathscr{L}}\boldsymbol{\sigma} := \mathbf{B}'\boldsymbol{\sigma} + \mathscr{L}^\perp$.

The kernels and the images of the reduced operators are given by

$$\operatorname{Ker}\mathbf{B}_{\mathscr{L}} = \operatorname{Ker}\mathbf{B} \cap \mathscr{L}; \quad \operatorname{Ker}\mathbf{B}'_{\mathscr{L}} = (\mathbf{B}')^{-1}\,\mathscr{L}^\perp = (\mathbf{B}\mathscr{L})^\perp; \quad \operatorname{Im}\mathbf{B}_{\mathscr{L}} = \mathbf{B}\mathscr{L}; \quad \operatorname{Im}\mathbf{B}'_{\mathscr{L}} = (\operatorname{Im}\mathbf{B}' + \mathscr{L}^\perp)/\mathscr{L}^\perp\,,$$

and we denote by

- $\mathscr{L}_{\mathbf{R}} := \operatorname{Ker}\mathbf{B} \cap \mathscr{L}$ the subspace of conforming rigid kinematisms and by
- $\mathscr{S}_{\text{self}} := (\mathbf{B}\mathscr{L})^\perp$ the subspace of self-equilibrated stresses (self-stresses).

A variational theory of structural models with linear external constraints requires that the fundamental form $\mathbf{b}$ is closed on $\mathscr{S} \times \mathscr{L}$. As shown below this property is in fact necessary and sufficient to express in variational form the problems of equilibrium and of kinematic compatibility.

We recall that by Banach's closed range theorem, Proposition A.3, the closedness of $\mathbf{b}$ on $\mathscr{S} \times \mathscr{L}$ can be stated in the equivalent forms

- orthogonality conditions:

$$\operatorname{Im} \mathbf{B}_{\mathscr{L}} = (\operatorname{Ker} \mathbf{B}'_{\mathscr{L}})^{\perp} , \qquad \operatorname{Im} \mathbf{B}'_{\mathscr{L}} = (\operatorname{Ker} \mathbf{B}_{\mathscr{L}})^{\perp} ,$$

- inequality conditions:

$$\|\mathbf{Bu}\|_{\mathscr{D}} \geq c_{\mathbf{b}} \|\mathbf{u}\|_{\mathscr{V}/(\operatorname{Ker} \mathbf{B} \cap \mathscr{L})} \qquad \forall \mathbf{u} \in \mathscr{L} \quad c_{\mathbf{b}} > 0 ,$$

$$\|\mathbf{B}'_{\mathscr{L}} \boldsymbol{\sigma}\|_{\mathscr{F}/\mathscr{L}^{\perp}} \geq c_{\mathbf{b}} \|\boldsymbol{\sigma}\|_{\mathscr{S}/(\mathbf{B}\mathscr{L})^{\perp}} \quad \forall \boldsymbol{\sigma} \in \mathscr{S} \quad c_{\mathbf{b}} > 0 ,$$

- inf-sup conditions:

$$\inf_{\boldsymbol{\sigma} \in \mathscr{S}} \sup_{\mathbf{v} \in \mathscr{L}} \frac{\mathbf{b}(\mathbf{v}, \boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/\operatorname{Ker} \mathbf{B}'_{\mathscr{L}}} \|\mathbf{v}\|_{\mathscr{V}/\operatorname{Ker} \mathbf{B}}} = \inf_{\mathbf{v} \in \mathscr{L}} \sup_{\boldsymbol{\sigma} \in \mathscr{S}} \frac{\mathbf{b}(\mathbf{v}, \boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/\operatorname{Ker} \mathbf{B}'_{\mathscr{L}}} \|\mathbf{v}\|_{\mathscr{V}/\operatorname{Ker} \mathbf{B}}} > 0 .$$

The closedness of $\mathbf{b}$ on $\mathscr{S} \times \mathscr{L}$ can be also expressed by requiring the closedness of the sum of two subspaces, as shown hereafter.

Proposition 4.1. Equivalent closedness properties: *Let $\mathscr{L}$ be a closed subspace of $\mathscr{V}$. Then we have*

$$\mathbf{B}\mathscr{L} \text{ closed in } \mathscr{D} \iff \operatorname{Im} \mathbf{B}' + \mathscr{L}^{\perp} \text{ closed in } \mathscr{F} .$$

If in addition $\operatorname{Im} \mathbf{B}$ *is closed in* $\mathscr{D}$ *the closedness properties above are equivalent to the closedness of* $\operatorname{Ker} \mathbf{B} + \mathscr{L}$ *in* $\mathscr{V}$.

Proof: The first result follows directly from the expressions of $\operatorname{Im} \mathbf{B}_{\mathscr{L}}$ and $\operatorname{Im} \mathbf{B}'_{\mathscr{L}}$ by recalling propositions A.3 and 2.2. The last statement is a simple consequence of proposition A.8. □

We can then state the main results.

Proposition 4.2. Equilibrium: *Let $l \in \mathscr{F}$ be an external force and $l_o = l + \mathscr{L}^{\perp} \in \mathscr{F}/\mathscr{L}^{\perp}$ the corresponding load on a constrained structural model. The property that $\mathbf{B}\mathscr{L}$ is closed in $\mathscr{D}$ is necessary and sufficient to ensure that the equilibrium problem*

$$\mathbf{B}'_{\mathscr{L}} \boldsymbol{\sigma} = l_o \quad \boldsymbol{\sigma} \in \mathscr{S} \iff \mathbf{B}' \boldsymbol{\sigma} = l + \mathbf{r} \quad \boldsymbol{\sigma} \in \mathscr{S},\ \mathbf{r} \in \mathscr{L}^{\perp}$$

admits a solution for every load satisfying the consistency condition

$$l_o \in (\operatorname{Ker} \mathbf{B}_{\mathscr{L}})^{\perp} \iff l \in (\operatorname{Ker} \mathbf{B} \cap \mathscr{L})^{\perp}$$

or in variational form $\langle l, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathscr{L}_{\mathbf{R}} = \operatorname{Ker} \mathbf{B} \cap \mathscr{L}$. The degeneracy condition $\mathscr{S}_{\text{self}} = \{\mathbf{o}\}$ is necessary and sufficient for the solution to be unique.

Proposition 4.3. Compatibility: *A kinematic pair $\{\boldsymbol{\varepsilon}, \mathbf{w}\}$ with $\boldsymbol{\varepsilon} \in \mathscr{D}$ and $\mathbf{w} \in \mathscr{V}$ is said to be compatible with the constraints if there exists a conforming kinematic field $\mathbf{v} \in \mathscr{L}$ such that*

$$\mathbf{Bv} = \boldsymbol{\varepsilon} - \mathbf{Bw} .$$

The property that $\mathbf{B}\mathscr{L}$ is closed in $\mathscr{D}$ is necessary and sufficient to ensure that the compatibility problem admits solution for every kinematic pair satisfying the consistency condition

$$\boldsymbol{\varepsilon} - \mathbf{Bw} \in (\operatorname{Ker} \mathbf{B}'_{\mathscr{L}})^{\perp} = (\mathscr{S}_{\text{self}})^{\perp}$$

or in variational form

$$((\boldsymbol{\sigma}, \boldsymbol{\varepsilon})) = ((\boldsymbol{\sigma}, \mathbf{Bw})) \quad \forall \boldsymbol{\sigma} \in \mathscr{S}_{\text{self}} .$$

The degeneracy of the subspace $\mathscr{L}_{\mathbf{R}}$ of rigid conforming kinematisms is necessary and sufficient in order that the solution be unique.

4.2 Elastic structures

A linearly elastic structure is characterized by a symmetric elastic operator $\mathbf{E} \in \operatorname{Lin}\{\mathscr{D}, \mathscr{S}\}$ which is $\mathscr{D}$-elliptic:

$$((\mathbf{E}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})) \geq c_{\mathbf{e}} \|\boldsymbol{\varepsilon}\|^2_{\mathscr{D}} \quad c_{\mathbf{e}} > 0 \quad \forall \boldsymbol{\varepsilon} \in \mathscr{D} .$$

The elastic strain energy in terms of kinematisms is provided by one-half the quadratic form associated with the positive symmetric bilinear form

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := ((\mathbf{EBu}, \mathbf{Bv})) \quad \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}$$

which is called the bilinear form of elastic strain energy.

The elastostatic problem for a constrained structural model consists in evaluating a conforming kinematism $\mathbf{u} \in \mathscr{L}$ such that the corresponding stress field $\boldsymbol{\sigma} = \mathbf{EBu}$ is in equilibrium with the prescribed load $l_o = l + \mathscr{L}^{\perp} \in \mathscr{F}/\mathscr{L}^{\perp}$.

In terms of elastic strain energy the problem is written as

$$\mathbf{a}(\mathbf{u},\,\mathbf{v}) = l(\mathbf{v}) \quad \mathbf{u} \in \mathscr{L} \quad \forall \mathbf{v} \in \mathscr{L}$$

and is well-posed if and only if $\mathbf{a}$ is closed on $\mathscr{L} \times \mathscr{L}$.

The elastic stiffness of the structure $\mathbf{A} = \mathbf{B}'\mathbf{E}\mathbf{B} \in \mathrm{Lin}\{\mathscr{V},\,\mathscr{F}\}$ is the symmetric bounded linear operator associated with $\mathbf{a}$ according to the formula

$$\langle \mathbf{A}\mathbf{u},\,\mathbf{v}\rangle = \mathbf{a}(\mathbf{u},\,\mathbf{v}) \quad \forall \mathbf{v} \in \mathscr{V}\,.$$

A direct verification of the closure property of $\mathbf{a}$ on $\mathscr{L} \times \mathscr{L}$ is often not possible in applications and hence it is natural to look for simpler sufficient conditions.

A key result is provided by the following

Proposition 4.4. Closedness of the elastic operator: *The closedness of* $\mathbf{B}\mathscr{L}$ *and the* $\mathscr{D}$*-ellipticity of the elastic operator* $\mathbf{E}$ *are sufficient to ensure the closedness of the bilinear form* $\mathbf{a}$ *on* $\mathscr{L} \times \mathscr{L}$.

Proof: From the inequalities

$$\begin{aligned} &\langle \mathbf{E}\boldsymbol{\varepsilon},\,\boldsymbol{\varepsilon}\rangle \ge c_{\mathbf{e}} \|\boldsymbol{\varepsilon}\|_{\mathscr{D}}^2 && \forall \boldsymbol{\varepsilon} \in \mathscr{D}\,,\\ &\|\mathbf{B}\mathbf{u}\|_{\mathscr{D}} \ge c_{\mathbf{b}} \|\mathbf{u}\|_{\mathscr{V}/(\mathrm{Ker}\,\mathbf{B} \cap \mathscr{L})} && \forall \mathbf{u} \in \mathscr{L} \end{aligned}$$

it follows that

$$\mathbf{a}(\mathbf{u},\,\mathbf{u}) \ge c_{\mathbf{a}} \|\mathbf{u}\|_{\mathscr{V}/(\mathrm{Ker}\,\mathbf{B} \cap \mathscr{L})}^2 \quad \forall \mathbf{u} \in \mathscr{L}\,,$$

where $c_{\mathbf{a}} = c_{\mathbf{e}}\, c_{\mathbf{b}}^2$. The strict positivity of $\mathbf{E}$ ensures that $\mathrm{Ker}\,\mathbf{a} = \mathrm{Ker}\,\mathbf{B}$ so that the inequality above can be written as

$$\mathbf{a}(\mathbf{u},\,\mathbf{u}) \ge c_{\mathbf{a}} \|\mathbf{u}\|_{\mathscr{V}/(\mathrm{Ker}\,\mathbf{a} \cap \mathscr{L})}^2 \quad \forall \mathbf{u} \in \mathscr{L}\,,$$

which by proposition 3.5 implies the closedness of $\mathbf{a}$ on $\mathscr{L} \times \mathscr{L}$. □

In applications the $\mathscr{D}$-ellipticity of the elastic operator $\mathbf{E}$ is easily checked so that the real task is to verify the closedness of $\mathbf{B}\mathscr{L}$.

Proposition 4.5. A closedness criterion: *Let* $\mathrm{Im}\,\mathbf{B}$ *be closed in* $\mathscr{D}$. *Then the subspace* $\mathbf{B}\mathscr{L}$ *is closed in* $\mathscr{D}$ *if the subspace* $\mathrm{Ker}\,\mathbf{B}$ *can be written as the sum of a finite dimensional subspace and of a subspace included in* $\mathscr{L}$:

$$\mathrm{Ker}\,\mathbf{B} = \mathscr{N} + \mathscr{L}_o \quad \dim \mathscr{N} < +\infty \quad \mathscr{L}_o \subseteq \mathscr{L}\,.$$

Proof: By proposition 4.1 we have to verify the closedness of the subspace $\mathrm{Ker}\,\mathbf{B} + \mathscr{L}$ in $\mathscr{V}$. The assumption ensures that $\mathrm{Ker}\,\mathbf{B} + \mathscr{L} = \mathscr{N} + \mathscr{L}$ with $\dim\,\mathscr{N} < +\infty$ and hence setting $\mathscr{A} = \mathscr{L}$ and $\mathscr{B} = \mathscr{N}$ in proposition A.7 we get the result. □

Remark 4.2: In most engineering applications the kernel of the kinematic operator $\mathbf{B}$ is finite dimensional so that the condition in proposition 4.5 is trivially fulfilled. A relevant exception is provided by the models of cable or membrane structures in which the subspace $\mathrm{Ker}\,\mathbf{B}$ of rigid kinematic fields is not finite dimensional. The condition in proposition 4.5 is however still met [13].

5. Mixed formulations

A more challenging problem concerns the elastic equilibrium of a structural model with a partially rigid constitutive behaviour and subject to external elastic constraints.

Rigid bilateral constraints, which have already been analysed, will not be explicitly considered to simplify the presentation. They can however be taken into account by substituting the kinematic operator $\mathbf{B} \in \mathrm{Lin}\{\mathscr{V},\,\mathscr{D}\}$ with the reduced operator $\mathbf{B}_{\mathscr{L}} \in \mathrm{Lin}\{\mathscr{L},\,\mathscr{D}\}$.

The analytical properties of the general model of elastic structure under investigation are described hereafter.

- The internal elastic compliance of the structure is expressed by a *continuous symmetric* bilinear form $\mathbf{c}(\boldsymbol{\sigma},\,\boldsymbol{\tau})$ which is *positive* and *closed* on $\mathscr{S} \times \mathscr{S}$, i.e.

$$\begin{aligned} &\text{i)} && \|\mathbf{c}\|_{\mathscr{S}} \|\boldsymbol{\sigma}\|_{\mathscr{S}} \|\boldsymbol{\tau}\|_{\mathscr{S}} \ge |\mathbf{c}(\boldsymbol{\sigma},\,\boldsymbol{\tau})| && \forall \boldsymbol{\sigma},\,\boldsymbol{\tau} \in \mathscr{S}\,,\\ &\text{ii)} && \mathbf{c}(\boldsymbol{\sigma},\,\boldsymbol{\tau}) = \mathbf{c}(\boldsymbol{\tau},\,\boldsymbol{\sigma}) && \forall \boldsymbol{\sigma},\,\boldsymbol{\tau} \in \mathscr{S}\,,\\ &\text{iii)} && \mathbf{c}(\boldsymbol{\sigma},\,\boldsymbol{\sigma}) \ge \mathbf{0} && \forall \boldsymbol{\sigma} \in \mathscr{S}\,,\\ &\text{iv)} && \inf_{\boldsymbol{\tau} \in \mathscr{S}} \sup_{\boldsymbol{\sigma} \in \mathscr{S}} \frac{\mathbf{c}(\boldsymbol{\sigma},\,\boldsymbol{\tau})}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/\mathrm{Ker}\,\mathbf{C}}\, \|\boldsymbol{\tau}\|_{\mathscr{S}/\mathrm{Ker}\,\mathbf{C}}} > 0\,. && \end{aligned}$$

The elastic compliance operator $\mathbf{C} \in \mathrm{Lin}\{\mathscr{S}, \mathscr{D}\}$ is defined by

$$((\mathbf{C}\boldsymbol{\sigma}, \boldsymbol{\tau})) := \mathbf{c}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathscr{S},$$

and $\mathrm{Im}\,\mathbf{C}$ is closed in $\mathscr{D}$ by virtue of iv). The elements of the kernel of $\mathbf{C}$ are the elastically ineffective stress fields.

- The external elastic stiffness of the structure is expressed by a *continuous symmetric* bilinear form $\mathbf{k}(\mathbf{u}, \mathbf{v})$ which is *positive* on $\mathscr{V} \times \mathscr{V}$, i.e.

$$\begin{array}{rll} \text{i)} & \|\mathbf{k}\|_{\mathscr{V}} \|\mathbf{u}|_{\mathscr{V}} \|\mathbf{v}\|_{\mathscr{V}} \geq |\mathbf{k}(\mathbf{u}, \mathbf{v})| & \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}, \\ \text{ii)} & \mathbf{k}(\mathbf{u}, \mathbf{v}) = \mathbf{k}(\mathbf{v}, \mathbf{u}) & \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}, \\ \text{iii)} & \mathbf{k}(\mathbf{u}, \mathbf{u}) \geq 0 & \forall \mathbf{u} \in \mathscr{V}. \end{array}$$

The external elastic stiffness operator $\mathbf{K} \in \mathrm{Lin}\{\mathscr{V}, \mathscr{F}\}$ is defined by

$$\langle \mathbf{K}\mathbf{u}, \mathbf{v} \rangle := \mathbf{k}(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}.$$

The elements of the kernel of $\mathbf{K}$ are kinematic fields which do not involve reactions of the external elastic constraints.

We emphasize that the form $\mathbf{k}$ is not assumed to be closed on $\mathscr{V} \times \mathscr{V}$. As we shall see this is important in applications and makes the static and the kinematic equations of the mixed formulation play different roles.

The mixed elastostatic problem is formulated in operator form as

$$\mathbb{M}) \quad \begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}'\boldsymbol{\sigma} = \mathbf{f} \\ \mathbf{B}\mathbf{u} - \mathbf{C}\boldsymbol{\sigma} = \boldsymbol{\delta} \end{cases} \quad \text{or} \quad \mathbf{S}\begin{vmatrix} \mathbf{u} \\ \boldsymbol{\sigma} \end{vmatrix} = \begin{bmatrix} \mathbf{K} & \mathbf{B}' \\ \mathbf{B} & -\mathbf{C} \end{bmatrix} \begin{vmatrix} \mathbf{u} \\ \boldsymbol{\sigma} \end{vmatrix} = \begin{vmatrix} \mathbf{f} \\ \boldsymbol{\delta} \end{vmatrix},$$

where $\mathbf{S} \in \mathrm{Lin}\{\mathscr{V} \times \mathscr{S}, \mathscr{F} \times \mathscr{D}\}$ is called the structural operator.

Equation $\mathbb{M}_1$ expresses the equilibrium condition in which

- $\mathbf{f} \in \mathscr{F}$ is the assigned load,
- $-\mathbf{K}\mathbf{u} \in \mathscr{F}$ is the reaction of the external elastic constraints,
- $\mathbf{B}'\boldsymbol{\sigma} \in \mathscr{F}$ is the total external force.

Equation $\mathbb{M}_2$ expresses the kinematic compatibility condition in which

- $\boldsymbol{\delta} \in \mathscr{D}$ is an imposed distorsion,
- $\mathbf{C}\boldsymbol{\sigma} \in \mathscr{D}$ is the elastic strain,
- $\mathbf{B}\mathbf{u} \in \mathscr{D}$ is the total strain field.

Imposed distorsions are often considered in engineering applications e.g. to simulate the effect of temperature fields in the structures.

The variational form of the mixed elastostatic problem is given by

$$\mathbb{M}) \quad \begin{cases} \mathbf{k}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, \boldsymbol{\sigma}) = \langle \mathbf{f}, \mathbf{v} \rangle & \mathbf{u} \in \mathscr{V} \quad \forall \mathbf{v} \in \mathscr{V}, \\ \mathbf{b}(\mathbf{u}, \boldsymbol{\tau}) - \mathbf{c}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \langle \boldsymbol{\delta}, \boldsymbol{\tau} \rangle & \boldsymbol{\sigma} \in \mathscr{S} \quad \forall \boldsymbol{\tau} \in \mathscr{S}. \end{cases}$$

Problems of this kind have been longly analysed in the literature (see e.g. the references in [14, 15, 16]) following the pioneering works by I. Babuška [1] and F. Brezzi [2]. A comprehensive presentation of the state of the art can be found in the book [3] by F. Brezzi and M. Fortin on Mixed and Hybrid F.E.M. formulations.

The approach proposed here is directly related to the original existence and uniqueness theorem by Brezzi [2]. His analysis was concerned with a mixed problem $\mathbb{M}$ in which the form $\mathbf{c}$ was taken to be zero and neither the symmetry nor the positivity of the form $\mathbf{k}$ were assumed.

A more general case in which a positive and symmetric form $\mathbf{c}$ is included has been recently addressed in [3], theorem II.1.2, by adopting a perturbation technique. A sufficient condition for the existence of a solution of the mixed problem is provided in [3] under a special assumption concerning the bilinear form $\mathbf{c}$ of elastic compliance.

However many engineering models of elastic structures fall outside the range of the existing results.

The analysis which we develop here is intended to provide a well-posedness result capable to encompass the usual engineering models in elasticity.

We preliminarily quote a result concerning the kernel of the structural operator $\mathbf{S}$.

Proposition 5.1. Representation of the kernel: *Let the forms* $\mathbf{c}$ *and* $\mathbf{k}$ *be symmetric and positive. The kernel of the structural operator* $\mathbf{S}$ *is then given by*

$$\mathrm{Ker}\,\mathbf{S} = \begin{vmatrix} \mathrm{Ker}\,\mathbf{B} \cap \mathrm{Ker}\,\mathbf{K} \\ \mathrm{Ker}\,\mathbf{B}' \cap \mathrm{Ker}\,\mathbf{C} \end{vmatrix}.$$

Proof: A pair $\{\mathbf{u},\, \boldsymbol{\sigma}\}$ belongs to $\operatorname{Ker}\mathbf{S}$ if and only if

$$\begin{cases} \mathbf{k}(\mathbf{u},\, \mathbf{v}) + \mathbf{b}(\mathbf{v},\, \boldsymbol{\sigma}) = 0 & \forall \mathbf{v} \in \mathscr{V}\,, \\ \mathbf{b}(\mathbf{u}, \boldsymbol{\tau}) - \mathbf{c}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 0 & \forall \boldsymbol{\tau} \in \mathscr{S} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}'\boldsymbol{\sigma} = 0\,, \\ \mathbf{B}\mathbf{u} - \mathbf{C}\boldsymbol{\sigma} = 0 \end{cases}$$

which imply that

$$\begin{cases} \mathbf{k}(\mathbf{u},\, \mathbf{u}) + \mathbf{b}(\mathbf{u}, \boldsymbol{\sigma}) = 0\,, \\ \mathbf{b}(\mathbf{u}, \boldsymbol{\sigma}) - \mathbf{c}(\boldsymbol{\sigma},\, \boldsymbol{\sigma}) = 0\,. \end{cases}$$

Subtracting we get $\mathbf{k}(\mathbf{u},\, \mathbf{u}) + \mathbf{c}(\boldsymbol{\sigma},\, \boldsymbol{\sigma}) = 0$ and the positivity of $\mathbf{k}$ and $\mathbf{c}$ implies that $\mathbf{k}(\mathbf{u},\, \mathbf{u}) = 0$ and $\mathbf{c}(\boldsymbol{\sigma},\, \boldsymbol{\sigma}) = 0$. Hence, being $\mathbf{u}$ and $\boldsymbol{\sigma}$ absolute minimum points of $\mathbf{k}$ and $\mathbf{c}$, their derivatives must vanish. By the symmetry of $\mathbf{k}$ and $\mathbf{c}$ these conditions are expressed by $\mathbf{K}\mathbf{u} = \mathbf{o}$ and $\mathbf{C}\boldsymbol{\sigma} = \mathbf{o}$. Substituting in the expression of the kernel we infer that $\mathbf{B}\mathbf{u} = \mathbf{o}$ and $\mathbf{B}'\boldsymbol{\sigma} = \mathbf{o}$. □

If a solution $\{\mathbf{u},\, \boldsymbol{\sigma}\} \in \mathscr{V} \times \mathscr{S}$ to problem $\mathbb{M}$ exists, the data $\{\mathbf{f},\, \boldsymbol{\delta}\} \in \mathscr{F} \times \mathscr{D}$ must necessarily meet the following variational conditions of admissibility:

$$\mathbf{f} \in (\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K})^{\perp}\,, \qquad \boldsymbol{\delta} \in (\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C})^{\perp}$$

which express the orthogonality of $\{\mathbf{f},\, \boldsymbol{\delta}\}$ to the kernel of the structural operator.

The engineers' confidence in finding solutions to elasticity problems is based upon the implicit assumption of well-posedness of the problem, a condition explicitly restated hereafter by recalling definition 3.1.

Definition 5.2. Well-posedness of the mixed problem: The mixed problem $\mathbb{M}$ is *well-posed* if the structural operator $\mathbf{S}$ has a closed range. The variational conditions of admissibility on the data $\{\mathbf{f},\, \boldsymbol{\delta}\} \in (\operatorname{Ker}\mathbf{S})^{\perp}$ are then also sufficient to ensure the existence of a solution, unique to within fields of the kernel $\operatorname{Ker}\mathbf{S}$ of the structural operator.

Remark 5.1: The well-posedness of the mixed problem $\mathbb{M}$ requires the validity of the orthogonality relations

$$\operatorname{Im}\mathbf{B}' + \operatorname{Im}\mathbf{K} = (\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K})^{\perp}\,, \qquad \operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{C} = (\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C})^{\perp}\,.$$

By remark A.2 the equalities above hold if and only if the sum of the two subspaces on the left hand sides is closed.

5.1 Solution strategy

Our aim is to provide a necessary and sufficient condition for the well-posedness of the mixed problem $\mathbb{M}$.

Planning the attack, we first try to transform the mixed problem $\mathbb{M}$ into a problem involving only kinematic fields.

To this end we must modify condition $\mathbb{M}_2$ of kinematic compatibility by inverting the elastic law to get an expression of the stress field $\boldsymbol{\sigma} \in \mathscr{S}$ in terms of the strain associated with the kinematic field $\mathbf{u} \in \mathscr{V}$. Since the internal elastic compliance operator $\mathbf{C} \in \operatorname{Lin}\{\mathscr{S},\, \mathscr{D}\}$ is singular, we have to pick up its non-singular part.

Due to the symmetry of $\mathbf{C}$ and the closedness of $\operatorname{Im}\mathbf{C}$, the subspace $\operatorname{Ker}\mathbf{C}$ of elastically ineffective stresses and the subspace $\operatorname{Im}\mathbf{C}$ of elastic strains fulfil the orthogonality conditions

$$\operatorname{Ker}\mathbf{C} = (\operatorname{Im}\mathbf{C})^{\perp} \quad \text{and} \quad \operatorname{Im}\mathbf{C} = (\operatorname{Ker}\mathbf{C})^{\perp}\,.$$

Recalling Remark 4.1 the spaces $\mathscr{D}$ and $\mathscr{S}$ can be identified without loss in generality. We can then perform the direct sum decomposition of the stress-strain space into complementary orthogonal subspaces:

$$\mathscr{D} = \mathscr{S} = \operatorname{Im}\mathbf{C} \oplus \operatorname{Ker}\mathbf{C}\,.$$

The reduced compliance operator $\mathbf{C}_o \in \operatorname{Lin}\{\operatorname{Im}\mathbf{C},\, \operatorname{Im}\mathbf{C}\}$, defined by

$$\mathbf{C}_o\boldsymbol{\sigma} = \mathbf{C}\,\boldsymbol{\sigma} \quad \forall \boldsymbol{\sigma} \in \operatorname{Im}\mathbf{C} \subseteq \mathscr{S}\,,$$

is positive definite and the operator $\mathbf{C}$ can be partitioned as follows:

$$\mathbf{C}\,\boldsymbol{\sigma} = \begin{bmatrix} \mathbf{C}_o & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{vmatrix} \boldsymbol{\sigma}^* \\ \boldsymbol{\sigma}_o \end{vmatrix} \quad \text{with} \quad \begin{cases} \boldsymbol{\sigma}^* \in \operatorname{Im}\mathbf{C} \\ \boldsymbol{\sigma}_o \in \operatorname{Ker}\mathbf{C} \end{cases}.$$

We also define in $\mathscr{S} = \mathscr{D}$ the symmetric orthogonal projector $\mathbf{P} = \mathbf{P}'$ onto the subspace $\operatorname{Ker}\mathbf{C}$ of elastically ineffective stresses so that

$$\operatorname{Im}\mathbf{P} = \operatorname{Ker}\mathbf{C}\,, \qquad \operatorname{Ker}\mathbf{P} = \operatorname{Im}\mathbf{C}\,.$$

The kernel of the product operator $\mathbf{P}\mathbf{B} \in \operatorname{Lin}\{\mathscr{V},\, \operatorname{Ker}\mathbf{C}\}$ is defined by

$$\operatorname{Ker}\mathbf{P}\mathbf{B} = \{\mathbf{u} \in \mathscr{V} \mid \mathbf{B}\mathbf{u} \in \operatorname{Im}\mathbf{C}\}$$

and its elements are the kinematic fields which generate elastic strain fields.

Remark 5.2: According to Remark 5.1 the closedness of $\operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{C} = \operatorname{Im}\mathbf{B} + \operatorname{Ker}\mathbf{P}$ is a necessary condition for the well-posedness of the mixed problem. Further, by proposition 2.3, this assumption is also equivalent to the closedness of $\operatorname{Im}\mathbf{B}'\mathbf{P}'$ in $\mathscr{F}$ and hence, by the closed range theorem, proposition A.3, to the closedness of $\operatorname{Im}\mathbf{PB}$.

Let us then assume that $\operatorname{Im}\mathbf{PB}$ is closed in $\mathscr{D}$ so that for any $\boldsymbol{\delta} \in (\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C})^{\perp}$ we can perform the decomposition

$$\boldsymbol{\delta} = \boldsymbol{\delta}_o + \boldsymbol{\delta}^* \quad \text{with} \quad \boldsymbol{\delta}_o \in \operatorname{Im}\mathbf{B} \quad \text{and} \quad \boldsymbol{\delta}^* \in \operatorname{Im}\mathbf{C}\,.$$

Choosing $\mathbf{u}_o \in \mathscr{V}$ such that $\mathbf{B}\mathbf{u}_o = \boldsymbol{\delta}_o$ the compatibility equation $\mathbb{M}_2$ can be rewritten as

$$\mathbf{B}\mathbf{u}^* = \mathbf{C}_o\boldsymbol{\sigma}^* + \boldsymbol{\delta}^*\,.$$

Denoting by $\mathbf{E}$ the inverse of $\mathbf{C}_o$ we can also write

$$\boldsymbol{\sigma}^* = \mathbf{E}(\mathbf{B}\mathbf{u}^* - \boldsymbol{\delta}^*)\,.$$

Substituting into the equilibrium equation $\mathbb{M}_1$ we get the following problem in the unknown fields $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ and $\boldsymbol{\sigma}_o \in \operatorname{Ker}\mathbf{C}$:

$$\mathbb{P})\quad (\mathbf{K} + \mathbf{B}'\mathbf{E}\mathbf{B})\,\mathbf{u}^* + \mathbf{B}'\boldsymbol{\sigma}_o = \mathbf{f} - \mathbf{K}\mathbf{u}_o + \mathbf{B}'\mathbf{E}\boldsymbol{\delta}^*\,.$$

Let us now define the bilinear form of the elastic energy:

$$\mathbf{a}(\mathbf{u}^*, \mathbf{v}) := (\mathbf{k}\mathbf{u}^*, \mathbf{v}) + ((\mathbf{E}\mathbf{B}\mathbf{u}^*, \mathbf{B}\mathbf{v})) \quad \forall \mathbf{u}^* \in \operatorname{Ker}\mathbf{PB} \quad \forall \mathbf{v} \in \mathscr{V}$$

and the effective load:

$$\langle l, \mathbf{v}\rangle := \langle \mathbf{f}, \mathbf{v}\rangle - \mathbf{k}(\mathbf{u}_o, \mathbf{v}) + ((\mathbf{E}\boldsymbol{\delta}^*, \mathbf{B}\mathbf{v})) \quad \forall \mathbf{v} \in \mathscr{V}\,.$$

The stiffness operator $\mathbf{A} = \mathbf{K} + \mathbf{B}'\mathbf{E}\mathbf{B}$ is defined by the identity

$$\langle \mathbf{A}\mathbf{u}^*, \mathbf{v}\rangle = \mathbf{a}(\mathbf{u}^*, \mathbf{v}) \quad \forall \mathbf{u}^* \in \operatorname{Ker}\mathbf{PB} \quad \forall \mathbf{v} \in \mathscr{V}\,.$$

The discussion above is summarized in the next statement.

Proposition 5.3. First equivalence property: *The closedness of* $\operatorname{Im}\mathbf{PB}$ *ensures that for any given* $\boldsymbol{\delta} \in (\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C})^{\perp}$ *the mixed problem*

$$\mathbb{M})\quad \begin{cases} \mathbf{k}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, \boldsymbol{\sigma}) = \langle \mathbf{f}, \mathbf{v}\rangle & \mathbf{u} \in \mathscr{V} \quad \forall \mathbf{v} \in \mathscr{V}\,, \\ \mathbf{b}(\mathbf{u}, \boldsymbol{\tau}) - \mathbf{c}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \langle \boldsymbol{\delta}, \boldsymbol{\tau}\rangle & \boldsymbol{\sigma} \in \mathscr{S} \quad \forall \boldsymbol{\tau} \in \mathscr{S} \end{cases}$$

in the unknown fields $\mathbf{u} \in \mathscr{V}$ *and* $\boldsymbol{\sigma} \in \mathscr{S}$ *is equivalent to the variational problem*

$$\mathbb{P})\quad \mathbf{a}(\mathbf{u}^*, \mathbf{v}) + ((\boldsymbol{\sigma}_o, \mathbf{B}\mathbf{v})) = \langle \ell, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathscr{V}$$

in the unknown fields $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ *and* $\boldsymbol{\sigma}_o \in \operatorname{Ker}\mathbf{C}$ *provided that the pair* $\{\mathbf{u}_o, \boldsymbol{\delta}^*\} \in \mathscr{V} \times \operatorname{Im}\mathbf{C}$ *is such that* $\boldsymbol{\delta} = \mathbf{B}\mathbf{u}_o + \boldsymbol{\delta}^*$*.*

The discussion of problem $\mathbb{P}$ is based on its equivalence to a classical one-field problem which is formulated by restricting the test fields $\mathbf{v} \in \mathscr{V}$ to range in the subspace $\operatorname{Ker}\mathbf{PB} \subseteq \mathscr{V}$.

Proposition 5.4. Second equivalence property: *The closedness of* $\operatorname{Im}\mathbf{PB}$ *ensures that the variational problem*

$$\mathbb{P})\quad \mathbf{a}(\mathbf{u}^*, \mathbf{v}) + ((\boldsymbol{\sigma}_o, \mathbf{B}\mathbf{v})) = \langle l, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathscr{V}$$

in the unknown fields $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ *and* $\boldsymbol{\sigma}_o \in \operatorname{Ker}\mathbf{C}$ *is equivalent to the reduced problem*

$$\mathbb{P}^*)\quad \mathbf{a}(\mathbf{u}^*, \mathbf{v}^*) = \langle \ell, \mathbf{v}^*\rangle \quad \forall \mathbf{v}^* \in \operatorname{Ker}\mathbf{PB}$$

in the unknown field $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$*.*

Proof: Clearly if $\{\mathbf{u}^*, \boldsymbol{\sigma}_o\} \in \operatorname{Ker}\mathbf{PB} \times \operatorname{Ker}\mathbf{C}$ is a solution of problem $\mathbb{P}$ then $\mathbf{u}^*$ will be solution of problem $\mathbb{P}^*$. In fact we have that $((\boldsymbol{\sigma}_o, \mathbf{B}\mathbf{v}^*)) = 0$ for all $\mathbf{v}^* \in \operatorname{Ker}\mathbf{PB}$ since $\boldsymbol{\sigma}_o \in \operatorname{Ker}\mathbf{C}$ and $\mathbf{B}\mathbf{v}^* \in \operatorname{Ker}\mathbf{P} = \operatorname{Im}\mathbf{C} = (\operatorname{Ker}\mathbf{C})^{\perp}$.

Conversely if $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ is solution of problem $\mathbb{P}^*$ the reactive force $\mathbf{r} \in \mathscr{F}$ defined by

$$\langle \mathbf{r}, \mathbf{v}\rangle := \mathbf{a}(\mathbf{u}^*, \mathbf{v}) - \langle l, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathscr{V}$$

will belong to $(\operatorname{Ker}\mathbf{PB})^{\perp}$. The assumption $\operatorname{Im}\mathbf{PB}$ closed ensures that $\operatorname{Im}\mathbf{B}'\mathbf{P}' = (\operatorname{Ker}\mathbf{PB})^{\perp}$ and hence for any $\mathbf{r} \in (\operatorname{Ker}\mathbf{PB})^{\perp}$ we can find a $\boldsymbol{\sigma}_o \in \operatorname{Im}\mathbf{P}' = \operatorname{Ker}\mathbf{C}$ such that $\mathbf{B}'\boldsymbol{\sigma}_o = \mathbf{r}$. Then $\langle \mathbf{r}, \mathbf{v}\rangle = ((\boldsymbol{\sigma}_o, \mathbf{B}\mathbf{v}))$ for all $\mathbf{v} \in \mathscr{V}$ and the pair $\{\mathbf{u}^*, \boldsymbol{\sigma}_o\}$ is solution of problem $\mathbb{P}$. The field $\boldsymbol{\sigma}_o$ is unique to within elements of the subspace $\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C}$ of elastically ineffective self-stresses. □

Remark 5.3: It is worth noting that the expression of the effective load l depends upon $\boldsymbol{\delta}^*$ and the field $\mathbf{u}_o$ which in turn is determined by $\boldsymbol{\delta}_o$ only to within an additional rigid field.

Further the additive decomposition of admissible distorsions $\boldsymbol{\delta}$ into the sum $\boldsymbol{\delta}_o + \boldsymbol{\delta}^*$ is unique only to within elements of $\operatorname{Im}\mathbf{B} \cap \operatorname{Im}\mathbf{C}$.

Anyway it can be easily shown that the solution $\mathbf{u} = \mathbf{u}_o + \mathbf{u}^*$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_o + \boldsymbol{\sigma}^*$ of the mixed problem $\mathbb{M}$ remains unaffected by this indeterminacy of l.

Let us now discuss the well-posedness of the reduced problem $\mathbb{P}^*$.

5.2 The reduced structural model

Problem $\mathbb{P}^*$ is the variational formulation of the elastostatic problem for a structural model subject to the rigid bilateral constraints defined by the subspace $\operatorname{Ker}\mathbf{PB} \subseteq \mathscr{V}$ of conforming kinematic fields. It is formally equivalent to the symmetric linear problems discussed in section 3.

Preliminarily we remark that by proposition A.2 the continuity of the elastic stiffness $\mathbf{E} = \mathbf{C}_o^{-1}$ is ensured by the continuity of $\mathbf{C}$ and the closedness of $\operatorname{Im}\mathbf{C}$. The continuity of $\mathbf{E} \in \operatorname{Lin}\{\operatorname{Ker}\mathbf{P}, \operatorname{Ker}\mathbf{P}\}$ implies the continuity of $\mathbf{A} = \mathbf{K} + \mathbf{B}'\mathbf{E}\mathbf{B}$ so that $\mathbf{A} \in \operatorname{Lin}\{\operatorname{Ker}\mathbf{PB}, \mathscr{F}\}$.

The bilinear form $\mathbf{a}$ is then continuous on $\operatorname{Ker}\mathbf{PB} \times \mathscr{V}$ and hence *a fortiori* on $\operatorname{Ker}\mathbf{PB} \times \operatorname{Ker}\mathbf{PB}$.

We then consider the canonical surjection $\boldsymbol{\Pi} \in \operatorname{Lin}\{\mathscr{F}, \mathscr{F}/(\operatorname{Ker}\mathbf{PB})^\perp\}$ and define

- the reduced elastic stiffness $\mathbf{A}_o := \boldsymbol{\Pi}\mathbf{A} \in \operatorname{Lin}\{\operatorname{Ker}\mathbf{PB}, \mathscr{F}/(\operatorname{Ker}\mathbf{PB})^\perp\}$,
- the reduced effective load $l_o := \boldsymbol{\Pi} l \in \mathscr{F}/(\operatorname{Ker}\mathbf{PB})^\perp$,

or explicitly

$$\mathbf{A}_o\mathbf{u}^* := \mathbf{A}\mathbf{u}^* + (\operatorname{Ker}\mathbf{PB})^\perp \quad \forall \mathbf{u}^* \in \operatorname{Ker}\mathbf{PB} \quad \text{and} \quad \ell_o := \ell + (\operatorname{Ker}\mathbf{PB})^\perp .$$

The following result is a direct consequence of the discussion carried out in section 3.

Proposition 5.5. Well-posedness of the reduced problem: *The symmetric linear problem*

$$\mathbb{P}^*) \quad \mathbf{A}_o\mathbf{u}^* = \ell_o , \qquad \mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$$

is well-posed if and only if $\operatorname{Im}\mathbf{A}_o$ *is closed in* $\mathscr{F}/(\operatorname{Ker}\mathbf{PB})^\perp$. *This closure property is equivalent to the closedness of the symmetric form* $\mathbf{a}$ *on* $\operatorname{Ker}\mathbf{PB} \times \operatorname{Ker}\mathbf{PB}$ *and is expressed by the inf-sup condition*

$$\inf_{\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}} \ \sup_{\mathbf{v}^* \in \operatorname{Ker}\mathbf{PB}} \frac{\mathbf{a}(\mathbf{u}^*, \mathbf{v}^*)}{\|\mathbf{u}^*\|_{\mathscr{V}/\operatorname{Ker}\mathbf{A}_o} \|\mathbf{v}^*\|_{\mathscr{V}/\operatorname{Ker}\mathbf{A}_o}} > 0 .$$

The existence of a solution is thus guaranteed if and only if $l_o \in (\operatorname{Ker}\mathbf{A}_o)^\perp$ *and the solution is unique to within elements of* $\operatorname{Ker}\mathbf{A}_o$.

The positivity of the elastic compliance $\mathbf{C}$ in $\mathscr{S}$ implies that the elastic stiffness $\mathbf{E} = \mathbf{C}_o^{-1}$ is positive definite on $\operatorname{Im}\mathbf{C}$. On this basis the next result provides an important formula for $\operatorname{Ker}\mathbf{A}_o$.

Proposition 5.6. Kernel of the reduced stiffness: *Let the forms* $\mathbf{c}$ *and* $\mathbf{k}$ *be symmetric and positive. The kernel of the reduced stiffness operator* $\mathbf{A}_o$ *is then given by*

$$\operatorname{Ker}\mathbf{A}_o = \operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K} .$$

Proof: By definition the elements of $\operatorname{Ker}\mathbf{A}_o$ are the kinematic fields $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ which meet the variational condition

$$\mathbf{k}(\mathbf{u}^*, \mathbf{v}^*) + ((\mathbf{E}\mathbf{B}\mathbf{u}^*, \mathbf{B}\mathbf{v}^*)) = 0 \quad \forall \mathbf{v}^* \in \operatorname{Ker}\mathbf{PB} .$$

Setting $\mathbf{v}^* = \mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ we get

$$\mathbf{k}(\mathbf{u}^*, \mathbf{u}^*) + ((\mathbf{E}\mathbf{B}\mathbf{u}^*, \mathbf{B}\mathbf{u}^*)) = 0 .$$

Both terms, being non negative, must vanish. Hence by the positive definiteness of $\mathbf{E}$ on $\operatorname{Im}\mathbf{C}$ we have that $\mathbf{u}^* \in \operatorname{Ker}\mathbf{B}$.

By the positivity of $\mathbf{k}$ in $\mathscr{V}$ and the condition $\mathbf{k}(\mathbf{u}^*, \mathbf{u}^*) = 0$ we infer that the field $\mathbf{u}^* \in \operatorname{Ker}\mathbf{PB}$ is an absolute minimum point of $\mathbf{k}$ in $\mathscr{V}$. Taking the directional derivative along an arbitrary direction $\mathbf{v} \in \mathscr{V}$ by the symmetry of $\mathbf{k}$ we get

$$\mathbf{k}(\mathbf{u}^*, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathscr{V} \quad \Longleftrightarrow \quad \mathbf{K}\mathbf{u}^* = \mathbf{o} \quad \Longleftrightarrow \quad \mathbf{u}^* \in \operatorname{Ker}\mathbf{K}$$

and the result is proved. □

By the representation formula of $\operatorname{Ker}\mathbf{A}_o$ provided in the previous proposition the admissibility condition on the data of problem $\mathbb{P}^*$ can be written

$$l_o \in (\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K})^\perp .$$

Now for any pair $\{\mathbf{u}_o,\,\boldsymbol{\delta}^*\} \in \mathscr{V} \times \operatorname{Im}\mathbf{C}$ we have

$$((\mathbf{E}\boldsymbol{\delta}^*,\,\mathbf{B}\mathbf{v})) - \mathbf{k}(\mathbf{u}_o,\,\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K}\,.$$

The admissibility condition on ℓ_o amounts then to the orthogonality requirement

$$\mathbf{f} \in (\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K})^{\perp}\,.$$

On the other hand, when the pair $\{\mathbf{u}_o,\,\boldsymbol{\delta}^*\}$ ranges in $\mathscr{V} \times \operatorname{Im}\mathbf{C}$, the corresponding distorsion $\boldsymbol{\delta} = \mathbf{B}\mathbf{u}_o + \boldsymbol{\delta}^*$ will range over the whole subspace $\operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{C}$ and this subspace, by the assumed closedness of $\operatorname{Im}\mathbf{PB}$, coincides with $(\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C})^{\perp}$.

In conclusion the admissibility condition

$$\mathbf{f} \in (\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K})^{\perp}\,, \qquad \{\mathbf{u}_o,\boldsymbol{\delta}^*\} \in \mathscr{V} \times \operatorname{Im}\mathbf{C}\,,$$

for the data of problem $\mathbb{P}^*$ coincides with the admissibility condition

$$\mathbf{f} \in (\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K})^{\perp}\,, \qquad \boldsymbol{\delta} \in (\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C})^{\perp}\,,$$

for the corresponding data of the mixed problem $\mathbb{M}$.

The previous results are summarized in the following theorem.

Proposition 5.7. Well-posedness conditions for the mixed problem: *Let the continuous bilinear form* $\mathbf{k}$ *be positive and symmetric on* $\mathscr{V} \times \mathscr{V}$ *and the continuous bilinear form* $\mathbf{c}$ *be positive, symmetric, and closed on* $\mathscr{S} \times \mathscr{S}$*. The mixed elastostatic problem* $\mathbb{M}$ *is well-posed if and only if the following two conditions are fulfilled:*

a_1) *The image of* $\mathbf{PB}$ *is closed in* $\mathscr{D}$*, or equivalently,* $\operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{C}$ *is closed in* $\mathscr{D}$*, i.e.*

$$\inf_{\mathbf{u}\in\mathscr{V}}\ \sup_{\boldsymbol{\sigma}\in\mathscr{S}} \frac{((\boldsymbol{\sigma},\,\mathbf{PBu}))}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/(\operatorname{Ker}\mathbf{B}'\mathbf{P}')}\ \|\mathbf{u}\|_{\mathscr{V}/(\operatorname{Ker}\mathbf{PB})}} = \inf_{\boldsymbol{\sigma}\in\mathscr{S}}\ \sup_{\mathbf{u}\in\mathscr{V}} \frac{((\boldsymbol{\sigma},\,\mathbf{PBu}))}{\|\boldsymbol{\sigma}\|_{\mathscr{S}/(\operatorname{Ker}\mathbf{B}'\mathbf{P}')}\ \|\mathbf{u}\|_{\mathscr{V}/(\operatorname{Ker}\mathbf{PB})}} > 0\,,$$

a_2) *the bilinear form of the elastic energy is closed on* $\operatorname{Ker}\mathbf{PB} \times \operatorname{Ker}\mathbf{PB}$*, i.e.*

$$\inf_{\mathbf{u}^*\in\operatorname{Ker}\mathbf{PB}}\ \sup_{\mathbf{v}^*\in\operatorname{Ker}\mathbf{PB}} \frac{\mathbf{k}(\mathbf{u}^*,\,\mathbf{v}^*) + ((\mathbf{EBu}^*,\,\mathbf{Bv}^*))}{\|\mathbf{u}^*\|_{\mathscr{V}/(\operatorname{Ker}\mathbf{B}\cap\operatorname{Ker}\mathbf{K})}\ \|\mathbf{v}^*\|_{\mathscr{V}/(\operatorname{Ker}\mathbf{B}\cap\operatorname{Ker}\mathbf{K})}} > 0\,.$$

In other terms conditions a_1) *and* a_2) *are equivalent to state that the structural operator* $\mathbf{S} \in \operatorname{Lin}\{\mathscr{V} \times \mathscr{S},\,\mathscr{F} \times \mathscr{D}\}$ *has a closed range so that the orthogonality condition* $\operatorname{Im}\mathbf{S} = (\operatorname{Ker}\mathbf{S})^{\perp}$ *holds.*

Applicable sufficient criteria for the fulfilment of the conditions a_1) and a_2) will be discussed in the next section.

6. Sufficient criteria

Proposition 5.7 provides a set of two necessary and sufficient conditions for the well-posedness of a general elastic problem.

More precisely condition a_1) states the equivalence of the mixed problem

$$\mathbb{M})\quad \begin{cases} \mathbf{k}(\mathbf{u},\,\mathbf{v}) + \mathbf{b}(\mathbf{v},\,\boldsymbol{\sigma}) = \langle \mathbf{f},\,\mathbf{v}\rangle & \mathbf{u} \in \mathscr{V} \quad \forall \mathbf{v} \in \mathscr{V}\,, \\ \mathbf{b}(\mathbf{u},\,\boldsymbol{\tau}) - \mathbf{c}(\boldsymbol{\sigma},\,\boldsymbol{\tau}) = \langle \boldsymbol{\delta},\,\boldsymbol{\tau}\rangle & \boldsymbol{\sigma} \in \mathscr{S} \quad \forall \boldsymbol{\tau} \in \mathscr{S} \end{cases}$$

to the reduced problem $\mathbb{P}^*$ and condition a_2) provides the well-posedness of problem $\mathbb{P}^*$.

Let us now discuss these two conditions in detail.

6.1 Discussion of condition a_1)

By remark 5.2 the condition a_1) can be stated in the equivalent forms

- the subspace $\operatorname{Im}\mathbf{PB}$ is closed in $\mathscr{D}$,
- the subspace $\operatorname{Im}\mathbf{B}'\mathbf{P}'$ is closed in $\mathscr{F}$,
- the sum $\operatorname{Ker}\mathbf{P} + \operatorname{Im}\mathbf{B} = \operatorname{Im}\mathbf{C} + \operatorname{Im}\mathbf{B}$ is closed in $\mathscr{D}$.

Condition a_1) is trivially fulfilled by the structural models belonging to one of the two extreme categories:

i) the elastic compliance is not singular, so that $\operatorname{Ker}\mathbf{C} = \{\mathbf{o}\}$ and $\mathbf{P} = \mathbf{O}$,
ii) the elastic compliance is null, so that $\operatorname{Ker}\mathbf{C} = \mathscr{S}$ and $\mathbf{P} = \mathbf{I}$.

Case i) corresponds to classical elasticity problems in which every stress field is elastically effective.

Case ii) corresponds to the opposite situation in which every stress field is elastically ineffective. The statics of a rigid structure resting on elastic supports is described by an elastic problem of this kind whose mixed formulation is

$$\mathbb{F})\quad \begin{cases} \mathbf{k}(\mathbf{u},\,\mathbf{v}) + \mathbf{b}(\mathbf{v},\,\boldsymbol{\sigma}) = \langle \mathbf{f},\,\mathbf{v}\rangle & \mathbf{u} \in \mathscr{V} \quad \forall \mathbf{v} \in \mathscr{V}\,, \\ \mathbf{b}(\mathbf{u},\,\boldsymbol{\tau}) = \langle \boldsymbol{\delta},\,\boldsymbol{\tau}\rangle & \boldsymbol{\sigma} \in \mathscr{S} \quad \forall \boldsymbol{\tau} \in \mathscr{S}\,. \end{cases}$$

This is exactly the saddle point problem first analysed by Brezzi in [2].

The existence and uniqueness proof contributed in [2] addressed the more general case in which the bilinear form $\mathbf{k}$ in problem $\mathbb{F}$ was neither positive nor symmetric.

A discussion of the general mixed problem

$$\mathbb{G})\quad \begin{cases} \mathbf{k}(\mathbf{u},\, \mathbf{v}) + \mathbf{h}(\mathbf{v},\, \boldsymbol{\sigma}) = \langle \mathbf{f},\, \mathbf{v}\rangle & \mathbf{u} \in \mathscr{U} \quad \forall \mathbf{v} \in \mathscr{V}\,, \\ \mathbf{b}(\mathbf{u},\, \boldsymbol{\tau}) - \mathbf{c}(\boldsymbol{\sigma},\, \boldsymbol{\tau}) = \langle \boldsymbol{\delta},\, \boldsymbol{\tau}\rangle & \boldsymbol{\sigma} \in \mathscr{S} \quad \forall \boldsymbol{\tau} \in \mathscr{S} \end{cases}$$

in which the bilinear forms $\mathbf{k}$ and $\mathbf{c}$ are neither positive nor symmetric, is carried out by ROMANO et al. in [17]. The results contributed in [17] include as special cases the existence and uniqueness theorem by BREZZI and its extensions due to NICOLAIDES [18] and BERNARDI et al. [19] in which the bilinear form $\mathbf{c}$ was absent.

Remark 6.1: The analysis performed in the previous section addressed the general case of an elastic mixed problem $\mathbb{M}$ with a possibly non-degenerate kernel of the structural operators $\mathbf{S}$. Structural problems in which the kernel of $\mathbf{S}$ in non-degenerate are usually dealt with in the engineering applications. An example is provided by elastic problems in which rigid kinematic fields not involving reactions of the elastic supports are admitted by the constraints.

To deal with the presence of a non-degenerate kernel, the symmetry of the governing operator $\mathbf{S}$ and the positivity of the elastic operators $\mathbf{K}$ and $\mathbf{C}$ seem however to be unavoidable assumptions. They play in fact an essential role in deriving the representation formulas for the kernels provided in propositions 5.1 and 5.6.

Remark 6.2: It is worth noting that, for two- or three-dimensional non rigid structural models with a singular elastic compliance, condition a_1) is difficult to be checked and is far from being verified as a rule.

A relevant exception is provided by the incompressibility constraint of Stokes problem ([20], [21]). We emphasize that a singularity of the elastic compliance $\mathbf{C}$ is equivalent to the imposition of constraints on the strain fields. Strain constraints in continua have been recently discussed by ANTMAN and MARLOW in [4, 5] and critically reviewed by ROMANO et al. in [6].

6.2 Discussion of condition a_2)

Under the assumption that the bilinear form $\mathbf{k}$ is $\operatorname{Ker}\mathbf{PB}$-semielliptic, and hence closed on $\operatorname{Ker}\mathbf{PB} \times \operatorname{Ker}\mathbf{PB}$, the next result yields a sufficient criterion for the fulfilment of condition a_2).

Proposition 6.1: *Condition* a_2) *is satisfied if the following properties hold:*

i) $\mathbf{k}(\mathbf{u},\, \mathbf{u}) \geq c_{\mathbf{k}} \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} \qquad c_{\mathbf{k}} > 0 \quad \forall \mathbf{u} \in \operatorname{Ker}\mathbf{PB}\,,$

ii) $((\mathbf{EBu},\, \mathbf{Bu})) \geq c \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \quad c > 0 \quad \forall \mathbf{u} \in \operatorname{Ker}\mathbf{PB}\,,$

iii) $\operatorname{Ker}\mathbf{B} + \operatorname{Ker}\mathbf{K}$ *closed*.

P r o o f: By proposition A.5 finite angle and remark 2.1 property (*iii*) is equivalent to

$$\|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} + \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \geq \boldsymbol{\alpha} \|\mathbf{u}\|^2_{\mathscr{V}/(\operatorname{Ker}\mathbf{K} \cap \operatorname{Ker}\mathbf{B})} \quad \forall \mathbf{u} \in \mathscr{V}$$

so that, summing up (i) and (ii) and suitably defining a positive constant $c_{\mathbf{a}}$ we get

$$\mathbf{k}(\mathbf{u},\, \mathbf{u}) + ((\mathbf{EBu},\, \mathbf{Bu})) \geq c_{\mathbf{a}} \|\mathbf{u}\|^2_{\mathscr{V}/(\operatorname{Ker}\mathbf{K} \cap \operatorname{Ker}\mathbf{B})} \quad \forall \mathbf{u} \in \operatorname{Ker}\mathbf{PB}$$

which implies the closedness condition a_2). □

Remark 6.3:

- Condition (i) is fulfilled in structural problems with discrete external elastic constraints. In fact when only a finite number of external elastic constraints are imposed, the subspace $\operatorname{Im}\mathbf{K}$ is finite dimensional and the constant $c_{\mathbf{k}}$ is provided by the smallest positive eigenvalue of the symmetric positive matrix associated with the restriction of the bilinear form $\mathbf{k}$ to $\mathscr{V}/\operatorname{Ker}\mathbf{K} \times \mathscr{V}/\operatorname{Ker}\mathbf{K}$. An example is provided by an elastic plate resting on a finite number of elastic supports, as shown in Fig. 6.1.

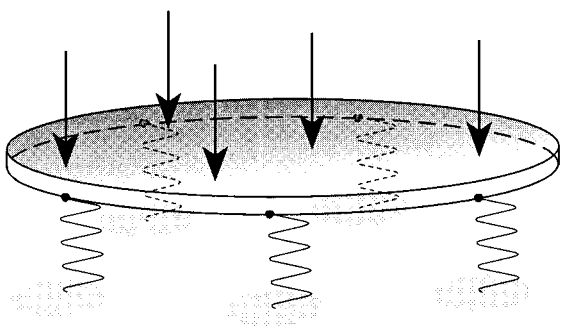


Fig. 6.1. Elastic plate on a finite number of elastic supports $\dim(\operatorname{Im})\,\mathbf{K} < +\infty$

- Condition (ii) follows from a standard ellipticity property of internal elasticity

$$((\mathbf{C}\boldsymbol{\sigma}, \boldsymbol{\sigma})) \geq c_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}\|^2_{\mathscr{S}/\operatorname{Ker}\mathbf{C}} \quad \forall \boldsymbol{\sigma} \in \mathscr{S},$$

  equivalent to

$$((\mathbf{E}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})) \geq c_{\mathbf{e}} \|\boldsymbol{\varepsilon}\|^2_{\mathscr{D}} \quad \forall \boldsymbol{\varepsilon} \in \operatorname{Ker}\mathbf{P} = \operatorname{Im}\mathbf{C},$$

  and from the closedness of the fundamental form $\mathbf{b}(\mathbf{u}, \boldsymbol{\sigma})$

$$\|\mathbf{B}\mathbf{u}\|_{\mathscr{D}} \geq \mathbf{c_b} \|\mathbf{u}\|_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \quad \forall \mathbf{u} \in \mathscr{V}.$$

  The positive constant in (ii) is given by $c = c_{\mathbf{e}} c_{\mathbf{b}}^2$.
- Condition (iii) is a consequence of the finite dimensionality of $\operatorname{Ker}\mathbf{B}$ in most structural models. More generally it follows from the closedness condition in proposition 4.5.

7. Elastic beds

Let us finally consider the general problem of the elastic equilibrium of a structural model in which

- the constitutive behaviour is partially rigid,
- the external elastic constraints include the presence of elastic beds so that $\operatorname{Im}\mathbf{K}$ is not finite dimensional in $\mathscr{F}$.

An example is provided by an elastic plate resting on an elastic bed, as in Fig. 7.1. Such a model is commonly adopted in engineering applications to simulate a foundation interacting with a supporting soil.

The difficulty connected with this kind as problems lies in the fact that the bilinear form of the external elastic energy is not semi-elliptic on $\mathscr{V} \times \mathscr{V}$ as required by condition i) of proposition 6.1.

To enlight the problem let us consider the model of an elastic beam resting on an elastic bed of springs (Winkler soil model). The flexural elastic energy of the beam is provided by one-half the integral of the squared second derivative of the transverse displacement. On the other hand, the elastic energy stored into the elastic springs is equal to one-half the integral of the squared transverse displacement. The kinematic space $\mathscr{V}$ is defined to be the Sobolev space $\mathscr{H}^2$ to ensure a finite value of the elastic energy. Considering a rapidly varying elastic curve of the beam, as depicted in 7.2, we get an extremely high value of the elastic energy in the beam and a negligible energy in the elastic bed.

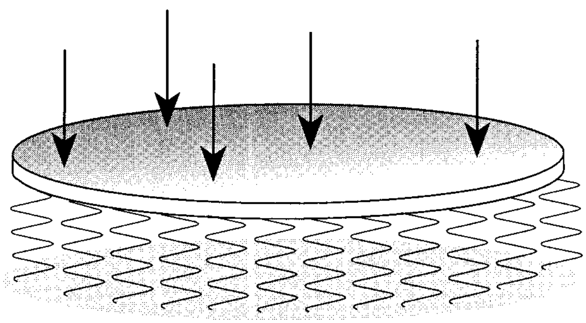


Fig. 7.1. Elastic plate resting on an elastic bed

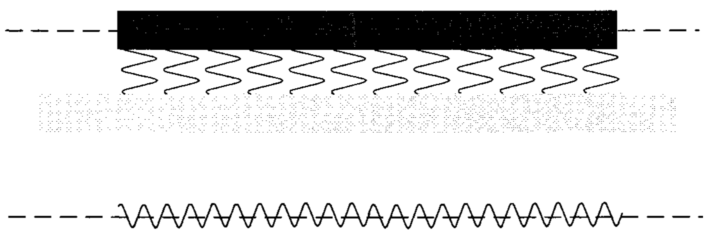


Fig. 7.2. Large elastic energy with small displacements

The discussion above leads to the conclusion that the semi-ellipticity condition on the bilinear form $\mathbf{k}$ of elastic constraints energy must be relaxed.

A by far less stringent requirement is the property that $\mathbf{k}$ is positive semi-definite on $\operatorname{Ker}\mathbf{PB} \times \operatorname{Ker}\mathbf{PB}$ and semi-elliptic only on $\operatorname{Ker}\mathbf{B} \times \operatorname{Ker}\mathbf{B}$, that is with respect to rigid kinematic fields, according to the inequalities

$$\text{i)} \ \mathbf{k}(\mathbf{u}, \mathbf{u}) \geq 0 \quad \forall \mathbf{u} \in \operatorname{Ker}\mathbf{PB},$$

$$\text{ii)} \ \mathbf{k}(\mathbf{u}, \mathbf{u}) \geq c_{\mathbf{k}} \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} \quad c_{\mathbf{k}} > 0 \quad \forall \mathbf{u} \in \operatorname{Ker}\mathbf{B}.$$

We remark that rigid kinematic fields cannot undergo very sauvage oscillations. In the case of the simple beam of 7.2 they are in fact affine functions. In general, when $\operatorname{Ker}\mathbf{B}$ is finite dimensional, property ii) above is a consequence of property i) since $c_{\mathbf{k}} > 0$ is the smallest positive eigenvalue of a non-null symmetric and positive matrix. We have now to prove that these less stringent assumptions on $\mathbf{k}$ are sufficient to ensure the fulfilment of condition a_2).

To this end we provide a preliminary result.

Proposition 7.1. The elastic bed inequality: *The assumptions*

i) $\mathbf{k}(\mathbf{u},\, \mathbf{u}) \geq \mathbf{0} \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{PB}\,,$

ii) $\mathbf{k}(\mathbf{u},\, \mathbf{u}) \geq c_{\mathbf{k}} \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{B}\,,$

iii) $((\mathbf{EBu},\, \mathbf{Bu})) \geq c_{\mathbf{e}} c^2_{\mathbf{b}} \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{PB}$

ensure the validity of the inequality

$$\mathbf{k}(\mathbf{u},\, \mathbf{u}) + ((\mathbf{EBu},\, \mathbf{Bu})) \geq c_{\boldsymbol{\pi}} \|\boldsymbol{\Pi}\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} \quad c_{\boldsymbol{\pi}} > 0 \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{PB}$$

where $\boldsymbol{\Pi}$ *denotes the orthogonal projector on* $\operatorname{Ker}\mathbf{B}$ *in* $\mathscr{V}$.

Proof: We proceed *per absurdum* by assuming that the inequality is false. Then, prescribing that $\|\boldsymbol{\Pi}\mathbf{u}\|_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} = 1$, the infimum of the first member would be zero. By taking a minimizing sequence $\{\mathbf{u}_n\}$ we have

$$\lim_{n\to\infty} \mathbf{k}(\mathbf{u}_n,\, \mathbf{u}_n) + ((\mathbf{EBu}_n,\, \mathbf{Bu}_n)) = 0\,.$$

By i) both terms of the sum are non-negative and then vanish at the limit. Hence from iii) we get

$$\lim_{n\to\infty} ((\mathbf{EBu}_n,\, \mathbf{Bu}_n)) = 0 \Rightarrow \lim_{n\to\infty} \|\mathbf{u}_n - \boldsymbol{\Pi}\mathbf{u}_n\|_{\mathscr{V}} = 0$$

and by the continuity of $\mathbf{k}$ and assumption ii)

$$\lim_{n\to\infty} \mathbf{k}(\mathbf{u}_n,\, \mathbf{u}_n) = 0 \Rightarrow \lim_{n\to\infty} \mathbf{k}(\boldsymbol{\Pi}\mathbf{u}_n,\, \boldsymbol{\Pi}\mathbf{u}_n) = 0 \Rightarrow \lim_{n\to\infty} \|\boldsymbol{\Pi}\mathbf{u}_n\|_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} = 0$$

contrary to the assumption that $\|\boldsymbol{\Pi}\mathbf{u}_n\|_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} = 1$. □

An applicable criterion for the validity of condition a_2) is now at hand.

Proposition 7.2: *Condition* a_2) *is satisfied if the following properties hold:*

i) $\mathbf{k}(\mathbf{u},\, \mathbf{u}) \geq \mathbf{0} \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{PB}\,,$

ii) $\mathbf{k}(\mathbf{u},\, \mathbf{u}) \geq c_{\mathbf{k}} \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{B}\,,$

iii) $((\mathbf{EBu},\, \mathbf{Bu})) \geq c_{\mathbf{e}} c^2_{\mathbf{b}} \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \quad \forall \mathbf{u} \in \operatorname{Ker} \mathbf{PB}\,,$

iv) $\operatorname{Ker}\mathbf{B} + \operatorname{Ker}\mathbf{K}$ *closed*.

Proof: Proposition 2.1 and Remark 2.1 ensure that property iv) implies the existence of a constant $\boldsymbol{\alpha} > 0$ such that

$$\|\boldsymbol{\Pi}\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{K}} + \|\mathbf{u}\|^2_{\mathscr{V}/\operatorname{Ker}\mathbf{B}} \geq \boldsymbol{\alpha} \|\mathbf{u}\|^2_{\mathscr{V}/(\operatorname{Ker}\mathbf{K} \cap \operatorname{Ker}\mathbf{B})} \quad \forall \mathbf{u} \in \mathscr{V}\,.$$

Condition a_2) then follows by adding inequality iii) and the one proved in proposition 7.1. □

7.1 A well-posedness criterion

Conditions i), ii), iii) of proposition 7.2 are always fulfilled by elastic structural models. Further, by remark A.2, the closedness of $\operatorname{Im}\mathbf{B}$ ensures that condition a_1) can be equivalently stated by requiring the closedness of $\operatorname{Ker}\mathbf{C} + \operatorname{Ker}\mathbf{B}'$. Then, to get a well posed mixed problem, what we really have to check is the fulfilment of the two properties concerning the kernels of the elastic operators, as stated in the next proposition.

Proposition 7.3. Well-posedness criterion: *Let* $\operatorname{Im}\mathbf{B}$ *be closed in* $\mathscr{D}$ *and conditions* i), ii), iii) *of proposition 7.2 be fulfilled. Then the closedness properties*
a_1) $\operatorname{Ker}\mathbf{B}' + \operatorname{Ker}\mathbf{C}$ *is closed in* $\mathscr{S}$,
b) $\operatorname{Ker}\mathbf{B} + \operatorname{Ker}\mathbf{K}$ *is closed in* $\mathscr{V}$
ensure that the mixed elastostatic problem $\mathbb{M}$ *is well posed.*

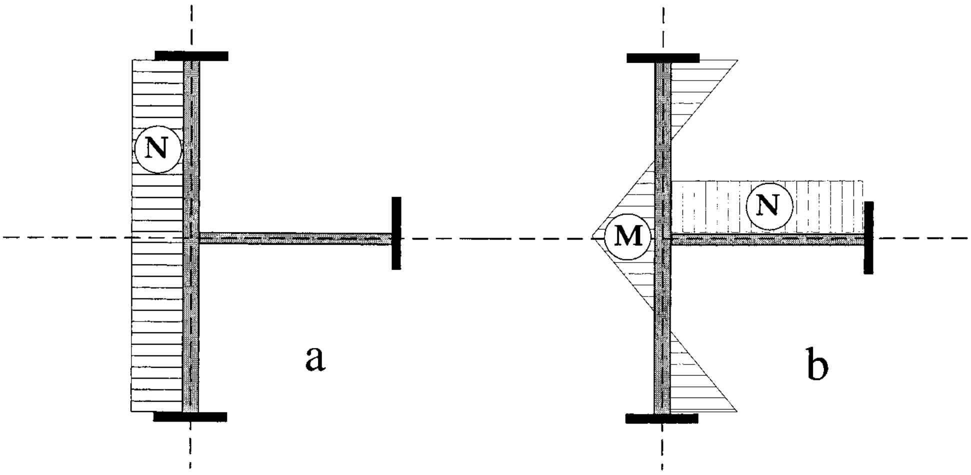
By virtue of proposition A.7 a relevant situation in which condition a_1) and b) are fulfilled is provided by the following family of structural models.

Definition 7.4. Simple structures: A structural model is said to be *simple* if the subspaces $\operatorname{Ker}\mathbf{B}$ of rigid kinematisms and $\operatorname{Ker}\mathbf{B}'$ of self-equilibrated stress fields are finite dimensional.

All one-dimensional engineering structural models composed by beam and bar elements belong to this class and hence the related elastic problems are always well-posed. A simple frame composed of two beams which are axially undeformable and flexurally elastic is depicted hereafter. The stress fields are pairs of diagrams of bending moments and axial forces.

Fig. 7.3a shows the diagram of axial forces in the vertical beam which corresponds to a self-equilibrated and elastically ineffective stress field. It cannot be evaluated by solving the elastic problem. Since this stress field generates the whole subspace $\operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C}$ the imposed distorsions δ must satisfy the related orthogonality condition which requires that the mean elongation of the vertical beam must vanish. Fig. 7.3b shows a diagram of axial forces and bending moments which is self-equilibrated but elastically effective.

A beam on elastic supports is sketched in Figs. 7.4 and 7.5 to show examples of kinematic fields which respectively belong to $\operatorname{Ker}\mathbf{K}$ and to $\operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K}$.



$$\begin{vmatrix} 0 \\ N \end{vmatrix} \in \operatorname{Ker}\mathbf{B}' \cap \operatorname{Ker}\mathbf{C}$$

$$\begin{vmatrix} M \\ N \end{vmatrix} \in \operatorname{Ker}\mathbf{B}'$$

Fig. 7.3. Flexurally elastic and axially rigid beams

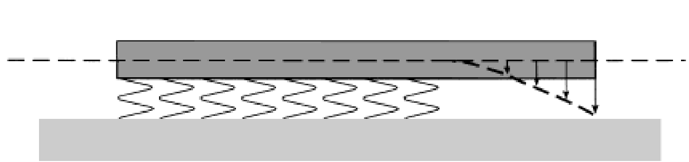


Fig. 7.4. $\mathbf{u} \in \operatorname{Ker}\mathbf{K}$

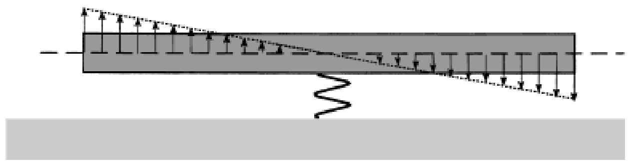


Fig. 7.5. $\mathbf{u} \in \operatorname{Ker}\mathbf{B} \cap \operatorname{Ker}\mathbf{K}$

8. Conclusions

The analytical properties of mixed formulations in elasticity have been investigated with an approach which provides a clear mechanical interpretation of the properties of the model and of the conditions for its well-posedness.

Necessary and sufficient conditions for the existence of a solution have been proved and effective criteria for their application have been contributed. In particular we have shown that all familiar one-dimensional engineering models of structural assemblies composed of bars and beams fulfil the well-posedness property in the presence of any singularity of the elastic compliance.

The case of two- or three-dimensional structural models drastically changes the scenary due to the infinite dimensionality of the subspace of self-stresses so that well-posedness of the mixed problem will be almost never fulfilled when the elastic compliance is singular. As relevant exceptions we quote structural models with either fully elastic or perfectly rigid behaviour.

The problem is strictly connected with the discussion of constrained structural models in which a linear constraint is imposed on the strain fields.

By means of simple counterexamples [5, 6] it can be shown that there is little hope to get well-posedness of a mixed problem when the elastic compliance is singular. In this respect Stokes' problem concerning the incompressible viscous flow of fluids, for which well-posedness is fulfilled, must be considered as an exception.

Although the analysis has been carried out with explicit reference to elastostatic problems, we observe that the results can as well be applied to the discussion of a number of interesting problems in mathematical physics modeled by analogous mixed formulations.

A variant of the proposed approach can also be applied to the discussion of problems in which linear constraints are imposed on the stress field, as shown in [22].

Appendix

To make the paper reasonably self-consistent and to provide a direct reference to known results of Functional Analysis, we collect here the most important theorems which are referred to in the paper.

The proofs of some results are explicitly reported in the simplest context of Hilbert space theory since they are usually formulated and proved in the more complex setting of Banach spaces.

First we recall the statement of Banach's open mapping theorem and closed range theorem (see [9] for a general proof in Frechét spaces. A much simpler proof of the closed range theorem in Hilbert spaces is provided in [13]). We also report some important consequences of the open mapping theorem and their specialization to Hilbert spaces where the projection theorem and the Riesz representation theorem provide fundamental analytical tools.

A linear operator $\mathbf{A}\colon \mathscr{X} \mapsto \mathscr{Y}$ between two Hilbert spaces is continuous if the counter-images under $\mathbf{A}$ of open sets in $\mathscr{Y}$ are open sets in $\mathscr{X}$. Continuity of linear operators is equivalent to boundedness which means that there exists a constant $C > 0$ such that

$$C\|\mathbf{x}\|_{\mathscr{X}} \geq \|\mathbf{A}\mathbf{x}\|_{\mathscr{Y}} \quad \forall \mathbf{x} \in \mathscr{X}\,.$$

On the basis of Baire-Hausdorff lemma (see [10] theorem II.1) the Polish mathematician S. BANACH proved a number of celebrated results which provide the foundation of modern Functional Analysis.

Most deep results of Functional Analysis rely upon the following theorem (see [10] theorem II.5).

Proposition A.1. The open mapping theorem: *Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces and $\mathbf{A} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{Y}\}$ a continuous linear operator which is surjective. Then there exists a constant $c > 0$ such that*

$$\|\mathbf{y}\|_{\mathscr{Y}} < c \Rightarrow \exists \mathbf{x} \in \mathscr{X} : \|\mathbf{x}\|_{\mathscr{X}} < 1\,, \qquad \mathbf{A}\mathbf{x} = \mathbf{y}\,.$$

As a consequence the operator $\mathbf{A}$ will map every open set of $\mathscr{X}$ onto an open set of $\mathscr{Y}$.

As a corollary it can be proved (see [10] corollary II.6) that the inverse of a continuous one-to-one linear map between two Hilbert spaces also enjoys the continuity property.

Proposition A.2. The continuous inverse theorem: *If a continuous linear operator $\mathbf{A} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{Y}\}$ establishes a one-to-one map between $\mathscr{X}$ and $\mathscr{Y}$ then the inverse operator is linear and continuous.*

In the sequel the symbol $\langle \bullet, \bullet \rangle$ will denote the duality pairing between dual Hilbert spaces.

We recall that, given a closed subspace $\mathscr{A}$ of a Banach space $\mathscr{X}$, the factor space $\mathscr{X}/\mathscr{A}$ is a Banach space when endowed with the norm

$$\|\mathbf{x}_{\mathscr{A}}\|_{\mathscr{X}/\mathscr{A}} := \inf\{\|\mathbf{x} - \bar{\mathbf{x}}\|_{\mathscr{X}} \mid \bar{\mathbf{x}} \in \mathscr{A}\}\,,$$

where $\mathbf{x}_{\mathscr{A}}$ denotes the equivalence class $\mathbf{x} + \mathscr{A}$. Let $\mathscr{X}$ be a Hilbert space and $\boldsymbol{\Pi}_{\mathscr{A}}$ be the orthogonal projector on the closed subspace $\mathscr{A} \subseteq \mathscr{X}$. The factor space $\mathscr{X}/\mathscr{A}$ is a Hilbert space for the inner product

$$(\mathbf{x}_{\mathscr{A}}, \mathbf{y}_{\mathscr{A}})_{\mathscr{X}/\mathscr{A}} := (\mathbf{x} - \boldsymbol{\Pi}_{\mathscr{A}}\mathbf{x}, \mathbf{y} - \boldsymbol{\Pi}_{\mathscr{A}}\mathbf{y})_{\mathscr{X}} \quad \forall \mathbf{x}_{\mathscr{A}}, \mathbf{y}_{\mathscr{A}} \in \mathscr{X}/\mathscr{A}\,; \qquad \mathbf{x} \in \mathbf{x}_{\mathscr{A}}, \mathbf{y} \in \mathbf{y}_{\mathscr{A}}$$

and the associated norm can be written as

$$\|\mathbf{x}_{\mathscr{A}}\|_{\mathscr{X}/\mathscr{A}} = \min\{\|\mathbf{x} - \bar{\mathbf{x}}\|_{\mathscr{X}} \mid \bar{\mathbf{x}} \in \mathscr{A}\} = \|\mathbf{x} - \boldsymbol{\Pi}_{\mathscr{A}}\mathbf{x}\|_{\mathscr{X}}\,.$$

For every element $\mathbf{x} \in \mathbf{x}_{\mathscr{A}}$ we shall set $\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}} := \|\mathbf{x}_{\mathscr{A}}\|_{\mathscr{X}/\mathscr{A}}$.

Given a pair of Hilbert spaces $\{\mathscr{X}, \mathscr{Y}\}$ a bilinear form $\mathbf{a}(\mathbf{x}, \mathbf{y})$ on $\mathscr{X} \times \mathscr{Y}$ is bounded if for a positive constant C the following inequality holds

$$C\|\mathbf{x}\|_{\mathscr{X}}\, \|\mathbf{y}\|_{\mathscr{Y}} \geq |\mathbf{a}(\mathbf{x}, \mathbf{y})| \quad \forall \mathbf{x} \in \mathscr{X}, \mathbf{y} \in \mathscr{Y}\,.$$

Denoting by $\{\mathscr{X}', \mathscr{Y}'\}$ the spaces in duality with $\{\mathscr{X}, \mathscr{Y}\}$, a pair of bounded linear operators $\mathbf{A} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{Y}'\}$ and $\mathbf{A}' \in \mathrm{Lin}\{\mathscr{Y}, \mathscr{X}'\}$ can be associated with $\mathbf{a}$ by the identity

$$\mathbf{a}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{A}'\mathbf{y}, \mathbf{x} \rangle \quad \forall \mathbf{x} \in \mathscr{X}, \mathbf{y} \in \mathscr{Y}\,.$$

The discussion of the well-posedness of variational formulations is founded upon the following fundamental result due to Banach. A proof in Banach spaces can found in [9], [10] and a simpler proof in Hilbert spaces in [13].

Proposition A.3. The closed range theorem: *Let $\{\mathscr{X}, \mathscr{Y}\}$ be a pair of Hilbert spaces, $\mathbf{a}(\mathbf{x}, \mathbf{y})$ a bounded bilinear form on $\mathscr{X} \times \mathscr{Y}$, and $\mathbf{A} \in \mathrm{Lin}\{\mathscr{X}, \mathscr{Y}'\}$ and $\mathbf{A}' \in \mathrm{Lin}\{\mathscr{Y}, \mathscr{X}'\}$ the bounded linear operators associated with $\mathbf{a}$. Then the following properties are equivalent:*

$$\begin{aligned}
&\text{i)}\ \ \mathrm{Im}\,\mathbf{A} \textit{ is closed in } \mathscr{Y}' \quad \Longleftrightarrow \quad \mathrm{Im}\,\mathbf{A} = (\mathrm{Ker}\,\mathbf{A}')^{\perp}\,,\\
&\text{ii)}\ \ \mathrm{Im}\,\mathbf{A}' \textit{ is closed in } \mathscr{X}' \quad \Longleftrightarrow \quad \mathrm{Im}\,\mathbf{A}' = (\mathrm{Ker}\,\mathbf{A})^{\perp}\,,\\
&\text{iii)}\ \ \|\mathbf{A}\mathbf{x}\|_{\mathscr{Y}'} \geq c\|\mathbf{x}\|_{\mathscr{X}/\mathrm{Ker}\,\mathbf{A}} \quad \forall \mathbf{x} \in \mathscr{X}\,,\\
&\text{iv)}\ \ \|\mathbf{A}'\mathbf{y}\|_{\mathscr{X}'} \geq c\|\mathbf{y}\|_{\mathscr{Y}/\mathrm{Ker}\,\mathbf{A}'} \quad \forall \mathbf{y} \in \mathscr{Y}\,,
\end{aligned}$$

where $c > 0$ is a positive constant.

Remark A.1: We recall that, by definition

$$\|\mathbf{A}\mathbf{x}\|_{\mathscr{Y}'} := \sup_{\mathbf{y} \in \mathscr{Y}} \frac{\mathbf{a}(\mathbf{x}, \mathbf{y})}{\|\mathbf{y}\|_{\mathscr{Y}}}\,, \qquad \|\mathbf{A}'\mathbf{y}\|_{\mathscr{X}'} := \sup_{\mathbf{x} \in \mathscr{X}} \frac{\mathbf{a}(\mathbf{x}, \mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}}}\,.$$

These expressions can be modified by observing that being

$$\begin{aligned}
&\mathbf{a}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}'\mathbf{y}, \mathbf{x} \rangle = 0 \quad \forall \mathbf{x} \in \mathscr{X} \quad \forall \mathbf{y} \in \mathrm{Ker}\,\mathbf{A}'\,,\\
&\mathbf{a}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = 0 \quad \forall \mathbf{y} \in \mathscr{Y} \quad \forall \mathbf{x} \in \mathrm{Ker}\,\mathbf{A}
\end{aligned}$$

we have

$$\|\mathbf{A}\mathbf{x}\|_{\mathscr{Y}'} = \sup_{\mathbf{y}\in\mathscr{Y}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{y}\|_{\mathscr{Y}}} = \sup_{\mathbf{y}\in\mathscr{Y}} \ \sup_{\mathbf{y}_o\in\operatorname{Ker}\mathbf{A}'} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{y}+\mathbf{y}_o\|_{\mathscr{Y}}} = \sup_{\mathbf{y}\in\mathscr{Y}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{y}\|_{\mathscr{Y}/\operatorname{Ker}\mathbf{A}'}}$$

and

$$\|\mathbf{A}'\mathbf{y}\|_{\mathscr{X}'} = \sup_{\mathbf{x}\in\mathscr{X}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}}} = \sup_{\mathbf{x}\in\mathscr{X}} \ \sup_{\mathbf{x}_o\in\operatorname{Ker}\mathbf{A}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}+\mathbf{x}_o\|_{\mathscr{X}}} = \sup_{\mathbf{x}\in\mathscr{X}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}}}\,.$$

Since the same constant $c > 0$ appears in iii) and iv) of proposition A.3, these inequalities are easily shown [13] to be equivalent to

$$\inf_{\mathbf{x}\in\mathscr{X}} \ \sup_{\mathbf{y}\in\mathscr{Y}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}}\,\|\mathbf{y}\|_{\mathscr{Y}/\operatorname{Ker}\mathbf{A}'}} = \inf_{\mathbf{y}\in\mathscr{Y}} \ \sup_{\mathbf{x}\in\mathscr{X}} \frac{\mathbf{a}(\mathbf{x},\,\mathbf{y})}{\|\mathbf{x}\|_{\mathscr{X}/\operatorname{Ker}\mathbf{A}}\,\|\mathbf{y}\|_{\mathscr{Y}/\operatorname{Ker}\mathbf{A}'}} > 0\,,$$

which will be referred to as the inf-sup conditions.

When the properties in proposition A.3 hold true, we shall say that the bilinear form $\mathbf{a}$ is *closed* on $\mathscr{X} \times \mathscr{Y}$.

Proposition A.1 implies the following result concerning the sum of two closed subspaces of a Banach space (see [10] theorem II.8):

Proposition A.4. A representation lemma: *Let $\mathscr{X}$ be a Banach space and $\mathscr{A} \subseteq \mathscr{X}$ and $\mathscr{B} \subseteq \mathscr{X}$ closed subspaces such that their sum $\mathscr{A} + \mathscr{B}$ is closed. Then there exists a constant $c > 0$ such that every $\mathbf{x} \in \mathscr{A} + \mathscr{B}$ admits a decomposition of the kind $\mathbf{x} = \mathbf{a} + \mathbf{b}$ with $\mathbf{a} \in \mathscr{A}$, $\mathbf{b} \in \mathscr{B}$, $\|\mathbf{a}\|_{\mathscr{X}} \le c\|\mathbf{x}\|_{\mathscr{X}}$ and $\|\mathbf{b}\|_{\mathscr{X}} \le c\|\mathbf{x}\|_{\mathscr{X}}$.*

Proof: By endowing the product space $\mathscr{X} \times \mathscr{X}$ with the norm $\|\{\mathbf{x},\,\mathbf{y}\}\|_{\mathscr{X}\times\mathscr{X}} := \|\mathbf{x}\|_{\mathscr{X}} + \|\mathbf{y}\|_{\mathscr{X}}$ the linear operator $\mathbf{A} \in \operatorname{Lin}\{\mathscr{X} \times \mathscr{X},\, \mathscr{X}\}$ defined by $\mathbf{A}\{\mathbf{x},\,\mathbf{y}\} := \mathbf{x} + \mathbf{y}$ is continuous and surjective. Then by proposition A.1 there exists a constant $c > 0$ such that every $\mathbf{x} \in \mathscr{A} + \mathscr{B}$ with $\|\mathbf{x}\|_{\mathscr{X}} < c$ can be written as $\mathbf{x} = \mathbf{a} + \mathbf{b}$ with $\mathbf{a} \in \mathscr{A}$, $\mathbf{b} \in \mathscr{B}$, and $\|\mathbf{a}\|_{\mathscr{X}} + \|\mathbf{b}\|_{\mathscr{X}} < 1$. Hence by homogeneity we get that $\mathbf{x} \in \mathscr{A} + \mathscr{B}$ admits the decomposition $\mathbf{x} = \mathbf{a} + \mathbf{b}$ with $\mathbf{a} \in \mathscr{A}$, $\mathbf{b} \in \mathscr{B}$, and $\|\mathbf{a}\|_{\mathscr{X}} + \|\mathbf{b}\|_{\mathscr{X}} \le c^{-1}\|\mathbf{x}\|_{\mathscr{X}}$. □

From lemma A.4 we get a useful characterization of the closedness of the sum of two closed subspaces (see [10] corollary II.9 for a proof in Banach spaces).

Proposition A.5. The finite angle property: *Let $\mathscr{X}$ be a Hilbert space and $\mathscr{A} \subseteq \mathscr{X}$ and $\mathscr{B} \subseteq \mathscr{X}$ closed subspaces such that their sum $\mathscr{A} + \mathscr{B}$ is closed. Then there exists a constant $c > 0$ such that*

$$\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le c(\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}} + \|\mathbf{x}\|_{\mathscr{X}/\mathscr{B}}) \quad \forall \mathbf{x} \in \mathscr{X}\,.$$

Proof: Let $\mathbf{a} \in \mathscr{A}$ and $\mathbf{b} \in \mathscr{B}$. Then by proposition A.4 there exist $\bar{\mathbf{a}} \in \mathscr{A}$, $\bar{\mathbf{b}} \in \mathscr{B}$, and a constant $k > 0$ such that

$$\mathbf{a} + \mathbf{b} = \bar{\mathbf{a}} + \bar{\mathbf{b}} \quad \text{and} \quad \|\bar{\mathbf{a}}\|_{\mathscr{X}} \le k\|\mathbf{a} + \mathbf{b}\|_{\mathscr{X}}\,, \qquad \|\bar{\mathbf{b}}\|_{\mathscr{X}} \le k\|\mathbf{a} + \mathbf{b}\|_{\mathscr{X}}\,.$$

By observing that $\mathbf{a} - \bar{\mathbf{a}} = \bar{\mathbf{b}} - \mathbf{b} \in \mathscr{A} \cap \mathscr{B}$ we have that $\forall \mathbf{a} \in \mathscr{A}$ and $\forall \mathbf{b} \in \mathscr{B}$

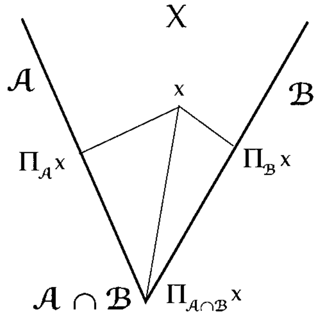
$$\|\mathbf{x}\|_{\mathscr{X}/\mathscr{A}\cap\mathscr{B}} \le \|\mathbf{x} - (\mathbf{a} - \bar{\mathbf{a}})\|_{\mathscr{X}} \le \|\mathbf{x} - \mathbf{a}\|_{\mathscr{X}} + \|\bar{\mathbf{a}}\|_{\mathscr{X}} \le \|\mathbf{x} - \mathbf{a}\|_{\mathscr{X}} + k\|\mathbf{a} + \mathbf{b}\|_{\mathscr{X}}\,.$$

We get the result by a further application of the triangle inequality

$$\|\mathbf{a} + \mathbf{b}\|_{\mathscr{X}} \le \|\mathbf{x} - \mathbf{a}\|_{\mathscr{X}} + \|\mathbf{x} - \mathbf{b}\|_{\mathscr{X}}\,,$$

taking the infimum with respect to $\mathbf{a} \in \mathscr{A}$ and $\mathbf{b} \in \mathscr{B}$ and setting $c = k + 1$. □

Fig. 9.1 provides a geometrical interpretation of proposition A.5.



$$\|\mathbf{x} - \boldsymbol{\Pi}_{\mathscr{A}}\mathbf{x}\|_{\mathscr{X}} + \|\mathbf{x} - \boldsymbol{\Pi}_{\mathscr{B}}\mathbf{x}\|_{\mathscr{X}} \ge c^{-1}\|\mathbf{x} - \boldsymbol{\Pi}_{\mathscr{A}\cap\mathscr{B}}\mathbf{x}\|_{\mathscr{X}} \quad \forall \mathbf{x} \in \mathscr{X}$$

Fig. 9.1. Geometrical interpretation of the finite angle property

The following lemma provides two basic orthogonality relations.

Lemma A.6. Orthogonality relations: *Let $\mathscr{A}$ and $\mathscr{B}$ be two subspaces of an Hilbert space $\mathscr{X}$, and $\mathscr{A}^\perp$ and $\mathscr{B}^\perp$ their orthogonal complements in the dual Hilbert space $\mathscr{X}'$. Then*

i) $(\mathscr{A} + \mathscr{B})^\perp = \mathscr{A}^\perp \cap \mathscr{B}^\perp$.

If $\mathscr{A}$ and $\mathscr{B}$ are closed subspaces we have also that

ii) $(\mathscr{A}^\perp + \mathscr{B}^\perp)^\perp = \mathscr{A} \cap \mathscr{B}$.

Proof: The equality i) is evident. To prove ii) we observe that $\mathscr{A} \cap \mathscr{B} \subseteq (\mathscr{A}^\perp + \mathscr{B}^\perp)^\perp$ since $\mathbf{x} \in \mathscr{A} \cap \mathscr{B}$ and $\mathbf{f} \in (\mathscr{A}^\perp + \mathscr{B}^\perp)$ implies $\langle \mathbf{f}, \mathbf{x} \rangle = 0$. The converse inclusion follows from $\mathscr{A}^\perp \subseteq \mathscr{A}^\perp + \mathscr{B}^\perp$ so that $(\mathscr{A}^\perp + \mathscr{B}^\perp)^\perp \subseteq \mathscr{A}^{\perp\perp} = \mathscr{A}$. Analogously $(\mathscr{A}^\perp + \mathscr{B}^\perp)^\perp \subseteq \mathscr{B}$ and hence $(\mathscr{A}^\perp + \mathscr{B}^\perp)^\perp \subseteq \mathscr{A} \cap \mathscr{B}$. □

Remark A.2: We recall that in a Hilbert space we have $\mathscr{A}^{\perp\perp} = \bar{\mathscr{A}}$ where $\bar{\mathscr{A}}$ denotes the closure of $\mathscr{A}$. Given any pair of subspaces $\mathscr{A}, \mathscr{B} \subseteq \mathscr{X}$ we have the inclusion $(\mathscr{A} \cap \mathscr{B})^\perp \supseteq \mathscr{A}^\perp + \mathscr{B}^\perp$. If $\mathscr{A}$ and $\mathscr{B}$ are closed, we get an equality if and only if $\mathscr{A}^\perp + \mathscr{B}^\perp$ is a closed subspace of $\mathscr{X}'$. In fact from property *ii*) of lemma A.6 we infer that $(\mathscr{A} \cap \mathscr{B})^\perp = (\mathscr{A}^\perp + \mathscr{B}^\perp)^{\perp\perp}$.

Further, by virtue of property *i*) of lemma A.6, any pair of subspaces $\mathscr{A}, \mathscr{B} \subseteq \mathscr{X}$ will meet the relation $(\mathscr{A} + \mathscr{B})^{\perp\perp} = (\mathscr{A}^\perp \cap \mathscr{B}^\perp)^\perp$. Hence the equality $\mathscr{A} + \mathscr{B} = (\mathscr{A}^\perp \cap \mathscr{B}^\perp)^\perp$ holds if and only if $\mathscr{A} + \mathscr{B}$ is closed in $\mathscr{X}$.

A useful criterion for the closedness of the sum of two closed subspaces is provided by the next result [13].

Proposition A.7. Closedness of the sum of two closed subspaces: *Let $\mathscr{A}$ and $\mathscr{B}$ be closed subspaces of a Hilbert space $\mathscr{X}$ with one of them finite dimensional. Then the subspace $\mathscr{A} + \mathscr{B}$ is closed.*

We can now prove a deep result which has been referred to in the paper (see [10] theorem II.15 for a proof valid in Banachs spaces).

Proposition A.8. Closedness of the sum of orthogonal complements: *Let us consider two closed subspaces $\mathscr{A}$ and $\mathscr{B}$ of an Hilbert space $\mathscr{X}$, and their orthogonal complements $\mathscr{A}^\perp$ and $\mathscr{B}^\perp$ in the dual Hilbert space $\mathscr{X}'$.*
Then $\mathscr{A} + \mathscr{B}$ is closed in $\mathscr{X}$ if and only if $\mathscr{A}^\perp + \mathscr{B}^\perp$ is closed in $\mathscr{X}'$.

Proof: By virtue of lemma A.6 the following equivalences hold true:

$$\begin{aligned} &\text{i)}\ \mathscr{A} + \mathscr{B} \quad \text{closed} \iff \text{ii)}\quad \mathscr{A} + \mathscr{B} = (\mathscr{A} + \mathscr{B})^{\perp\perp} = (\mathscr{A}^\perp \cap \mathscr{B}^\perp)^\perp\,, \\ &\text{iii)}\ \mathscr{A}^\perp + \mathscr{B}^\perp \quad \text{closed} \iff \text{iv)}\quad \mathscr{A}^\perp + \mathscr{B}^\perp = (\mathscr{A}^\perp + \mathscr{B}^\perp)^{\perp\perp} = (\mathscr{A} \cap \mathscr{B})^\perp\,. \end{aligned}$$

Let us now show that *i*) ⇒ *iv*).

Being $(\mathscr{A} \cap \mathscr{B})^\perp = (\mathscr{A}^\perp + \mathscr{B}^\perp)^{\perp\perp} \supseteq \mathscr{A}^\perp + \mathscr{B}^\perp$ it suffices to prove the converse inclusion $(\mathscr{A} \cap \mathscr{B})^\perp \subseteq \mathscr{A}^\perp + \mathscr{B}^\perp$.

Since $\mathscr{A} + \mathscr{B}$ is closed, proposition A.4 ensures that there exists a constant $c > 0$ such that any $\mathbf{x} \in \mathscr{A} + \mathscr{B}$ admits a decomposition of the kind

$$\mathbf{x} = \mathbf{a} + \mathbf{b} \quad \text{with} \quad \mathbf{a} \in \mathscr{A}\,; \qquad \mathbf{b} \in \mathscr{B}\,; \qquad \|\mathbf{a}\|_{\mathscr{X}} \le c\|\mathbf{x}\|_{\mathscr{X}}\,; \qquad \|\mathbf{b}\|_{\mathscr{X}} \le c\|\mathbf{x}\|_{\mathscr{X}}\,.$$

Now let $\mathbf{f} \in (\mathscr{A} \cap \mathscr{B})^\perp$. Then we can define the linear functional $\boldsymbol{\phi}$ on $\mathscr{A} + \mathscr{B}$:

$$\langle \boldsymbol{\phi}, \mathbf{x} \rangle \coloneqq \langle \mathbf{f}, \mathbf{a} \rangle \quad \forall \mathbf{x} \in \mathscr{A} + \mathscr{B}$$

since the definition does not depend on the decomposition of $\mathbf{x}$. Further $\boldsymbol{\phi}$ is continuous since

$$|\langle \boldsymbol{\phi}, \mathbf{x} \rangle| = |\langle \mathbf{f}, \mathbf{a} \rangle| \le \|\mathbf{f}\|_{\mathscr{X}'}\, \|\mathbf{a}\|_{\mathscr{X}} \le c\|\mathbf{f}\|_{\mathscr{X}'}\, \|\mathbf{x}\|_{\mathscr{X}} \quad \forall \mathbf{x} \in \mathscr{A} + \mathscr{B}\,.$$

Let $\boldsymbol{\Pi}$ be the orthogonal projector on $\mathscr{A} + \mathscr{B}$ in $\mathscr{X}$. The continuous linear functional $\boldsymbol{\varphi} \in \mathscr{X}'$ defined by

$$\langle \boldsymbol{\varphi}, \mathbf{x} \rangle \coloneqq \langle \boldsymbol{\phi}, \boldsymbol{\Pi}\mathbf{x} \rangle \quad \forall \mathbf{x} \in \mathscr{X}$$

is such that

$$\boldsymbol{\varphi} \in \mathscr{B}^\perp\,; \qquad \mathbf{f} - \boldsymbol{\varphi} \in \mathscr{A}^\perp\,.$$

The implication iii) ⇒ ii) is proved in an analogous way. □

References

1 BABUŠKA, I.: The finite element method with lagrangian multipliers. Numer. Math. **20** (1973), 179–192.
2 BREZZI, F.: On the existence, uniqueness and approximation of saddle point problems arising from lagrangian multipliers. R.A.I.R.O. Anal. Numer. **8** (1974), 129–151.
3 BREZZI, F.; FORTIN, M.: Mixed and hybrid finite element methods. Springer-Verlag, New York 1991.
4 ANTMAN, S. S.; MARLOW, R. S.: Material constraints, Lagrange multipliers, and compatibility. Applications to rod and shell theories. Arch. Rat. Mech. Anal. **116** (1991), 257–299.
5 MARLOW, R. S.: Global material constraints and their associated reactive stress. J. Elasticity **33** (1993), 203–212.
6 ROMANO, G.; MAROTTI DE SCIARRA, F.; DIACO, M.: Mixed formulation, localization and LBB condition. Proc. SACAM '98 Univ. Cape Town, South Africa 1998.
7 FICHERA, G.: Existence theorems in elasticity. In: Handbuch der Physik. Vol. VI/a. Springer-Verlag, Berlin 1972.
8 DUVAUT, G.; LIONS, J. L.: Les inéquations en mécanique et en physique. Dunod, Paris 1972.
9 YOSIDA, K.: Functional analysis. Springer-Verlag, New York 1980.
10 BREZIS, H.: Analyse fonctionnelle. Théorie et applications. Masson Editeur, Paris 1983.
11 BREZZI, F.; GILARDI, G.: FEM mathematics. In: Finite-Element Handbook. Kardestuncer editor, McGraw-Hill, 1987, 1.1–1.76.
12 PANAGIOTOPOULOS, P. D.: Inequality problems in mechanics and applications. Birkhäuser, Basel 1985.
13 ROMANO, G.; DIACO, M.: A variational theory of linear structural problems. Internat. J. Solids Struct. (submitted).
14 BABUŠKA, I.; AZIZ, A. K.: The mathematical foundations of the finite element method with application to partial differential equations. Part I. Academic Press, New York 1972.
15 ODEN, J. T.: Problems with constraints. In: The Finite Element Handbook. 1987. Ch. 9.4, 2.321–2.326.
16 ROBERTS, J. E.; THOMAS, J. M.: Mixed and hybrid methods. In: CIARLET, P. G.; LIONS, J. L. (ed.): Handbook of Numerical Analysis. Vol. II. Finite Element Methods. Part I. North-Holland, Amsterdam 1989.
17 ROMANO, G.; ROSATI, L.; DIACO, M.: A general theory of mixed problems with applications to elasticity. Math. Mech. Solids (submitted).

18 Nicolaides, R. A.: Existence, uniqueness and approximation for generalized saddle point problems. SIAM J. Numer. Anal. **19** (1982), 349–357.
19 Bernardi, C.; Canuto, C.; Maday, Y.: Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem. SIAM J. Numer. Anal. **25** (1988), 1237–1271.
20 Ladyszhenskaya, O. A.: The mathematical theory of viscous incompressible flow. Gordon and Breach, New York 1969.
21 Temam, R.: Navier-Stokes equations. North-Holland, Amsterdam 1977.
22 Romano, G.; Diaco, M.: A discussion on stress-constrained continua. J. Elasticity (submitted).

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