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# A GEOMETRIC APPROACH TO THE ALGORITHMIC TANGENT STIFFNESS

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**Abstract.** *A geometrical approach is proposed to estimate the algorithmic tangent stiffness in the evolutive analysis of elastoplastic structures based on the elastic-trial, plastic-correction method. The algorithmic tangent stiffness is evaluated in terms of the derivative of the nonlinear projector on the yield locus in complementary elastic energy norm. A geometric formula is provided to express the derivative of the nonlinear projector in terms of the linear projector on the tangent hyperplane and of the shape operator of the parallel folium to the yield hypersurface thru the trial stress point. The geometric expression reveals that the algorithmic tangent stiffness is a symmetric operator and that its evaluation does not require operator inversions. An effective approximation is provided by substituting the parallel folium with the level set of the yield function passing thru the trial stress point, still retaining the asympthotic quadratic converge of the iteration algorithm.*

## 1 Introduction

The elastoplastic tangent stiffness is the linear operator which provides the stress rate corresponding to a prescribed strain rate. As such, it plays a central role in the computational aspects of elastoplastic problems. According to the usual approach to the nonlinear evolutive analysis of elastoplastic models, a finite time-step is considered and the evolution law describing the constitutive behavior is reformulated as a finite step flow rule. The solution of the nonlinear problem is then performed by an iterative scheme in which the respect of the plastic flow rule is imposed at each iteration after a linearized guess of the constitutive behavior. At the initial iteration a linear elastic stiffness is assumed, while in the subsequent ones a reduced stiffness is adopted to take into account the plastic response due to overstresses. At each iteration the resulting trial stress-state is projected on the yield hypersurface, according to the normality rule, to impose the respect of the plastic condition. Normality is meant with respect to the metric in complementary elastic energy. This leads to the evaluation of a residual system of generalized forces which provides the inital datum for the next step of the iterative loop. The fixed point of the algorithm leads to the solution of the nonlinear elastoplastic equilibrium problem. In each iteration, the incremental response is governed by a linear law which relates the strain rate to the stress rate. In a sharp geometric picture, the strain rate is a vector whose base point is the trial stress-state, while the stress rate is a vector tangent to the yield hypersurface at the projected stress-state.

The relevant linear operator is dubbed the algorithmic tangent stiffness, first introduced by Simo and Taylor in [1]. They showed that the adoption of the algorithmic tangent stiffness leads to a significant improvement of the asympthotic convergence rate. The expression of the algorithmic tangent stiffness provided in [1], and in all the subsequent references to their contribution, was based on an explicit formulation of the elastoplastic constitutive law in terms of a plastic scalar multiplier. The details of their analysis will be reproduced hereafter, for sake of comparison with the proposed geometric approach. The merits of the geometric approach are twofold. It provides a more direct formula for the algorithmic tangent stiffness, showing that it differs from the rate tangent stiffness by a corrective subtractive term, which depends on the curvature of the yield hypersurface. No matrix inversions, appearing in the original formula developed in [1], are required. It is important to grasp the geometrical interpretation of the difference between the rate elastoplastic tangent stiffness and the algorithmic elastoplastic tangent stiffness. The former applies to an ideal time-continuous scheme of iterations and relates the strain rate, based at the current stress state on the yield hypersurface, to the stress rate which is tangent to the yield hypersurface at the same point. As will be shown below, the algorithmic tangent stiffness is not greater than the rate tangent stiffness and the two tangent stiffnesses coincide if and only if the yield hypersurface is locally flat. This reveals why and when a better convergence is got by adopting the algorithmic one.

The geometrical analysis developed in the present paper is based on a formulation of the constitutive problem in terms of the nonlinear projector, in complementary elastic energy, on the convex elastic domain. The algorithmic tangent stiffness is evaluated as the composition between the derivative of the nonlinear projector and the elastic stiffness. The key point consists in the evaluation of the derivative of the nonlinear projector. A direct geometric argument, based on hypersurface theory, shows that the derivative can be expressed as the difference between the linear projector on the hyperplane tangent at the trial stress point and the shape operator of the parallel hypersurface passing thru the trial stress-state, multiplied by the distance between the trial stress and the projected stress, evaluated in the complementary elastic norm.

The composition of the linear projector with the elastic stiffness is in fact the rate elastoplastic tangent stiffness. Since the analytic expression of the parallel hypersurface thru the trial stress point is available only in special cases, an effective procedure consists in substituting it with the level set of the yield function passing thru the trial stress point. In this way, the exact expression of the algorithmic stiffness is got when the level sets of the yield function are homothetic hypersurfaces, as in von Mises plasticity criterion, and a simple useful approximation is obtained in the general case. Indeed the proposed procedure greatly simplifies the computations while preserving the benefit of an improved convergence rate, since it takes effectively into account the curvature of the yield hypersurface, thus leading to a reduced tangent stiffness, in comparison with the rate tangent stiffness.

## 2 Elastoplastic constitutive law

Let us consider a general model of a continuous structural model with an elastoplastic constitutive behavior. The constitutive relation is written in terms of stress and strain vectors belonging to a finite dimensional linear space $\mathbb{S}$ endowed with an inner product $\mathbf{g} \in BL(\mathbb{S}\,;\mathbb{S})$ .

The elastic behavior is governed by the linear symmetric and positive definite operators $\mathbf{C} \in BL(\mathbb{S}\,;\mathbb{S})$ and $\mathbf{E} \in BL(\mathbb{S}\,;\mathbb{S})$ , respectively the elastic compliance and the elastic stiffness, with $\mathbf{C} = \mathbf{E}^{-1}$ which induce in $\mathbb{S}$ two metric tensors and the associated norms:

$$\mathbf{g}_{\mathbf{C}}(\mathbf{h}_1, \mathbf{h}_2) := \mathbf{g}(\mathbf{C}\mathbf{h}_1, \mathbf{h}_2)\,, \quad \|\mathbf{h}\,\|^2_{\mathbf{C}} := \mathbf{g}_{\mathbf{C}}(\mathbf{h}, \mathbf{h})\,, \qquad \forall\, \mathbf{h}, \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{S}\,,$$

$$\mathbf{g}_{\mathbf{E}}(\mathbf{h}_1, \mathbf{h}_2) := \mathbf{g}(\mathbf{E}\mathbf{h}_1, \mathbf{h}_2)\,, \quad \|\mathbf{h}\,\|^2_{\mathbf{E}} := \mathbf{g}_{\mathbf{E}}(\mathbf{h}, \mathbf{h})\,, \qquad \forall\, \mathbf{h}, \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{S}\,.$$

The elastoplastic problem is assumed to be characterized by a regular convex elastic domain $\mathcal{K}$ of the stress space $\mathbb{S}$, defined in terms of a twice differentiable convex yield function $\varphi \in \mathrm{C}^2(\mathbb{S}\,;\mathfrak{R})$ :

$$\mathcal{K} := \{\, \boldsymbol{\sigma} \in \mathbb{S} \mid \varphi(\boldsymbol{\sigma}) \le 0 \,\}\,,$$

Let us consider a discrete time integration scheme and denote by $\boldsymbol{\varepsilon}_0, \mathbf{p}_0, \boldsymbol{\sigma}_0 \in \mathbb{S}$ the total strain, the plastic strain and the stress state provided by the solution of the elastoplastic problem at the end of a time-step. In the subsequent time-step, the iterative algorithm for the solution of the nonlinear elastoplastic problem provides a sequence of total strains, starting with a purely elastic initial guess of the structural response. We denote by $\boldsymbol{\varepsilon} \in \mathbb{S}$ the driving total strain predicted at the current iteration and by the pair $\mathbf{p}, \boldsymbol{\sigma} \in \mathbb{S}$ the corresponding solution of the constitutive problem in terms of plastic strain and stress state.

Approximating the flow rule according to a fully implicit integration scheme, the elastoplastic constitutive problem is written as

$$\begin{cases} \mathbf{E}\,\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\,\mathbf{p}\,, \\[2ex] \mathbf{p} - \mathbf{p}_0 \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})\,, \end{cases}$$

where $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})$ is the outward convex normal cone in $\{\mathbb{S}\,, \mathbf{g}\}$ to the elastic domain at the point $\boldsymbol{\sigma} \in \mathcal{K}$ , given by

$$\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}) := \{\, \mathbf{h} \in \mathbb{S} \,:\, \mathbf{g}(\mathbf{h}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \le 0 \quad \forall\, \boldsymbol{\tau} \in \mathcal{K} \,\}\,.$$

At points internal to $\mathcal{K}$ the normal cone $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})$ degenerates to the null set and at regular points of the boundary $\partial\mathcal{K}$ it reduces to an orthogonal straight half-line.

Denoting by $\mathbb{S}_{\mathbf{C}} := \{\mathbb{S}, \mathbf{g}_{\mathbf{C}}\}$ the stress space endowed with the inner product in complementary elastic energy and setting:

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\,, \qquad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0\,, \qquad \Delta\boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_0\,,$$

the elastoplastic constitutive problem may be rewritten as (see Fig. 1):

$$\begin{cases} \mathbf{E}\,\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\sigma} + \mathbf{E}\,\Delta\mathbf{p}\,, \\ \Delta\mathbf{p} \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})\,, \end{cases} \iff \begin{cases} \boldsymbol{\sigma}_0 + \mathbf{E}\,\Delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\,\Delta\mathbf{p}\,, \\ \mathbf{E}\,\Delta\mathbf{p} \in \mathcal{N}^{\mathbf{C}}_{\mathcal{K}}(\boldsymbol{\sigma})\,, \end{cases}$$

where $\mathcal{N}^{\mathbf{C}}_{\mathcal{K}}(\boldsymbol{\sigma})$ is the outward normal cone at $\boldsymbol{\sigma} \in \mathcal{K}$ in $\mathbb{S}_{\mathbf{C}}$:

$$\mathcal{N}^{\mathbf{C}}_{\mathcal{K}}(\boldsymbol{\sigma}) := \{\, \mathbf{h} \in \mathbb{S} \,:\, \mathbf{g}_{\mathbf{C}}(\mathbf{h}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \leq 0 \quad \forall\, \boldsymbol{\tau} \in \mathcal{K} \,\}\,.$$

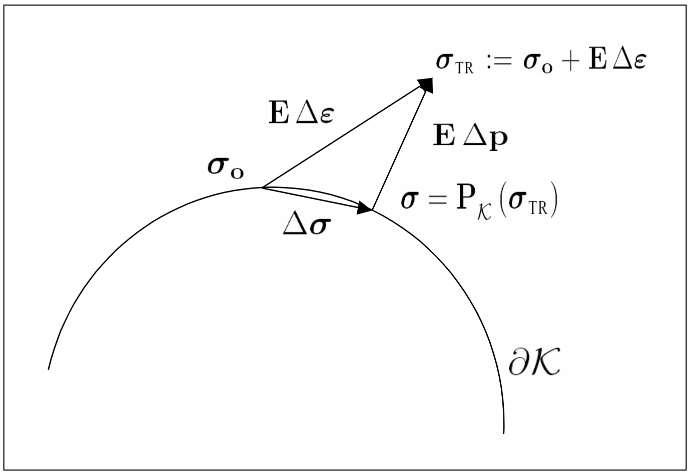


Figure 1: Projection in $\mathbb{S}_{\mathbf{C}}$

Denoting by $\mathbf{P}_{\mathcal{K}}$ the orthogonal projector in $\mathbb{S}_{\mathbf{C}}$ onto $\mathcal{K}$ and defining the trial stress by $\boldsymbol{\sigma}_{\mathrm{TR}} := \boldsymbol{\sigma}_{\mathbf{o}} + \mathbf{E}\,\Delta\boldsymbol{\varepsilon}$, the constitutive problem may be conveniently rewritten in the geometric form

$$\boldsymbol{\sigma} = \mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})\,, \qquad \Delta\mathbf{p} = \mathbf{C}\,(\boldsymbol{\sigma}_{\mathrm{TR}} - \boldsymbol{\sigma})\,.$$

In most treatments the finite step elasto-plastic constitutive law is formulated in terms of a scalar plastic multiplier and of the gradient of the yield function, fulfilling a complementarity rule.

Let us recall that, if $d_{\mathbf{h}}\varphi(\boldsymbol{\sigma})$ is the directional derivative at $\boldsymbol{\sigma} \in \mathbb{S}$ along the vector $\mathbf{h} \in \mathbb{S}$, the gradient $\nabla\varphi(\boldsymbol{\sigma}) \in \mathbb{S}$ at $\boldsymbol{\sigma} \in \mathbb{S}$ according to the metric $\mathbf{g}$ is defined by

$$\mathbf{g}\,(\nabla\varphi(\boldsymbol{\sigma}), \mathbf{h}) = d_{\mathbf{h}}\varphi(\boldsymbol{\sigma})\,, \quad \forall\, \mathbf{h} \in \mathbb{S}\,.$$

The finite step elasto-plastic constitutive law may then be written as

$$\begin{cases} \Delta\mathbf{p} = \lambda\,\nabla\varphi(\boldsymbol{\sigma}) \\ \mathbf{C}\,\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\varepsilon} - \Delta\mathbf{p} \\ \lambda \geq 0 \quad \varphi(\boldsymbol{\sigma}) \leq 0 \quad \lambda\,\varphi(\boldsymbol{\sigma}) = 0\,, \end{cases}$$

and, in a plastic loading process, i.e. when $\varphi(\boldsymbol{\sigma}_{\text{TR}}) \geq 0$, it gives

$$\begin{cases} \mathbf{C}\,\boldsymbol{\sigma} + \lambda\,\nabla\varphi(\boldsymbol{\sigma}) = \mathbf{C}\,\boldsymbol{\sigma}_{\text{TR}}\,, \quad \lambda \geq 0 \\ \varphi(\boldsymbol{\sigma}) = 0\,. \end{cases}$$

For further reference, we note that

$$\boldsymbol{\sigma}_{\text{TR}} - \boldsymbol{\sigma} = \mathbf{E}\,\Delta\mathbf{p} = \lambda\,\mathbf{E}\,\nabla\varphi(\boldsymbol{\sigma})\,.$$

## 3 Algorithmic tangent stiffness

Taking the derivative with respect to the evolution parameter, from the formula $\boldsymbol{\sigma} = \mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\text{TR}})$, with $\boldsymbol{\sigma}_{\text{TR}} := \boldsymbol{\sigma}_{\mathbf{o}} + \mathbf{E}\,\Delta\boldsymbol{\varepsilon}$, we get the incremental law

$$\dot{\boldsymbol{\sigma}} = (d\,\mathbf{P}_{\mathcal{K}}\,(\boldsymbol{\sigma}_{\text{TR}}))\,\mathbf{E}\,\dot{\boldsymbol{\varepsilon}}\,.$$

The linear operator $\mathbf{K} := (d\,\mathbf{P}_{\mathcal{K}}\,(\boldsymbol{\sigma}_{\text{TR}}))\,\mathbf{E}$ is the algorithmic elastoplastic tangent stiffness.

The classical expression of the algorithmic tangent stiffness provided in [1] is based on the finite step elastoplastic constitutive law formulated in terms of a plastic multiplier. Indeed, taking the derivative with respect to the evolution parameter, from the formula $\mathbf{C}\,\boldsymbol{\sigma} + \lambda\,\nabla\varphi(\boldsymbol{\sigma}) = \mathbf{C}\,\boldsymbol{\sigma}_{\text{TR}}$, we get [3]:

$$\mathbf{C}\,\dot{\boldsymbol{\sigma}} + \dot{\lambda}\,\nabla\varphi(\boldsymbol{\sigma}) + \lambda\,\nabla^2\varphi(\boldsymbol{\sigma})\,\dot{\boldsymbol{\sigma}} = \mathbf{C}\,\dot{\boldsymbol{\sigma}}_{\text{TR}} = \dot{\boldsymbol{\varepsilon}}\,,$$

where $\nabla^2\varphi(\boldsymbol{\sigma})$ is the hessian, defined by

$$\mathbf{g}\,(\nabla^2\varphi(\boldsymbol{\sigma})\,\mathbf{h}_1, \mathbf{h}_2) = d_{\mathbf{h}_2}\mathbf{g}\,(\nabla\varphi(\boldsymbol{\sigma}), \mathbf{h}_1)\,, \quad \forall\,\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{S}\,.$$

Under plastic loading the stress point is bound to move along the boundary of the elastic domain, so that

$$\dot{\boldsymbol{\sigma}} \in \mathbb{T}_{\partial\mathcal{K}}(\boldsymbol{\sigma}) \quad \Longleftrightarrow \quad \mathbf{g}\,(\dot{\boldsymbol{\sigma}}, \nabla\varphi(\boldsymbol{\sigma})) = 0\,.$$

Substituting in the previous expression we get the formula for the plastic multiplier rate:

$$\dot{\lambda} = \frac{\mathbf{g}\,(\mathbf{H}\,\dot{\boldsymbol{\varepsilon}}, \nabla\varphi(\boldsymbol{\sigma}))}{\mathbf{g}\,(\mathbf{H}\,\nabla\varphi(\boldsymbol{\sigma}), \nabla\varphi(\boldsymbol{\sigma}))}\,.$$

Hence, setting

$$\begin{aligned} \mathbf{H} \; &:= [\mathbf{C} + \lambda\,\nabla^2\varphi(\boldsymbol{\sigma})]^{-1}\,, \\ \mathbf{N}_{\mathbf{H}} &:= \mathbf{H}\,\nabla\varphi(\boldsymbol{\sigma})\,, \\ \beta \; &:= \mathbf{g}\,(\mathbf{H}\,\nabla\varphi(\boldsymbol{\sigma}), \nabla\varphi(\boldsymbol{\sigma}))\,, \end{aligned}$$

the algorithmic tangent stiffness is given by

$$\mathbf{K} = \mathbf{H} - \frac{\mathbf{N}_{\mathbf{H}} \otimes \mathbf{N}_{\mathbf{H}}}{\beta}\,.$$

Apparently the evaluation of $\mathbf{H}$ requires a matrix inversion.

We propose here an alternative procedure for the evaluation of the algorithmic tangent stiffness which avoids matrix inversions. To this end, we start from the geometric formula for the algorithmic elastoplastic tangent stiffness:

$$\mathbf{K} = (d\,\mathbf{P}_{\mathcal{K}}\,(\boldsymbol{\sigma}_{\text{TR}}))\,\mathbf{E}\,.$$

To provide an expression for the derivative of the projector $\mathbf{P}_{\mathcal{K}}$ it is expedient to consider the foliation of the space $\mathbb{S}_{\mathbf{C}}$ induced by the boundary $\partial\mathcal{K}$ of the convex elastic domain $\mathcal{K}$. Each folium of the foliation is a hypersurface parallel to $\partial\mathcal{K}$, obtained by shifting its points outward in the normal direction of a fixed amount, as depicted in Fig. 2.

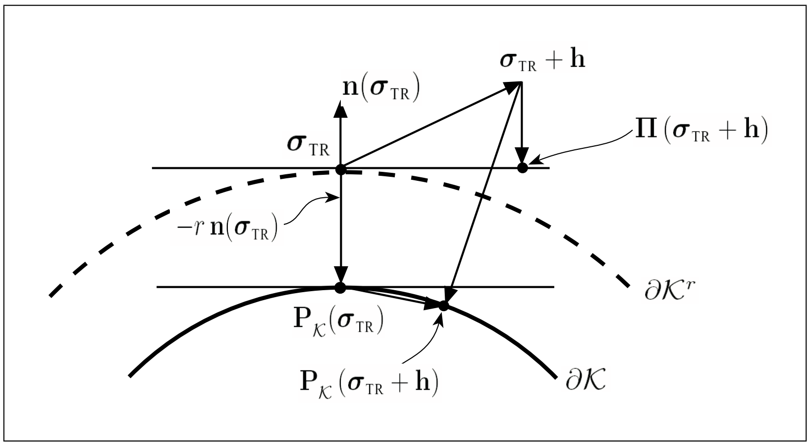


Figure 2: Projectors

The derivative of the projector $\mathbf{P}_{\mathcal{K}}$ is conveniently computed by considering only tangent directions to the relevant folium since the derivative vanishes along the normal direction. To this end we denote by

$$r(\mathbf{s}) := \|\mathbf{s} - \mathbf{P}_{\mathcal{K}}(\mathbf{s})\,\|_{\mathbf{C}}\,,$$

the distance in complementary elastic energy between a stress point $\mathbf{s} \in \mathbb{S}_{\mathbf{C}}$ and its projection on $\mathcal{K}$, and by $\partial\mathcal{K}^r$ the folium passing thru $\mathbf{s}$.

Hence we have that

$$\mathbf{s} = \mathbf{P}_{\mathcal{K}}(\mathbf{s}) + r(\mathbf{s})\,\mathbf{n}(\mathbf{s})\,, \quad \mathbf{s} \in \partial\mathcal{K}^r\,.$$

where $\mathbf{n}(\mathbf{s})$ is the outward unit normal in $\mathbb{S}_{\mathbf{C}}$ to the folium $\partial\mathcal{K}^r$.

Denoting by $\mathbb{T}_{\partial\mathcal{K}^r}(\mathbf{s})$ the tangent hyperplane to the folium $\partial\mathcal{K}^r$ at the point $\mathbf{s} \in \partial\mathcal{K}^r$, taking the derivative along tangents vectors to the folium and observing that $r(\mathbf{s})$ is a constant function on $\partial\mathcal{K}^r$, we get

$$\mathbf{h} = d\,\mathbf{P}_{\mathcal{K}}(\mathbf{s})\mathbf{h} + r(\mathbf{s})\,\mathbf{S}(\mathbf{s})\mathbf{h}\,, \quad \forall\,\mathbf{h} \in \mathbb{T}_{\partial\mathcal{K}^r}(\mathbf{s})\,,$$

where $\mathbf{S}(\mathbf{s})$ is the shape operator of the folium passing thru $\mathbf{s} \in \mathbb{S}_{\mathbf{C}}$ and $d\,\mathbf{P}_{\mathcal{K}}(\mathbf{s})\mathbf{n}(\mathbf{s}) = 0$ and $\mathbf{S}(\mathbf{s})\mathbf{n}(\mathbf{s}) = 0$ (see section 5).

Denoting by $\boldsymbol{\Pi}(\mathbf{s})$ the linear orthogonal projector in $\mathbb{S}_{\mathbf{C}}$ on $\mathbb{T}_{\partial\mathcal{K}^r}(\mathbf{s})$ and by $\otimes$ the tensor product in $\mathbb{S}_{\mathbf{C}}$, we have that

$$\boldsymbol{\Pi}(\mathbf{s})\mathbf{h} = (\mathbf{I} - \mathbf{n}(\mathbf{s}) \otimes \mathbf{n}(\mathbf{s}))\mathbf{h} = \mathbf{h} - \mathbf{g}_{\mathbf{C}}\,(\mathbf{h}, \mathbf{n}(\mathbf{s}))\mathbf{n}(\mathbf{s}) = \mathbf{h}\,, \quad \forall\,\mathbf{h} \in \mathbb{T}_{\partial\mathcal{K}^r}(\mathbf{s})\,,$$

and hence we get the formula

$$\boldsymbol{\Pi}(\mathbf{s}) = d\,\mathbf{P}_{\mathcal{K}}(\mathbf{s}) + r(\mathbf{s})\,\mathbf{S}(\mathbf{s})\,.$$

Accordingly, the algorithmic tangent stiffness is given by

$$\mathbf{K} = (d\,\mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\rm TR}))\,\mathbf{E} = (\mathbf{\Pi}(\boldsymbol{\sigma}_{\rm TR}) - r(\boldsymbol{\sigma}_{\rm TR})\,\mathbf{S}(\boldsymbol{\sigma}_{\rm TR}))\,\mathbf{E}\,,$$

with

$$r(\boldsymbol{\sigma}_{\rm TR}) := \|\boldsymbol{\sigma}_{\rm TR} - \mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\rm TR})\,\|_{\mathbf{C}} = \|\mathbf{E}\Delta\mathbf{p}\,\|_{\mathbf{C}} = \|\Delta\mathbf{p}\,\|_{\mathbf{E}} = \lambda\,\|\mathbf{E}\,\nabla\varphi(\mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\rm TR}))\,\|_{\mathbf{C}}\,,$$

while the rate tangent stiffness is the term

$$\mathbf{K}_{\rm RATE} = \mathbf{\Pi}(\boldsymbol{\sigma}_{\rm TR})\,\mathbf{E}\,.$$

From these geometrically motivated formulas, it is apparent that the algorithmic tangent stiffness is not greater than the rate tangent stiffness. Indeed the shape operator is the hessian of the distance function from the yield hypersurface and the convexity of the elastic domain ensures that the hessian is a positive operator (see section 5).

The two tangent stiffnesses coincide if and only if the yield hypersurface is locally flat, since then the shape operator vanishes.

This reveals why and when a better convergence is got by adopting the algorithmic tangent stiffness instead of the rate tangent stiffness.

## 4 Computational issues

To provide an approximate explicit expression of the algorithmic elastoplastic tangent stiffness, the shape operator $\mathbf{S}(\boldsymbol{\sigma}_{\rm TR})$ must be evaluated in terms of the yield function.

An exact evaluation would require the analytical expression of the parallel hypersurface $\partial\mathcal{K}^r$ passing through the trial stress point.

We propose here a strategy which does not require this analytical expression, preserves the main advantages of the algorithmic tangent stiffness, while avoiding matrix inversions.

The trick consists in the substitution of the parallel hypersurface $\partial\mathcal{K}^r$ with the corresponding level set of the yield function (the one passing thru the trial stress point) (see Fig. 3).

This substitution provides the exact expression of the algorithmic tangent stiffness when the level sets of the yield function are homothetic hypersurfaces, as in von Mises plasticity criterion, and a simple effective approximation otherwise.

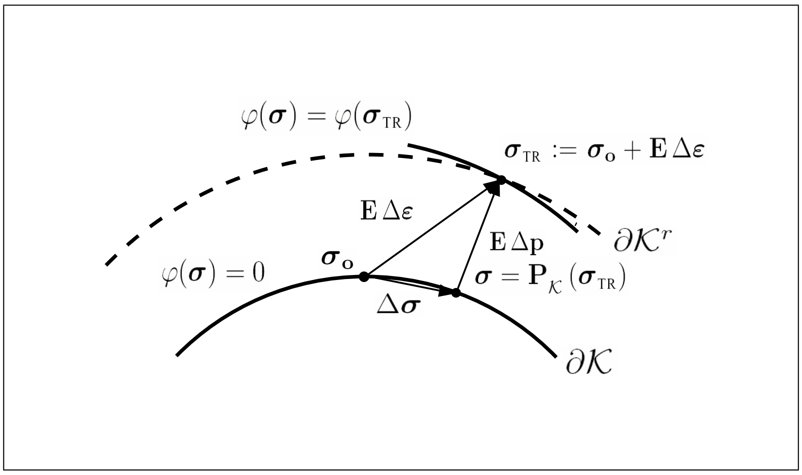


Figure 3: Parallel hypersurface

To evaluate the expression of the shape operator of a level set of $\boldsymbol{\varphi}$ in the space $\mathbb{S}_{\mathbf{C}}$, we recall that $\mathbf{S}_{\varphi}(\mathrm{s}) = \nabla_{\mathbf{C}}\mathbf{n}_{\varphi}(\mathrm{s})$ where

$$\mathbf{n}_{\varphi}(\mathbf{s}) := \frac{\nabla_{\mathbf{C}}\varphi(\mathbf{s})}{\|\nabla_{\mathbf{C}}\varphi(\mathbf{s})\|_{\mathbf{C}}},$$

is the unit normal in $\mathbb{S}_{\mathbf{C}}$ to the level set.

Let us denote by $\boldsymbol{\Pi}_{\varphi}(\mathbf{s})$ the linear orthogonal projector in $\mathbb{S}_{\mathbf{C}}$ on the tangent hyperplane to the level set of $\boldsymbol{\varphi}$ at $\mathrm{s} \in \mathbb{S}_{\mathbf{C}}$.

The directional derivative of $\mathbf{n}_{\varphi}(\mathrm{s})$ along a tangent direction $\boldsymbol{\Pi}_{\varphi}(\mathrm{s})\,\mathbf{h}$ is then given by:

$$\begin{aligned}
&\nabla_{\mathbf{C}}\mathbf{n}_{\varphi}(\mathbf{s})\cdot\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\mathbf{h} = \\
&\quad = \left(\frac{\nabla^2_{\mathbf{C}}\varphi(\mathbf{s})\cdot\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\mathbf{h}}{\|\nabla_{\mathbf{C}}\varphi(\mathbf{s})\|_{\mathbf{C}}} - \frac{\mathbf{g}\left(\nabla_{\mathbf{C}}\varphi(\mathbf{s}), \nabla^2_{\mathbf{C}}\varphi(\mathbf{s})\cdot\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\mathbf{h}\right)}{\|\nabla_{\mathbf{C}}\varphi(\mathbf{s})\|^3_{\mathbf{C}}}\,\nabla_{\mathbf{C}}\varphi(\mathbf{s})\right) \\
&\quad = \frac{1}{\|\nabla_{\mathbf{C}}\varphi(\mathbf{s})\|_{\mathbf{C}}}\Big(\mathbf{I} - \mathbf{n}_{\varphi}(\mathbf{s})\otimes\,\mathbf{n}_{\varphi}(\mathbf{s})\Big)\,\nabla^2_{\mathbf{C}}\varphi(\mathbf{s})\cdot\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\mathbf{h} \\
&\quad = \frac{1}{\|\nabla_{\mathbf{C}}\varphi(\mathbf{s})\|_{\mathbf{C}}}\,\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\nabla^2_{\mathbf{C}}\varphi(\mathbf{s})\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\mathbf{h}\,,
\end{aligned}$$

where $\otimes$ is the tensor product in $\mathbb{S}_{\mathbf{C}}$.

The expression of the shape operator $\mathbf{S}_{\varphi}(\mathrm{s}) = \nabla_{\mathbf{C}}\mathbf{n}_{\varphi}(\mathrm{s})\cdot\boldsymbol{\Pi}_{\varphi}(\mathrm{s})$ of the level set is then:

$$\mathbf{S}_{\varphi}(\mathbf{s}) = \frac{1}{\|\nabla_{\mathbf{C}}\varphi(\mathbf{s})\|_{\mathbf{C}}}\,\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\nabla^2_{\mathbf{C}}\varphi(\mathbf{s})\,\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,,$$

which is apparently a $\mathbf{g}_{\mathbf{C}}$-symmetric operator. Taking into account the relations between gradients and hessians according to the metrics $\mathbf{g}$ and $\mathbf{g}_{\mathbf{C}}$:

$$\begin{aligned}
&\mathbf{g}\left(\nabla\varphi(\mathbf{s}), \mathbf{h}\right) = \mathbf{g}\left(\mathbf{C}\,\nabla_{\mathbf{C}}\varphi(\mathbf{s}), \mathbf{h}\right) &&\Longleftrightarrow\quad \nabla_{\mathbf{C}}\varphi(\mathbf{s}) = \mathbf{E}\,\nabla\varphi(\mathbf{s}) \\
&\mathbf{g}\left(\nabla^2\varphi(\mathbf{s})\mathbf{h}_1, \mathbf{h}_2\right) = \mathbf{g}\left(\mathbf{C}\,\nabla^2_{\mathbf{C}}\varphi(\mathbf{s})\mathbf{h}_1, \mathbf{h}_2\right) &&\Longleftrightarrow\quad \nabla^2_{\mathbf{C}}\varphi(\mathbf{s}) = \mathbf{E}\,\nabla^2\varphi(\mathbf{s})\,,
\end{aligned}$$

the shape operator may also be written as

$$\mathbf{S}_{\varphi}(\mathbf{s}) = \frac{1}{\|\nabla\varphi(\mathbf{s})\|_{\mathbf{E}}}\,\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,\mathbf{E}\,\nabla^2\varphi(\mathbf{s})\,\boldsymbol{\Pi}_{\varphi}(\mathbf{s})\,.$$

The approximate expression of the algorithmic tangent stiffness is given by

$$\mathbf{K}_{\varphi} = \left(d\,\mathbf{P}_{\mathcal{K}}\left(\boldsymbol{\sigma}_{\text{TR}}\right)\right)\mathbf{E} = \left(\boldsymbol{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\text{TR}}) - r(\boldsymbol{\sigma}_{\text{TR}})\,\mathbf{S}_{\varphi}(\boldsymbol{\sigma}_{\text{TR}})\right)\mathbf{E}\,.$$

We remark that the algorithmic tangent stiffness is $\mathbf{g}$-symmetric due to the $\mathbf{g}$-symmetry property of $\boldsymbol{\Pi}_{\varphi}\mathbf{E}$, which follows from the $\mathbf{g}_{\mathbf{C}}$-symmetry of $\boldsymbol{\Pi}_{\varphi}$ since

$$\begin{aligned}
&\mathbf{g}_{\mathbf{C}}\left(\boldsymbol{\Pi}_{\varphi}\mathbf{h}_1, \mathbf{h}_2\right) = \mathbf{g}\left(\mathbf{C}\boldsymbol{\Pi}_{\varphi}\mathbf{h}_1, \mathbf{h}_2\right) = \mathbf{g}\left(\mathbf{h}_1, \mathbf{C}\boldsymbol{\Pi}_{\varphi}\mathbf{h}_2\right), \\
&\mathbf{g}\left(\boldsymbol{\Pi}_{\varphi}\mathbf{E}\mathbf{h}_1, \mathbf{h}_2\right) = \mathbf{g}\left(\boldsymbol{\Pi}_{\varphi}\mathbf{E}\mathbf{h}_1, \mathbf{C}\mathbf{E}\mathbf{h}_2\right) = \mathbf{g}\left(\mathbf{C}\boldsymbol{\Pi}_{\varphi}\mathbf{E}\mathbf{h}_1, \mathbf{E}\mathbf{h}_2\right) = \mathbf{g}\left(\mathbf{h}_1, \boldsymbol{\Pi}_{\varphi}\mathbf{E}\mathbf{h}_2\right).
\end{aligned}$$

## 5 Appendix

We provide here some definition and results from surface theory, referred to above.

Let $\Sigma$ be a hypersurface bounding a convex domain in the finite dimensional linear space $\{\mathbb{S}, \mathbf{g}\}$. The distance function from $\Sigma$ is the scalar function $f \in \mathrm{C}^2(\mathbb{S}; \Re)$ fulfilling the property:

$$\|\nabla f(\mathbf{s})\| = 1, \quad \forall \mathbf{s} \in \mathbb{S}.$$

Its level sets generate a foliation of the space $\{\mathbb{S}, \mathbf{g}\}$ into a family of parallel folii, that is hypersurfaces parallel to $\Sigma$ [2, 4].

The unit normal to a folium is the gradient of the distance function:

$$\mathbf{n}(\mathbf{s}) = \nabla f(\mathbf{s}).$$

The shape operator $\mathbf{S}(\mathbf{s})$ of a folium is the hessian of the distance function, defined by the relation:

$$\mathbf{g}(\mathbf{S}(\mathbf{s})\mathbf{h}_1, \mathbf{h}_2) := \mathbf{g}(\nabla^2 f(\mathbf{s})\mathbf{h}_1, \mathbf{h}_2) = \mathbf{g}(\nabla \mathbf{n}(\mathbf{s})\mathbf{h}_1, \mathbf{h}_2), \quad \forall \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{S}.$$

The symmetry of the second derivative ensures the symmetry of $\mathbf{S}(\mathbf{s})$ according to any metric tensor. Hence we have that

$$\mathbf{g}(\mathbf{S}\mathbf{n}, \mathbf{h}) = \mathbf{g}(\mathbf{S}\mathbf{h}, \mathbf{n}) = \mathbf{g}(\partial_{\mathbf{h}}\mathbf{n}, \mathbf{n}) = \frac{1}{2}\,\partial_{\mathbf{h}}\,\mathbf{g}(\mathbf{n}, \mathbf{n}) = 0, \quad \forall \mathbf{h} \in \mathbb{S},$$

so that $\mathbf{S}(\mathbf{s})\,\mathbf{n}(\mathbf{s}) = 0$.

## REFERENCES

[1] J.C. Simo and R.L. Taylor, *Consistent tangent operators for rate-independent elastoplasticity*. Comp. Meth. Appl. Mech. Engrg. , **48**, 101–118, 1985.

[2] P. Petersen, *Riemannian Geometry*. Springer-Verlag, New York, 1998.

[3] G. Romano, R. Barretta and C. Sellitto, *On the evaluation of the elastoplastic tangent stiffness*. Computational Plasticity Fundamentals and Applications, COMPLAS 2005, Barcelona, 5-8 September 2005.

[4] G. Romano, C. Sellitto and R. Barretta, *Nonlinear theory of shells as foliations of submanifolds*. Submitted to Journal of Mechanics of Materials and Structures, 2005.