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First Principle of Thermodynamics and Virtual Thermal-Work Theorem

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Abstract. The First Principle of continuum Thermodynamics is formulated as a variational condition in which the test fields are piecewise constant virtual temperatures. A strict analogy is thus shown to exist with the variational condition of equilibrium in continuum mechanics. This suggests to apply the Lagrange multipliers theorem to relax the constraint of piecewise constant virtual temperatures, thus leading to infer the existence of a square summable vector field of cold flow through the body. The variational statement, analogous to the theorem of virtual work in mechanics, is named the theorem of virtual thermal-work. In boundary value problems the cold flow has a piecewise square integrable divergence and Green's formula, upon localization, leads to the standard differential and jump conditions analogous to Cauchy equilibrium equations in continuum mechanics. A similar procedure can be applied to any balace law of mathematical physics to detect the associated vector field.

Key words: Continuum thermodynamics, Lagrange multipliers, Virtual temperatures, Cold flow

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1 Introduction

Duality is a basic concept in Mathematical physics and the master dual objects in continuum mechanics are velocity fields and force systems which interplay in the axiomatic formulation of dynamical equilibrium. The notion of a stress field in a continuous body in equilibrium was introduced in celebrated papers by Cauchy [1–3] but, according to Truesdell and Toupin [4], it was Piola [5] who first applied Lagrange's multiplier method to introduce the notion of stress field in a connected body. This brilliant intuition was however not properly evaluated and the more *easy-to-follow* geometrical method by Cauchy has been reproduced almost without exceptions in the subsequent literature, also in the continuum mathematical physics literature. To the best of our knowledge, the supremacy of Piola's approach has not been fully claimed until quite recently [6,7]. The motivations for this supremacy are twofold. On one hand, duality plays a basic role in Lagrange's method so that naturally the players coming into the scene are properly defined and detected. On the other hand, Lagrange's method for continuous problems is now a theorem, since, with a precise definition of functional spaces and operators involved, a rigorous existence proof for the field of multipliers can be given by relying on the powerful tools of Functional Analysis [8]. More precisely, dual players are in dual Banach spaces (or, more in general, in dual Banach vector bundles) and the linear differential operator providing the implicit representation of the constraint are such that the image of any closed linear subspace is closed. The result is then inferred from Banach's closed range theorem. In continuum mechanics, the constraint dealt with by Lagrange's method is the requirement of infinitesimal isometry of virtual displacement fields, which according to Euler [9], is expressed by a vanishing symmetric part of the covariant derivative of virtual vector fields, equal to onehalf the Lie derivative of the metric tensor along the virtual flow (see e.g [10]). The closure of the image by Euler's operator of any closed linear subspace of a suitable Sobolev space, is implied by Korn's inequality [11,12], and Lagrange's method is called the theorem of virtual work [7]. A main task in Mathematical Physics is to pursue the recognition of formal analogies between theories dealing with seemingly different physical contexts, in order to apply to them the same mathematical methods and results. It was then natural to ask if this powerful, modern treatment of the basics of continuum mechanics could be shared with other fields of continuum physics. The answer is positive, even if not completely straightforward, and requires a simple but tricky reformulation of classical statements. However, in the standard formulation of balance laws, no constraint is explicitly involved and Lagrange's method seems to be inapplicable, at a first sight. The key observation is that balance laws are required to hold for any part of a given continuous body and this fact may be equivalently formulated in variational terms by means of piecewise constant scalar test-fields. They play the role of virtual displacements in mechanics, with Euler's operator being replaced by the gradient operator. The closure of the image, by the gradient operator, of any closed linear subspace in a Sobolev space is a well-known result following from Poincaré inequality (see e.g. [13]). In this way it is shown that balance laws, expressing general principles for continuous bodies in Mathematical Physics, are susceptible of being treated by Lagrange's method and this leads directly to establish the existence and the basic properties of the field interplaying with the constraint. The topic is explicitly treated here with reference to the balance law expressing the First Principle of Thermodynamics, which is shown to be equivalent to a virtual thermal-work theorem providing the existence and the basic properties of the cold flow through the body.

2 The First Principle of Thermodynamics

The First Principle of Thermodynamics is a balance law prescribing an energy conservation rule to be fulfilled by any body undergoing any thermodynamical process. The principle states that given a body $\mathcal{B}$ at a placement $\boldsymbol{\Omega} = \boldsymbol{\varphi}(\mathcal{B})$ the time-rate of change of the internal energy $\dot{\mathcal{E}}(\mathcal{P})$ of any sub-body $\mathcal{P} \subseteq \boldsymbol{\Omega}$ is equal to the mechanical power $\mathcal{M}(\mathcal{P})$ plus the heat power $\mathcal{Q}(\mathcal{P})$ supplied to the body (see e.g. [14–16]):

$$\dot{\mathcal{E}}(\mathcal{P}) = \mathcal{M}(\mathcal{P}) + \mathcal{Q}(\mathcal{P})\,.$$

It is convenient to define the energy-rate gap $\mathcal{G}(\mathcal{P}) := \mathcal{M}(\mathcal{P}) + \mathcal{Q}(\mathcal{P}) - \dot{\mathcal{E}}(\mathcal{P})$ and to rewrite the first principle as $\mathcal{G}(\mathcal{P}) = 0$. To formulate the First Principle of Thermodynamics as a variational principle, we preliminarily provide a definition of virtual temperature fields.

2.1 Virtual temperatures

In the sequel $\textrm{P}\scriptstyle{\textrm{AT}}(\boldsymbol{\Omega})$ denotes a patchwork of $\boldsymbol{\Omega}$, that is a finite family of open connected, non-overlapping subsets of $\boldsymbol{\Omega}$, say $\mathcal{P} \in \mathrm{Pat}(\boldsymbol{\Omega})$ called parts or elements, such that the union of their closures is a covering for $\boldsymbol{\Omega}$. The set of all patchworks of $\boldsymbol{\Omega}$ is a directed set with the partial order relation *finer than*; the coarsest patchwork finer than $\mathrm{Pat}_1(\boldsymbol{\Omega})$ and $\mathrm{Pat}_2(\boldsymbol{\Omega})$ is the *grid*: $\mathrm{Pat}_1(\boldsymbol{\Omega}) \wedge \mathrm{Pat}_2(\boldsymbol{\Omega})$ whose elements are nonempty pairwise intersections of their elements. The Hilbert spaces of square integrable scalar fields (functions), vector fields and tensor fields on $\mathcal{P}$ will be denoted by $\mathrm{Sqif}(\mathcal{P})$, $\mathrm{Sqiv}(\mathcal{P})$ and $\mathrm{Sqit}(\mathcal{P})$. The linear space Temp of virtual temperatures is composed by *Green regular* scalar fields, i.e. square integrabile fields $\theta \in \mathrm{Sqif}(\boldsymbol{\Omega})$ whose distributional derivatives are piecewise square integrable in $\boldsymbol{\Omega}$ according to a regularity patchwork $\mathrm{Pat}_\theta(\boldsymbol{\Omega})$, i.e. $\nabla\theta \in \mathrm{Sqiv}(\mathrm{Pat}_\theta(\boldsymbol{\Omega}))$. The linear space Temp of virtual temperature fields is a pre-Hilbert space when endowed with the inner product and norm given by:

$$
\begin{aligned}
(\theta_1, \theta_2)_{\mathrm{Temp}} &:= \int_{\boldsymbol{\Omega}} \theta_1\, \theta_2\, \boldsymbol{\mu} + \int_{\mathrm{Pat}_{\theta_{12}}(\boldsymbol{\Omega})} \mathbf{g}(\nabla\theta_1, \nabla\theta_2)\, \boldsymbol{\mu}\,,\\
\|\theta\|^2_{\mathrm{Temp}} &:= \int_{\boldsymbol{\Omega}} \theta^2\, \boldsymbol{\mu} + \int_{\mathrm{Pat}_{\theta}(\boldsymbol{\Omega})} \|\nabla\theta\|^2\, \boldsymbol{\mu}\,,
\end{aligned}
$$

where $\mathrm{Pat}_{\theta_{12}}(\boldsymbol{\Omega}) := \mathrm{Pat}_{\theta_1}(\boldsymbol{\Omega}) \wedge \mathrm{Pat}_{\theta_2}(\boldsymbol{\Omega})$ is the grid of the involved patchworks, $\mathbf{g} \in BL(V^2; \mathcal{R})$ is the euclidean metric tensor and $\boldsymbol{\mu} \in BL(V^3; \mathcal{R})$ is the associated volume form. Sometimes we omit the argument when $\mathcal{P} = \boldsymbol{\Omega}$. In each element $\mathcal{P}$ of the regulatity patchwork, the distributional gradient fulfills the equivalence:

$$
\|\nabla\theta\|_0 + \|\theta\|_0 \simeq \|\theta\|_1\,, \quad \forall\, \theta \in H^1(\mathcal{P}; \mathcal{R})\,, \quad \mathcal{P} \in \mathrm{Pat}(\boldsymbol{\Omega})\,,
$$

where $\|\cdot\|_k$ is the mean square norm on $\mathcal{P}$ of the field and of all its derivatives up to the order k and $H^1(\mathcal{P}; \mathcal{R}) = W^{1,2}(\mathcal{P}; \mathcal{R})$ is a Sobolev space [17]. Indeed, the trivial norm equivalence holds:

$$
\|\theta\|_1^2 := \|\nabla\theta\|_0^2 + \|\theta\|_0^2 \le (\|\nabla\theta\|_0 + \|\theta\|_0)^2 \le 2\,(\|\nabla\theta\|_0^2 + \|\theta\|_0^2)\,.
$$

The fields in the pre-Hilbert space Temp will then have well-defined boundary values on each $\partial\mathcal{P}$, since in the Sobolev space $H^1(\mathcal{P}; \mathcal{R})$ boundary values are well-defined. The regular part of the distributional gradient of a virtual temperature field is the cartesian product of the distributional gradients evaluated on the restriction of a field to the elements of a regularity patchwork.

It is a linear bounded operator $\nabla \in BL(\textsc{Temp}\,;\textsc{Sqiv})$ from $\textsc{Temp}$ into the Hilbert space $\textsc{Sqiv}$ of square integrable vector fields in $\boldsymbol{\Omega}$ which fulfils the equivalence: $\|\nabla\theta\|_0 + \|\theta\|_0 \simeq \|\theta\|_1\,, \quad \forall\, \theta \in \textsc{Sob}^1_\theta$, where $\textsc{Sob}^1_\theta$ is the product of the Sobolev spaces $H^1(\mathcal{P}\,;\mathcal{R})$ for $\mathcal{P} \in \textsc{Pat}_\theta$. We denote by $\textsc{Conf} \subset \textsc{Temp}$ the space of conforming virtual temperatures, which is a closed linear subspace of virtual temperature fields which share a common regularity patchwork $\textsc{Pat}$ and fulfil continuous linear constraints on the boundary of the patchwork. The space $\textsc{Conf} \subset \textsc{Temp}$ is a Hilbert space for the topology inherited by $\textsc{Temp}$. The restriction $\nabla \in BL(\textsc{Conf}\,;\textsc{Sqiv})$ of the regular part of the distributional gradient $\nabla \in BL(\textsc{Temp}\,;\textsc{Sqiv})$ to conforming virtual temperatures, is a bounded linear operator between Hilbert spaces. By an equivalence Lemma [18–20] it follows that the range: $\nabla(\textsc{Conf}) \subset \textsc{Sqiv}$ is closed and the kernel: $\ker\nabla \cap \textsc{Conf}$ is finite dimensional. Denoting by an exponent $*$ the dual topological vector spaces, Banach's closed range theorem [21] ensures that the adjoint operator $\nabla^* \in BL(\textsc{Sqiv}^*\,;\textsc{Temp}^*)$, defined by the identity $\langle \nabla^*\boldsymbol{\omega}, \theta \rangle = \langle \boldsymbol{\omega}, \nabla\theta \rangle\,, \quad \forall\, \boldsymbol{\omega} \in \textsc{Sqiv}^*\,, \quad \forall\, \theta \in \textsc{Temp}$, is such that $\nabla^*(\textsc{Sqiv}^*) = (\ker\nabla \cap \textsc{Conf})^\circ$ where:

$$(\ker\nabla \cap \textsc{Conf})^\circ := \{\, f \in \textsc{Conf}^* : \langle f, \theta \rangle = 0\,,\ \forall\, \theta \in \ker\nabla \cap \textsc{Conf}\,\}\,.$$

The fields $\theta \in \ker\nabla \subset \textsc{Temp}$ are characterized by the property that the regular part $\nabla\theta$ of their distributional gradient vanishes on each element of the regularity patchwork $\textsc{Pat}_\theta(\boldsymbol{\Omega})$. It follows that they are piecewise constant virtual temperature fields (see e.g. [13], Proposition II.5.3).

2.2 Variational form of the First Principle

For any $\theta \in \textsc{Temp}$, we consider the characteristic functions of the elements $\mathcal{P}$ of the patchwork $\textsc{Pat}_\theta(\boldsymbol{\Omega})$:

$$1_{\mathcal{P}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathcal{P} \\ \\ 0 & \mathbf{x} \in \boldsymbol{\Omega}\backslash\mathcal{P} \end{cases}$$

with $\mathcal{P} \in \textsc{Pat}_\theta(\boldsymbol{\Omega})$, and define the functionals:

$$\mathcal{F}_{\dot{\mathcal{E}}}(1_{\mathcal{P}}) := \dot{\mathcal{E}}(\mathcal{P})\,,$$

$$\mathcal{F}_{\mathcal{M}}(1_{\mathcal{P}}) := \mathcal{M}(\mathcal{P})\,,$$

$$\mathcal{F}_{\mathcal{Q}}(1_{\mathcal{P}}) := \mathcal{Q}(\mathcal{P})\,,$$

which associate with the characteristic function $1_{\mathcal{P}}$, respectively, the time-rate of change of internal energy, the mechanical power and the heat power, evaluated on the element $\mathcal{P}$. Performing an extension by linearity, we define the linear functionals $\mathcal{F}_{\dot{\mathcal{E}}}$, $\mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{Q}}$ on the linear subspace $\ker \nabla \subseteq$ $\mathrm{T{\small EMP}}$ of piecewise constant virtual temperature fields. By HAHN's extension theorem these bounded linear functionals can be extended (non-univocally) to bounded linear functionals on $\mathrm{T{\small EMP}}$ without increasing their norm (see e.g. [21]). The First Principle of Thermodynamics can then be reformulated in variational terms as:

$$\langle \mathcal{F}_{\dot{\mathcal{E}}}, \theta \rangle = \langle \mathcal{F}_{\mathcal{M}}, \theta \rangle + \langle \mathcal{F}_{\mathcal{Q}}, \theta \rangle , \quad \forall\, \theta \in \ker \nabla \, .$$

Recalling the definition of energy-rate gap $\mathcal{G} := \mathcal{M} + \mathcal{Q} - \dot{\mathcal{E}}$, and introducing the thermal force $\mathcal{F}_{\mathcal{G}} \in \mathrm{T{\small EMP}}^*$ as the linear functional given by:

$$\mathcal{F}_{\mathcal{G}} := \mathcal{F}_{\mathcal{M}} + \mathcal{F}_{\mathcal{Q}} - \mathcal{F}_{\dot{\mathcal{E}}} \, ,$$

the energy conservation law $\mathcal{G} = 0$ takes the variational form

$$\langle \mathcal{F}_{\mathcal{G}}, \theta \rangle = 0 \, , \quad \forall\, \theta \in \ker \nabla \quad \Longleftrightarrow \quad \mathcal{F}_{\mathcal{G}} \in (\ker \nabla)^{\circ} \, .$$

This condition, which is analogous to the axiom of dynamical equilibrium in mechanics [7], is called the axiom of thermal equilibrium and can be stated by saying that the thermal virtual work of the *thermal force* $\mathcal{F}_{\mathcal{G}}$ must vanish for any piecewise constant virtual temperature field. The restriction of $\mathcal{F}_{\mathcal{G}}$ to conforming virtual temperatures will be called a *thermal load*. The closed range property of the regular part of the distributional gradient $\nabla \in BL(\mathrm{C{\small ONF}}\,; \mathrm{S{\small QIV}})$ leads to the following existence result.

Theorem 1 (Virtual thermal-work). *The Axiom of thermal equilibrium:*

$$\langle \mathcal{F}_{\mathcal{G}}, \theta \rangle = 0 \, , \quad \forall\, \theta \in \ker \nabla \cap \mathrm{C{\small ONF}} \, ,$$

is equivalent to the existence of a square integrable vector field $\mathbf{q} \in \mathrm{S{\small QIV}}$*, the cold-flow vector field, which performs, for the regular part of the distributional gradient of a conforming virtual temperature field, a virtual thermal-work equal to the one that the thermal load performes for the conforming virtual temperature field:*

$$\langle \mathcal{F}_{\mathcal{G}}, \theta \rangle = \int_{\mathrm{P{\small AT}}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q} \, , \nabla\theta) \, \boldsymbol{\mu} \, , \quad \forall\, \theta \in \mathrm{C{\small ONF}} \, .$$

Proof. Since $\mathcal{F}_{\mathcal{G}} \in (\ker \nabla)^{\circ} \subset (\ker \nabla \cap \mathrm{C{\small ONF}})^{\circ} = \nabla^{*}(\mathrm{S{\small QIV}}^{*})$ there exists a one-form $\boldsymbol{\omega} \in \mathrm{S{\small QIV}}^{*}$, the cold-flow one-form, such that $\mathcal{F}_{\mathcal{G}} = \nabla^{*}\boldsymbol{\omega}$. Moreover, by Riesz-Fréchet representation theorem (see e.g. [21,13]) we infer that there exists a cold-flow vector field $\mathbf{q} \in \mathrm{S{\small QIV}}$ such that

$$\langle \mathcal{F}_{\mathcal{G}}, \theta \rangle = \langle \nabla^{*}\boldsymbol{\omega}\,, \theta \rangle = \langle \boldsymbol{\omega}, \nabla\theta \rangle = \int_{\mathrm{P{\small AT}}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q}\,, \nabla\theta)\,\boldsymbol{\mu}\,, \quad \forall\, \theta \in \mathrm{C{\small ONF}}\,.$$

■

3 Boundary value problems

The basic tool in boundary value problems governed by a linear partial differential operator $\mathrm{D{\small IFF}}$ of order n, is GREEN's formula of integration by parts, which formally may be written as:

$$\int_{\mathrm{P{\small AT}}(\boldsymbol{\Omega})} \langle \bullet, \mathrm{D{\small IFF}} \circ \rangle \boldsymbol{\mu} = \int_{\mathrm{P{\small AT}}(\boldsymbol{\Omega})} \langle \mathrm{A{\small DJ}D{\small IFF}} \bullet, \circ \rangle \boldsymbol{\mu} + \oint_{\partial \mathrm{P{\small AT}}(\boldsymbol{\Omega})} \langle \mathrm{F{\small LUX}} \bullet, \mathrm{V{\small AL}} \circ \rangle \partial\boldsymbol{\mu}\,,$$

where $\mathrm{P{\small AT}}(\boldsymbol{\Omega})$ is a fixed patchwork, $\partial\mathrm{P{\small AT}}(\boldsymbol{\Omega})$ is its boundary, $\partial\boldsymbol{\mu}$ is the volume $(n-1)$-form induced on the surfaces $\partial\mathrm{P{\small AT}}(\boldsymbol{\Omega})$ and all the integrals are assumed to take a finite value. The differential operator $\mathrm{A{\small DJ}D{\small IFF}}$ of order n is the *formal adjoint* of $\mathrm{D{\small IFF}}$. The boundary integral acts on the duality pairing between the two fields $\mathrm{F{\small LUX}}\bullet$ and $\mathrm{V{\small AL}}\circ$ with the differential operators $\mathrm{F{\small LUX}}$ and $\mathrm{V{\small AL}}$ being n-tuples of normal derivatives of order from 0 to $n-1$ in inverse sequence, to that the duality pairing is the sum of n terms, whose k-th term is the pairing of normal derivatives of two fields respectively of order k and $n-1-k$.

Boundary value problems are characterized by the property that the closed linear subspace $\mathrm{C{\small ONF}}$ of conforming test fields includes the whole linear subspace $\ker(\mathrm{V{\small AL}})$ of test fields in $\mathrm{T{\small EMP}}$ with vanishing boundary values on $\partial\mathrm{P{\small AT}}(\boldsymbol{\Omega})$, i.e.

$$\ker(\mathrm{V{\small AL}}) \subseteq \mathrm{C{\small ONF}}\,.$$

Let us assume that the heat power $\mathcal{Q}(\boldsymbol{\Omega})$ is expressed in terms of a bulk density $q \in \mathrm{S{\small QIF}}(\boldsymbol{\Omega})$ and of a superficial density $\partial q \in \mathrm{S{\small QIF}}(\partial\mathrm{P{\small AT}}(\boldsymbol{\Omega}))$:

$$\mathcal{Q}(\boldsymbol{\Omega}) = \langle \mathcal{F}_{\mathcal{Q}}, 1_{\boldsymbol{\Omega}} \rangle := \int_{\boldsymbol{\Omega}} q\,\rho_t\,\boldsymbol{\mu} + \int_{\partial\mathrm{P{\small AT}}(\boldsymbol{\Omega})} \partial q\,\partial\boldsymbol{\mu}\,.$$

At dynamical equilibrium, the mechanical power is expressed in terms of the stress tensor field $\mathbf{T} \in \textsc{Sqit}(\boldsymbol{\Omega})$ and of the Euler operator evaluated on the velocity field $\mathbf{v} \in \textsc{Sob}^1_{\mathbf{v}}$ of the body:

$$\mathcal{M}(\boldsymbol{\Omega}) = \langle \mathcal{F}_{\mathcal{M}}, 1_{\boldsymbol{\Omega}} \rangle := \int_{\boldsymbol{\Omega}} \langle \mathbf{T}, \operatorname{sym} \nabla \mathbf{v} \rangle \, \boldsymbol{\mu} \,,$$

where $\textsc{Sob}^1_{\mathbf{v}}$ is the product of the Sobolev spaces $H^1(\mathcal{P}\,; V)$ for $\mathcal{P} \in \textsc{Pat}_{\mathbf{v}}$. The bounded linear functionals $\mathcal{F}_{\dot{\mathcal{E}}}$, $\mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{Q}}$ may then be univocally defined on the whole space $\textsc{Temp}$ by setting:

$$\begin{aligned}
\langle \mathcal{F}_{\dot{\mathcal{E}}}, \theta \rangle &:= \int_{\boldsymbol{\Omega}} \rho_t \, \dot{\varepsilon} \, \theta \, \boldsymbol{\mu} \,, \\
\langle \mathcal{F}_{\mathcal{M}}, \theta \rangle &:= \int_{\boldsymbol{\Omega}} \langle \mathbf{T}, \operatorname{sym} \nabla \mathbf{v} \rangle \, \theta \, \boldsymbol{\mu} \,, \\
\langle \mathcal{F}_{\mathcal{Q}}, \theta \rangle &:= \int_{\boldsymbol{\Omega}} \rho_t \, q \, \theta \, \boldsymbol{\mu} + \int_{\partial \textsc{Pat}_\theta(\boldsymbol{\Omega})} \langle \partial q, \textsc{Val}\, \theta \rangle \, \partial \boldsymbol{\mu} \,,
\end{aligned}$$

for any $\theta \in \textsc{Temp}$. Defining the bulk energy-rate gap field as

$$p := -\rho_t \, \dot{\varepsilon} + \langle \mathbf{T}, \operatorname{sym} \nabla \mathbf{v} \rangle + \rho_t \, q \,,$$

the energy-rate gap $\mathcal{F}_{\mathcal{G}} \in \textsc{Conf}^*$ is given by:

$$\langle \mathcal{F}_{\mathcal{G}}, \theta \rangle = \int_{\boldsymbol{\Omega}} \langle p, \theta \rangle \, \boldsymbol{\mu} + \int_{\partial \textsc{Pat}_\theta(\boldsymbol{\Omega})} \langle \partial q, \textsc{Val}\, \theta \rangle \, \partial \boldsymbol{\mu} \,, \quad \forall \, \theta \in \textsc{Temp} \,.$$

Due to the basic property of boundary value problems: $\ker(\textsc{Val}\,) \subseteq \textsc{Conf}$, the variational condition stated in the virtual thermal-work theorem can be localized to yield differential and jump conditions. Although the procedure is standard, we report hereafter the statement and the proof of the basic results for sake of completeness. Moreover a detailed presentation of these results at this level of generality is not easily found in the literature.

Cold flow vector fields whose distributional divergence is piecewise representable by a square integrable field, i.e. such that $\textsc{Div}\, \mathbf{q} \in \textsc{Sqif}(\textsc{Pat}_{\mathbf{q}}(\boldsymbol{\Omega}))$ with regularity patchwork $\textsc{Pat}_{\mathbf{q}}$, are said to be *Green regular* and we will write $\mathbf{q} \in \textsc{Cold}$. The $\textsc{Green}$ regularity of cold flows and virtual temperature fields ensures that all the terms in the relevant $\textsc{Green}$'s formula are well defined:

$$\begin{aligned}
\int_{\textsc{Pat}_{\theta\mathbf{q}}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q}, \nabla\theta) \, \boldsymbol{\mu} &= \int_{\textsc{Pat}_{\theta\mathbf{q}}(\boldsymbol{\Omega})} \langle -\textsc{Div}\, \mathbf{q}, \theta \rangle \, \boldsymbol{\mu} \\
&+ \int_{\partial \textsc{Pat}_{\theta\mathbf{q}}(\boldsymbol{\Omega})} \langle \mathbf{g}(\mathbf{q}, \mathbf{n}), \textsc{Val}\,(\theta) \rangle \, \partial \boldsymbol{\mu} \,, \quad \begin{cases} \forall \, \theta \in \textsc{Temp} \,, \\ \forall \, \mathbf{q} \in \textsc{Cold} \,. \end{cases}
\end{aligned}$$

where $\textsc{Pat}_{\theta\mathbf{q}} = \textsc{Pat}_\theta \wedge \textsc{Pat}_{\mathbf{q}}$, $\textsc{Flux}\, \mathbf{q} = \mathbf{g}(\mathbf{q}, \mathbf{n})$, with $\mathbf{n}$ outward unit normal to the boundary $\partial \textsc{Pat}_{\theta\mathbf{q}}(\boldsymbol{\Omega})$ and $\textsc{Val}\,(\theta)$ boundary value of the field $\theta \in \textsc{Temp}$ on $\partial \textsc{Pat}_{\theta\mathbf{q}}(\boldsymbol{\Omega})$.

This formula may be also written in terms of jumps at the interfaces of a sufficiently fine patchwork. Indeed a boundary $\partial\textsc{Pat}(\boldsymbol{\Omega})$ may be considered as the union of the boundary of $\boldsymbol{\Omega}$ and of the pairs of positive and negative faces of each interface between elements of $\textsc{Pat}(\boldsymbol{\Omega})$, i.e.: $\partial\textsc{Pat}(\boldsymbol{\Omega}) = \partial\boldsymbol{\Omega} + \textsc{If}(\textsc{Pat})$. In the sequel we shall denote by $\textsc{Pat}_\infty$ a patchwork sufficiently fine for the statement at hand. Then, setting $\mathbf{n} = \mathbf{n}^+$ outward normal to the $+$ face, defining the jump:

$$[[\mathbf{g}(\mathbf{q},\mathbf{n})]] := \mathbf{g}(\mathbf{q}^+,\mathbf{n}^+) + \mathbf{g}(\mathbf{q}^-,\mathbf{n}^-) = \mathbf{g}(\mathbf{q}^+,\mathbf{n}^+) - \mathbf{g}(\mathbf{q}^-,\mathbf{n}^+)\,,$$

across the interfaces, considering the boundary $\partial\boldsymbol{\Omega}$ as a $+$ face, and taking a patchwork $\textsc{Pat}_\infty$ finer than $\textsc{Pat}_{\theta\mathbf{q}}$ the $\textsc{Green}$'s formula may be written as:

$$\begin{aligned}\int_{\textsc{Pat}_\infty(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q},\nabla\theta)\,\boldsymbol{\mu} &= \int_{\textsc{Pat}_\infty(\boldsymbol{\Omega})} \langle\, -\textsc{Div}\,\mathbf{q},\theta\,\rangle\,\boldsymbol{\mu} + \oint_{\partial\boldsymbol{\Omega}} \langle \mathbf{g}(\mathbf{q},\mathbf{n}),\textsc{Val}\,(\theta)\,\rangle\,\partial\boldsymbol{\mu} \\ &+ \int_{\textsc{If}(\textsc{Pat}_\infty(\boldsymbol{\Omega}))} \langle\,[[\mathbf{g}(\mathbf{q},\mathbf{n})]],\textsc{Val}\,(\theta)\,\rangle\,\partial\boldsymbol{\mu}\,, \quad \begin{cases} \forall\,\theta\in\textsc{Temp}\,, \\ \forall\,\mathbf{q}\in\textsc{Cold}\,. \end{cases}\end{aligned}$$

Then we have the following result.

Theorem 2 (Localization). *In a boundary value problem, a cold flow vector field* $\mathbf{q}$ *in thermal equilibrium with a thermal load* $\mathcal{F}_{\mathcal{G}}$ *, i.e. fulfilling the identity in the virtual thermal-work theorem:*

$$\int_{\boldsymbol{\Omega}} \langle\, p,\theta\,\rangle\,\boldsymbol{\mu} + \int_{\partial\textsc{Pat}(\boldsymbol{\Omega})} \langle\,\partial q,\textsc{Val}\,\theta\,\rangle\,\partial\boldsymbol{\mu} = \int_{\textsc{Pat}(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q},\nabla\theta)\,\boldsymbol{\mu}\,, \quad \forall\,\theta\in\textsc{Conf}\,,$$

has a distributional divergence $\textsc{Div}\,\mathbf{q}$ *whose restriction to each element* $\mathcal{P}\in\textsc{Pat}_\infty(\boldsymbol{\Omega})$ *of the patchwork is* $\mathbf{g}$*-square integrable with*

$$-\textsc{Div}\,\mathbf{q} = p\,, \quad \text{in} \quad \textsc{Pat}_\infty(\boldsymbol{\Omega})\,,$$

and the jump $[[\mathbf{g}(\mathbf{q},\mathbf{n})]]$ *flux across the boundary of the domain* $\boldsymbol{\Omega}$ *and across the interfaces of the patchwork* $\textsc{Pat}_\infty(\boldsymbol{\Omega})$ *fulfills the conditions:*

$$\mathbf{g}(\mathbf{q},\mathbf{n}) \in \partial q + \textsc{Conf}^\circ\,, \qquad \text{on} \quad \partial\boldsymbol{\Omega}\,,$$

$$[[\mathbf{g}(\mathbf{q},\mathbf{n})]] \in \partial q^+ + \partial q^- + \textsc{Conf}^\circ\,, \quad \text{on} \quad \textsc{If}(\textsc{Pat}_\infty(\boldsymbol{\Omega}))\,,$$

where the fields ∂q *of surficial heat supply are taken to be zero outside their domain of definition.*

Proof. By Green's formula, the statement of the virtual thermal-work theorem may be written as:

$$\begin{aligned}\int_{\boldsymbol{\Omega}} \langle\, p,\theta\,\rangle\,\boldsymbol{\mu} + \int_{\partial\textsc{Pat}_\infty(\boldsymbol{\Omega})} \langle\,\partial q,\textsc{Val}\,\theta\,\rangle\,\partial\boldsymbol{\mu} &= \int_{\textsc{Pat}_\infty(\boldsymbol{\Omega})} \langle\, -\textsc{Div}\,\mathbf{q},\theta\,\rangle\,\boldsymbol{\mu} \\ &+ \int_{\partial\textsc{Pat}_\infty(\boldsymbol{\Omega})} \langle \mathbf{g}(\mathbf{q},\mathbf{n}),\textsc{Val}\,(\theta)\,\rangle\,\partial\boldsymbol{\mu}\,, \quad \begin{cases} \forall\,\theta\in\textsc{Conf}\,, \\ \forall\,\mathbf{q}\in\textsc{Cold}\,. \end{cases}\end{aligned}$$

Due to the density of the linear space $\mathrm{C}_0^\infty(\mathcal{P})$ of infinitely differentiable scalar fields with compact support in the space $\mathrm{S{\scriptsize QIF}}(\mathcal{P})$ of square integrable fields on each element $\mathcal{P} \in \mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})$, and being $\mathrm{C}_0^\infty(\mathcal{P}) \subset \ker \mathrm{V{\scriptsize AL}}\,(\mathcal{P})$, it follows that $\ker \mathrm{V{\scriptsize AL}}\,(\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega}))$ is dense in $\mathrm{S{\scriptsize QIF}}(\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega}))$. Moreover, for boundary value problems it is: $\ker \mathrm{V{\scriptsize AL}}\,(\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})) \subset \mathrm{C{\scriptsize ONF}}$, so that:

$$\int_{\boldsymbol{\Omega}} \langle p, \theta \rangle\, \boldsymbol{\mu} = \int_{\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle -\mathrm{D{\scriptsize IV}}\,\mathbf{q}, \theta \rangle\, \boldsymbol{\mu}\,, \quad \begin{cases} \forall\, \theta \in \ker \mathrm{V{\scriptsize AL}}\,(\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega}))\,, \\ \forall\, \mathbf{q} \in \mathrm{C{\scriptsize OLD}}\,, \end{cases}$$

and hence, by density: $p + \mathrm{D{\scriptsize IV}}\,\mathbf{q} = 0$ in $\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})$. Then

$$\int_{\partial \mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle \partial q, \mathrm{V{\scriptsize AL}}\,\theta \rangle\, \partial\boldsymbol{\mu} = \int_{\partial \mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle \mathbf{g}(\mathbf{q}, \mathbf{n}), \mathrm{V{\scriptsize AL}}\,(\theta) \rangle\, \partial\boldsymbol{\mu}\,, \quad \begin{cases} \forall\, \theta \in \mathrm{C{\scriptsize ONF}}\,, \\ \forall\, \mathbf{q} \in \mathrm{C{\scriptsize OLD}}\,, \end{cases}$$

and the result follows. ■

Let us now observe that the virtual thermal-work

$$\int_{\mathrm{P{\scriptsize AT}}_\theta(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q}, \nabla\theta)\, \boldsymbol{\mu}\,, \quad \theta \in \mathrm{T{\scriptsize EMP}}\,,$$

is well-defined for any (even nonconforming) temperature field $\theta \in \mathrm{T{\scriptsize EMP}}$. Then, by GREEN's formula, we define the reactive thermal force $\mathcal{R}(\partial q, p, \mathbf{q}) \in \mathrm{T{\scriptsize EMP}}^*$, associated with a bulk energy-rate gap p, a surficial heat supply ∂q and a cold flow field $\mathbf{q} \in \mathrm{C{\scriptsize OLD}}$, by the relation:

$$\begin{aligned} \langle \mathcal{R}, \theta \rangle &:= \int_{\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \mathbf{g}(\mathbf{q}, \nabla\theta)\, \boldsymbol{\mu} - \int_{\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle p, \theta \rangle\, \boldsymbol{\mu} - \int_{\partial \mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle \partial q, \mathrm{V{\scriptsize AL}}\,(\theta) \rangle\, \partial\boldsymbol{\mu} \\ &= \int_{\mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle -\mathrm{D{\scriptsize IV}}\,\mathbf{q} - p, \theta \rangle\, \boldsymbol{\mu} + \int_{\partial \mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle \mathbf{g}(\mathbf{q}, \mathbf{n}) - \partial q, \mathrm{V{\scriptsize AL}}\,(\theta) \rangle\, \partial\boldsymbol{\mu} \\ &= \int_{\partial \mathrm{P{\scriptsize AT}}_\infty(\boldsymbol{\Omega})} \langle \mathbf{g}(\mathbf{q}, \mathbf{n}) - \partial q, \mathrm{V{\scriptsize AL}}\,(\theta) \rangle\, \partial\boldsymbol{\mu}\,, \end{aligned}$$

for any $\theta \in \mathrm{T{\scriptsize EMP}}$.

Reactive thermal forces are the ones that do not perform virtual thermal-work for conforming virtual temperature fields:

$$\langle \mathcal{R}, \theta \rangle = 0\,, \quad \forall\, \theta \in \mathrm{C{\scriptsize ONF}} \quad \Longleftrightarrow \quad \mathcal{R} \in \mathrm{C{\scriptsize ONF}}^\circ\,.$$

Defining the reactive surficial heat supply as $\partial r(\mathbf{q}, \partial q) := \mathbf{g}(\mathbf{q}, \mathbf{n}) - \partial q$, the virtual thermal-work theorem ensures that

$$\int_{\partial \mathrm{P{\scriptsize AT}}(\boldsymbol{\Omega})} \partial r(\mathbf{q}, \partial q)\, \mathrm{V{\scriptsize AL}}\,(\theta)\, \partial\boldsymbol{\mu} = 0\,, \quad \forall\, \theta \in \mathrm{C{\scriptsize ONF}}\,.$$

Hence, in particular, $\partial r(\mathbf{q}, \partial q) = 0$ on any piece of boundary where virtual temperatures are not prescribed to vanish.

The bulk equation $\mathrm{D{\scriptstyle IV}}\,\mathbf{q} = -p$ is known in literature as the reduced equation of conservation of the energy. The boundary equation $\partial q = \mathbf{g}(\mathbf{q}, \mathbf{n})$, in absence of reactive surficial heat supply, is known as the heat flow principle of FOURIER-STOKES.

4 Conclusions

The first principle of the thermodynamics has been reformulated, by a simple but tricky reasoning, as a variational condition in which the test fields are piecewise constant virtual temperature fields. A specialization of the theorem of Lagrange multipliers yields the virtual thermal-work theorem which provides the existence of a cold flow vector field in the body. In all classical treatments (e.g. [14–16] the existence of a heat flow vector field is instead assumed as a separate axiom of continuum thermodynamics. For boundary value problems, Green's formula and a localization procedure lead to differential and boundary conditions.

A similar treatment may be applied to any balance law in continuum physics. For instance, the principle of mass conservation leads to a variational principle in which the lagrangian multipliers are vector fields describing the mass flow through a control volume [22].

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