

A numerical approach to nonlinear two-point boundary value problems for ODEs

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Abstract

In this paper we propose a numerical approach to solve some problems connected with the implementation of the Newton type methods for the resolution of the nonlinear system of equations related to the discretization of a nonlinear two-point BVPs for ODEs with mixed linear boundary conditions by using the finite difference method.

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1. Introduction

The theory of the boundary value problems is an extremely important and interesting area of research in differential equations (see [1–36]). In this paper we consider the following two-point boundary value problem, which occurs in applied mathematics, theoretical physics, engineering, control and optimization theory

$$\begin{cases} y'' = f(x, y, y'), \\ l_a y(a) - m_a y'(a) = v_a, \\ l_b y(b) + m_b y'(b) = v_b, \end{cases} \quad x \in [a, b], \quad (1)$$

where $f \in C([a, b] \times \mathfrak{R}^2, \mathfrak{R})$ is a nonlinear function, l_a, m_a, l_b , and m_b are given non-negative constants and v_a, v_b assigned constants.

We face this problem by using analytical and/or numerical approximation methods since, generally, the solution cannot be exhibited in a *closed form* even when it exists. Usually, the adopted integration methods for (1), are the finite difference method [9,7,8,32,33,35,34], the shooting method [9,7,8,31,30], the monotone iterative method [15, 18,20,23,21,22], and the quasilinearization method [13,17,19,24,25,27,28]. The analysis contained in Section 2 puts in evidence the advantages and the open problems of all these methods. As a consequence, we are forced to use the **finite difference method** since we are able to overcome the open problems of this approach in many cases. It is well

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known that the finite difference method reduces the problem (1) to the following *discrete problem*

$$\begin{cases} \mathcal{D}^2 y_k = f(x_k, y_k, \mathcal{D}y_k), & k = 1, \dots, n-1 \\ l_a y_0 - m_a \mathcal{D}y_0 = v_a, \\ l_b y_n + m_b \mathcal{D}y_n = v_b, \end{cases} \quad (2)$$

where $h = (b-a)/n$ is the step size of the grid points $x_k = a + kh$, $y_k = y(x_k)$, and $\mathcal{D}y_k$, $\mathcal{D}^2 y_k$ are the centered difference quotients, for $k = 0, \dots, n$.

The problems connected with the resolution of system (2) are:

- It is essential to determine under which conditions problem (2) admits a solution converging to the solution $y(x)$ of problem (1) since the property of existence and uniqueness of the solution does not necessarily transfer from the continuous problem (1) to the discrete problem (2).
- System (2), when it is nonlinear, can be solved by using Newton-like methods. These iterative methods generate a sequence $\{y_k^v\}_{v \in \mathcal{N}}$ converging to y_k provided that suitable initial data y_k^0 , $k = 0, \dots, n$ are assigned.

About the first item, we recall that in 1974 Gaines in [1] proved that the discrete BVP (2) could admit *spurious solutions* which become unbounded and irrelevant for the corresponding *continuous problem* (1) when the grid size h tends to zero. Moreover, in [2] Agarwal exhibited an example in which the continuous problem has a solution, whereas the discrete problem does not. Finally, Henderson and Thompson in [4,3,5,6] extended the existence results of Gaines' to the case of nonlinear boundary conditions $\mathbf{G}((y_a, y_b), (y'_a, y'_b)) = \mathbf{0}$, proving convergence theorems for the solution of the discrete problem when the grid size h goes to zero. However, it is not easy to verify the hypotheses of the cited theorems of existence, uniqueness and convergence since they refer to *discrete Green's functions*, the lower and upper solutions, the degree theory, and a suitable *compatibility conditions* on $\mathbf{G}$. Differently, in Numerical Analysis, the *convergence properties* of a finite difference scheme is, generally, based on the concepts of *consistency* and *stability* (see [8]). Generally, it has to be noted that the convergence theorems of the BVPs with mixed linear or nonlinear boundary conditions refer to the convergence properties of the *linearized difference scheme* around an *isolated solution* of the BVP. However, in [7], the *convergence properties* are directly proved for the finite difference scheme (2). In any case, the convergence of the finite difference method is strongly conditioned by the *start points* $\mathbf{y}_\pi^0 = (y_0^0, \dots, y_n^0)$ we have chosen in the Newton-like method used for the solution of the discrete problem in the unknowns $\mathbf{y}_\pi = (y_0, \dots, y_n)$, for any choice of the mesh $\pi : a = x_0 < x_2 < \dots < x_n = b$. We remark that if $\mathbf{y}_\pi^0$ is not "close" to $\mathbf{y}_\pi$, then the promised *convergence* could not be reached, and even if there is eventually a convergence of the nonlinear iteration, it could not be realized before the end of the iterative process.

In order to solve this last important problem, we propose a scheme in which a family of BVPs A_j , $j = 0, \dots, s$, is considered such that, for $j = 0$, the corresponding problem can be analytically integrated, whereas, for $j = s$, we are again faced with the original problem (see [9] for a particular class of the BVPs with linear boundary conditions). The proposed numerical scheme finds the start points $\mathbf{y}_\pi^0$ "sufficiently close" to the analytic exact solution of the considered BVP provided that the following conditions are satisfied:

- (1) An existence and uniqueness theorem must hold both for the assigned problem and the whole class of problems A_j , $j = 0, \dots, s-1$. Moreover, a constant $\chi > 0$ exists, such that:

$$|y(x, j) - y(x)| \leq \left(1 - \frac{j}{s}\right) \chi, \quad \forall x \in [a, b],$$

where $y(x, j)$ is the unique solution of A_j .

- (2) The finite difference method has to be valid both for the assigned problem and the class of the problems A_j , $j = 0, \dots, s-1$.

In this paper, we propose sufficient conditions to make requests (1) and (2) satisfied for the problem (1). Moreover, the proposed scheme will be formally extended to BVPs with nonlinear boundary conditions without verifying that conditions (1) and (2) are satisfied.

This paper is organized in five sections including the introduction. In Section 2, the usually used numerical and analytical methods for the BVP (1) are critically described. Moreover, we recall some known convergence results of the finite difference method. In Section 3 the proposed scheme is widely described for the BVP (1) and the conditions

to verify that the requests (1) and (2) are exposed. In the Section 4 the numerical scheme is formally extended to BVPs with fully nonlinear boundary conditions. Finally, in Section 5 some meaningful applications are presented.

Although the contents of this paper have been focused on second-order equations, various computational experiments show that the method efficiently operates also in higher dimensions. Therefore, an interesting research project looks at the generalization of the method to PDEs, as long as the space domain can be properly discretized, see [37,38], so that the boundary value problem (with boundary values both for the time and space variable) can be transformed into a boundary value problem for a system of ordinary differential equations [39].

2. Survey of known results for BVPs

In this section we describe some analytical and numerical methods used to find an approximate solution of the BVPs for ODEs. In particular, we describe in detail the finite difference method highlighting the open problems and recalling some known convergence results.

The **monotone iterative methods**, originally introduced by Picard [15], are based on the idea of building sequences of approximated solutions which converge monotonically to the solution of the BVP (1). Generally, these methods use the lower and upper solutions,¹ generating sequences of approximations $\{\alpha_n\}_{n \in \mathcal{N}}$ and $\{\beta_n\}_{n \in \mathcal{N}}$ defined by a suitable iterative scheme. In [21] and [22] the authors consider the Dirichlet problem

$$\begin{cases} y'' = f(x, y, y'), & x \in [a, b], \\ y(a) = y(b) = 0, \end{cases} \quad (3)$$

under suitable hypothesis on f , proposing for it the following iterative schemes

$$\begin{cases} -\alpha_n'' + \lambda \alpha_n = -f(x, \alpha_{n-1}, \alpha_{n-1}') + \lambda \alpha_{n-1}, \\ \alpha_n(a) = \alpha_n(b) = 0, \end{cases} \quad (4)$$

$$\begin{cases} -\alpha_n'' + 2k|\alpha_n' - \alpha_{n-1}'| + \lambda \alpha_n = -f(x, \alpha_{n-1}, \alpha_{n-1}') + \lambda \alpha_{n-1}, \\ \alpha_n(a) = \alpha_n(b) = 0, \end{cases} \quad (5)$$

where k, λ are suitable constants, and α_0 is a lower solution of (3). We remark that approach (4) does not explicitly give computable approximation α_n since the nonlinear function f depends on α_n' , whereas (5) is a linear problem. In [20], under suitable hypotheses on f , the Neumann problem

$$\begin{cases} y'' = f(x, y, y'), & x \in [a, b], \\ y'(a) = y'(b) = 0, \end{cases} \quad (6)$$

is examined introducing the sequences $\{\alpha_n\}_{n \in \mathcal{N}}$ and $\{\beta_n\}_{n \in \mathcal{N}}$ defined by

$$\begin{cases} -\alpha_n'' + \lambda \alpha_n = -f(x, \alpha_{n-1}, \alpha_{n-1}') + \lambda \alpha_{n-1}, \\ -\beta_n'' + \lambda \beta_n = -f(x, \beta_{n-1}, \beta_{n-1}') + \lambda \beta_{n-1}, \\ \alpha_n'(a) = \alpha_n'(b) = 0, & \beta_n'(a) = \beta_n'(b) = 0, \end{cases} \quad (7)$$

where α_0 and β_0 are the lower and upper solutions of (6) with $\alpha_0 \leq \beta_0$. Moreover, the sequences $\{\alpha_n\}_{n \in \mathcal{N}}$ and $\{\beta_n\}_{n \in \mathcal{N}}$ are monotone increasing and decreasing, respectively, and they converge punctually to solutions u and v of (6) and are such that $\alpha_n \leq u \leq v \leq \beta_n$ for all n . The use of these methods exhibits many difficulties. The lower and upper solutions of BVPs represent the start points of the iterative schemes, but there is no clue to finding them. Moreover, the iterative schemes do not supply constructive algorithms to approximate the solutions of the BVPs. If this happens (see [23] for periodic problems), then it is necessary to give an efficient numerical algorithm to approximate the solutions by monotone sequences. In this case the error analysis is only possible “a posteriori”.

The **quasilinearization method** (QLM) was originally introduced by Bellman and Kalaba [16] as a generalization of the Newton–Raphson method. This method generates sequences of approximate solutions of linear problems which

¹ We recall that a function α (β) $\in C^2([a, b])$ is a lower (upper) solution for (1) if $\alpha''(x) - f(x, \alpha(x), \alpha'(x)) \geq 0$ ($f(x, \beta(x), \beta'(x)) - \beta''(x) \geq 0$) $\forall x \in [a, b]$.

quadratically, often monotonically, converge to a solution of the original nonlinear BVP provided that the functions involved are convex [17,19]. The aim of QLM is to solve the nonlinear BVP (1) as a limit of the following sequence of linear BVP

$$\begin{cases} y''_{r+1} = f(x, y_r, y'_r) + \sum_{s=0}^1 (y_{r+1}^{(s)} - y_r^{(s)}) f_{y^{(s)}}(x, y_r, y'_r), & x \in [a, b], \\ l_a y_{r+1}(a) - m_a y'_{r+1}(a) = v_a, & l_b y_{r+1}(b) + m_b y'_{r+1}(b) = v_b, \end{cases}$$

where $y^{(0)} = y$, $y^{(1)} = y'$ and $f_{y^{(s)}} = \partial f / \partial y^{(s)}$. We remark that the zeroth approximation y_0 is based on mathematical or physical considerations, and only a “sufficiently good” initial guess y_0 generates a rapid convergence of the method. For these reasons, the QLM gives excellent results when applied to nonlinear BVP in physics [12,29].

The **generalized quasilinearization method** (GQL) adopts the technique of lower and upper solutions combined with QLM and it generates lower and upper monotone sequences whose elements are the solutions of the corresponding linear problems. Both the sequences converge rapidly to the solution of the considered BVP [13,24,25,27,28].

The **shooting method** consists in reducing the BVPs to a family $\mathcal{F}$ of initial value problems (IVPs) for the same equation, where $\mathcal{F}$ is chosen in such a way to contain the solution of the given BVP [9,7,8]. If we denote by $y(x, r)$ the solution of the following IVP

$$F_s : \begin{cases} y'' = f(x, y, y'), & x \in [a, b], \\ y(a, r) = (v_a + m_a r) / l_a, \\ y'(a, r) = r, \end{cases} \quad (8)$$

the problem is reduced to finding $r = r_*$ which solves the equation

$$\phi(r) \equiv l_b y(b, r_*) + m_b y'(b, r_*) - v_b = 0. \quad (9)$$

In other words, in the shooting method we solve (8) for different choices of r until condition (9) is satisfied. We remark that each evaluation of (9) involves again the solution of the IVP (8). Then, we cannot hope to exactly evaluate $\phi(r)$. Therefore, to approximate the root of (9) we should try to use very rapidly converging iterative schemes such as Newton's method. Finally, to obtain a general code for solving nonlinear BVPs by the shooting method, it is necessary to use a library routine for solving nonlinear equations and a standard IVPs solver. We note that the IVPs in (8) could be *ill-conditioned*, even if the BVP is *well-conditioned*. In this case, the shooting method is *unstable*. In the presence of the nonlinear BVP, there is another potential trouble: when the shooting method starts from wrong initial values r_m , the IVP solution $y(x, r_m)$ could exist only in $[a, c]$, where $c < b$. In such cases, it is not known how to correct r_m , because the nonlinear iteration of (9) cannot be completed, and therefore Newton's method (or any of its variants) fails (see [31,30] and the references therein).

Finally, we expose in detail the **finite difference methods** [9,7,8]. It is well known that the fundamental steps of this method can be described as follows:

- (1) The interval $[a, b]$ is substituted by a discrete set of its points $x_k, k = 0, \dots, n$, where $x_0 = a$ and $x_n = b$, i.e. we introduce the mesh $\pi : a = x_1 < x_2 < \dots < x_n = b$.
- (2) Instead of the function $y(x)$, depending on the continuous variable x , a *grid function* $\mathbf{y}_\pi = (y_0, \dots, y_n)$, where $y_k = y(x_k)$, of the discrete variable x_k is considered. The derivatives appearing in the differential equations and in the boundary conditions are approximated by suitable algebraic expressions $\mathcal{D}y_k, \mathcal{D}^2y_k$ containing the unknown values y_k .
- (3) The resulting finite system S

$$\begin{cases} \mathcal{D}^2y_k = f(x_k, y_k, \mathcal{D}y_k), & k = 1, \dots, n-1 \\ l_a y_0 - m_a \mathcal{D}y_0 = v_a, \\ l_b y_n + m_b \mathcal{D}y_n = v_b, \end{cases} \quad (10)$$

in the unknown numerical values y_k is solved in order to approximate $y(x)$ with the function interpolating the values $(x_k, y_k), k = 0, \dots, n$.

We recall that the approximate formulae of the derivatives are often given by the Lagrangian interpolating polynomials [32], the Chebyshev polynomials [33] or by the finite difference expressions of order four [35], six and eight [34].

In order to highlight the problems connected with the implementation of the finite difference method it is necessary to recall some definitions.

We define the differential operator

$$\mathcal{L}y = -y'' + f(x, y, y') \quad (11)$$

and a corresponding difference operator

$$\mathcal{L}_\pi y_i = -\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right), \quad (12)$$

where h is the step size of the uniform mesh π . The difference operator $\mathcal{L}_\pi$ is said to be *consistent of order p with $\mathcal{L}$* if, for every smooth function $v(x)$, positive constants c and h_0 exist such that for all meshes π with $h \leq h_0$, we have that

$$\mathcal{T}_\pi v = \|\mathcal{T}_i v\|_\infty \leq ch^p,$$

where $\mathcal{T}_i v = \mathcal{L}_\pi v(x_i) - (\mathcal{L}v)(x_i)$ is the *local truncation error*.

Moreover, from Taylor's theorem, we can derive the relation

$$\mathcal{T}_i v = -\frac{h^2}{12} \left[v^{(4)}(\xi_1) - 2f_z(x_i, v_i, v'(\xi_2)) v'''(\xi_3) \right], \quad (13)$$

for any grid function $v \in C^4([a, b])$ and $\xi_i \in [x_{i-1}, x_{i+1}]$, $i = 1, 2, 3$. Then, $\mathcal{L}_\pi$ has a second-order accuracy in approximating $\mathcal{L}$ for any function v having continuous fourth derivatives on $[a, b]$ (see [7,36]). Finally, the scheme (10) is said to be *converging* to the solution $y = y(x)$ of the BVP (1) if for any grid function v a constant $h_0 > 0$ exists such that for all the meshes π with $h \leq h_0$ we have

$$\lim_{h \rightarrow 0} \|v(x_i) - y(x_i)\|_\infty = 0.$$

Another important concept of the Numerical Analysis is the *stability* of the difference scheme (10) which requires that the inverse of the difference operator, including the boundary conditions, is suitably bounded. Moreover, for both the linear and nonlinear BVPs, the consistency and the stability of the finite difference scheme assure the convergence (see [8]).

The definition and proof of stability for the difference scheme

$$\begin{cases} \mathcal{D}^2 y_k = f(x_k, y_k, \mathcal{D}y_k), & k = 1, \dots, n-1 \\ y_0 = v_a, & y_n = v_b, \end{cases} \quad (14)$$

relative to the Dirichlet problem

$$\begin{cases} y'' = f(x, y, y'), \\ y(a) = v_a, & y(b) = v_b, \end{cases} \quad x \in [a, b], \quad (15)$$

are contained in the following theorem.

Theorem 1. *If we assume that*

- (1) $f(x, y, z) \in C^1([a, b] \times \mathfrak{R}^2, \mathfrak{R})$
- (2) $\exists K_1, K_2, L_2$ such that $0 < K_1 \leq f_y \leq K_2$, $|f_z| \leq L_2$, on $[a, b] \times \mathfrak{R}^2$,

then, for all h such that $hL_2 \leq 2$, the scheme (14) is stable in the sense that for any two grid functions $v = \{v_n\}$, $w = \{w_n\}$ we have that

$$\|v - w\|_\infty \leq M \{ \max(|v_0 - w_0|, |v_n - w_n|) + \|\mathcal{L}_\pi v - \mathcal{L}_\pi w\|_\infty \},$$

where $M = \max(1, 1/K_1)$. Moreover, for any function $v \in C^4([a, b])$ it is $\|v - y\|_\infty \leq M\mathcal{T}_\pi y$, and the solution of (14) can be computed by a Newton-like method.

Remark 2. The above theorem requires that the function f is smooth, f_z is bounded and f_y is bounded and positive. These assumptions are usually the hypotheses of the existence and uniqueness theorems of BVP (15). Nevertheless, it is worthwhile to note that in the applications we frequently face with smooth functions that do not satisfy one or more of these conditions.

Remark 3. We have already observed that the convergence of the finite difference scheme, which is assured by the Theorem 1, is subjected to a “good” choice of the *start points* $\mathbf{y}_\pi^0 = (y_1^0, \dots, y_n^0)$. However, how to choose the initial guess $\mathbf{y}_\pi^0$ “close” to the exact solution remains an open problem. Usually, for solving (14) the start points y_i^0 are chosen in such a way that the points $P_i \equiv (x_i, y_i^0)$ belong to the segment of extrema $A \equiv (a, y_a)$ and $B \equiv (b, y_b)$. In other cases the choice $y_i^0 = 0$ for all i is preferred. However, as it is proved in this paper, the choice of the initial guess is strictly connected to the assigned BVP and a wrong choice could compromise the results obtained with the implementation of the finite difference method.

Generally, a unified theory of stability and convergence for the nonlinear BVPs with mixed linear or nonlinear boundary conditions, is given in terms of first-order systems of ODEs (see [7,8]). In addition, the consistency and the order of accuracy are defined as before, whereas the stability is defined only in the proximity of an *isolated* solution. The known result that the convergence is assured by the consistence and stability is adopted in [8], where the authors prove the convergence theorems under the assumption that the linearized finite difference scheme, around the (unknown) isolated solution, is consistent and stable (see [8], p. 207). We rather prefer to follow Keller’s approach (see [7]) in which the consistency and convergence of the finite difference scheme is proved by the following theorem (it has been reformulated in order to fit the formulation of BVP (1)).

Theorem 4. Let $f(x, y, z)$ satisfy in $R = \{(x, y, z) : x \in [a, b], \quad |y|, |z| < \infty\}$, the conditions

- (1) $f(x, y, z) \in C^1(R, \mathfrak{R})$,
- (2) $\exists K$ such that $\left\| \begin{pmatrix} 0 & 1 \\ f_y & f_z \end{pmatrix} \right\|_\infty \leq K$,
- (3) $\sigma := m_a l_b + m_b l_a \neq 0$,
- (4) for some $\lambda \in (0, 1)$ and $m = \frac{1}{\sigma} \max \{m_a l_b, m_a m_b, l_a l_b, m_b l_a\}$,

$$K|b - a| \leq \ln \left(1 + \frac{\lambda}{m} \right). \quad (16)$$

Then, both the BVP (1) and, for all h , the finite difference problem (10) have solutions which are unique. In particular, the solution of the difference equations (10) is the limit of the sequence $\{\mathbf{y}_\pi^v\}_v$ defined by

$$\begin{cases} \mathcal{D}^2 y_k^{(v+1)} = f(x_k, y_k^v, \mathcal{D} y_k^v), & k = 1, \dots, n-1 \\ l_a y_0^{(v+1)} - m_a \mathcal{D} y_0^{(v+1)} = v_a, \\ l_b y_n^{(v+1)} + m_b \mathcal{D} y_n^{(v+1)} = v_b, \end{cases}$$

where $\{\mathbf{y}_\pi^0\}$ are arbitrary. Moreover, if $f(x, y, z) \in C^2(R, \mathfrak{R})$, then the solution $y(x)$ of the BVP (1) and the approximate solution $y_\pi(x)$ defined by the difference problem (10) satisfy the condition

$$\|y(x_k) - y_\pi(x_k)\|_\infty = O(h^2).$$

Even in this case, the Remarks 2 and 3 still hold. In the hypotheses of the Theorems 1 and 4, the finite difference method is consistent and convergent and the problems (15) and (1) can be solved with the accuracy order $O(h^2)$. The next section is devoted to the description of a particular numerical scheme for solving the problem of the starting points.

3. The proposed scheme for BVP with linear boundary conditions

In order to overcome the problems related to the implementation of the finite difference method for (1), we first search for a function $y_*(x)$ such that

$$f(x, y_*(x), y'_*(x)) \neq 0, \quad \text{and/or} \quad \begin{cases} l_a y_*(a) - m_a y'_*(a) - v_a \neq 0, \\ l_b y_*(b) + m_b y'_*(b) - v_b \neq 0, \end{cases} \quad (17)$$

and then we introduce the following family of BVPs A_j , $j = 0, \dots, s$,

$$A_j : \begin{cases} y'' = f(x, y, y') - \left(1 - \frac{j}{s}\right) f_*(x), & x \in [a, b], \\ l_a y(a) - m_a y'(a) - v_a = \left(1 - \frac{j}{s}\right) g_{*1}, \\ l_b y(b) + m_b y'(b) - v_b = \left(1 - \frac{j}{s}\right) g_{*2}, \end{cases} \quad (18)$$

where $f_*(x) = f(x, y_*(x), y'_*(x))$, $g_{*1} = l_a y_*(a) - m_a y'_*(a) - v_a$, $g_{*2} = l_b y_*(b) + m_b y'_*(b) - v_b$.

Let us suppose that each BVP (18) admits a solution which can be evaluated by the finite difference method. This means that Theorems 1 and 4 for the problems (15) and (1), respectively, hold.

If $y(x, j)$ denotes the solution of the BVP A_j , for $j = 0, \dots, s$, then the problem A_0 admits only the known solution $y(x, 0) = y_*(x)$, whereas the problem A_s coincides with the original problem (1). To each A_j in (18) is associated the following finite system S_j

$$S_j : \begin{cases} \mathcal{D}^2 y_k = f(x_k, y_k, \mathcal{D} y_k) - \left(1 - \frac{j}{s}\right) f_*(x_k), & k = 1, \dots, n-1 \\ l_a y_0 - m_a \mathcal{D} y_0 - v_a = \left(1 - \frac{j}{s}\right) g_{*1}, \\ l_b y_n + m_b \mathcal{D} y_n - v_b = \left(1 - \frac{j}{s}\right) g_{*2}. \end{cases} \quad (19)$$

The system S_0 admits the exact solution $y(x, 0) = y_*(x)$, for $j = 1$, the nonlinear system S_1 can be solved with an iterative method that uses the solution $y_*(x_k)$, $k = 0, \dots, n$, as start points of Newton-like methods. Similarly, the problem S_j , $j > 1$, can be solved using the approximate solution obtained at the step $j - 1$. Finally, for $j = s$, the nonlinear system S_s , corresponding to BVP (1), is solved by using the approximate solution evaluated at the step $j = s - 1$. The only aim of this procedure is to assign the starting values $\mathbf{y}_\pi^0$ of the sequence whose limit is the required solution of (19) for $j = s$. We presume that the chosen starting points are very close to the values $y_{k,j} = y(x_k, j)$ which, for $j = s$, represent approximate values of the solution $y(x)$ of the BVP (1) at the points x_k . This conjecture is simply verified for the problem (1) when *Green's functions* are used. Indeed, it is well known that if $f \in C([a, b] \times \mathfrak{R}^2, \mathfrak{R})$ and the following conditions hold

$$l_a m_b + l_b m_a \neq 0, \quad \text{or} \quad l_a + l_b > 0, m_a + m_b > 0, \quad (20)$$

then the BVP (1) has a solution $y(x)$ in the form (see f.i. [10,1,11])

$$y(x) = \int_a^b G(x, \tau) f(\tau, y(\tau), y'(\tau)) d\tau + \phi(x), \quad (21)$$

where $\phi(x)$ is the solution of the BVP $y'' = 0$, (1)_{2,3}, and *Green's function* $G(x, \tau)$ is

$$G(x, \tau) = \begin{cases} (1/c) u(x) v(\tau), & a \leq \tau \leq x \leq b, \\ (1/c) u(\tau) v(x), & a \leq x \leq \tau \leq b, \end{cases} \quad (22)$$

where $c = u(x)v'(x) - v(x)u'(x)$, and $u(x)$, $v(x)$ are two linearly independent solutions of the following problems, respectively

$$\begin{cases} y'' = 0, \\ l_a y(a) - m_a y'(a) = 0, \end{cases} \quad \begin{cases} y'' = 0, \\ l_b y(b) + m_b y'(b) = 0, \end{cases} \quad x \in [a, b].$$

In particular, the following existence result for the BVPs (1) holds.

Proposition 5. Let $f \in C([a, b] \times \mathfrak{R}^2, \mathfrak{R})$ be bounded on $[a, b] \times \mathfrak{R}^2$, then, the BVP (1) has a solution, whenever (20) holds.

It is easy to believe that if an existence theorem for the BVP (1) is valid, then any other BVP in (18) has a solution $y(x, j)$ “close” to the solution $y(x)$ of (1). In fact, the solution $y(x, j)$, $j = 0, \dots, s-1$, of any BVPs (18) writes

$$y(x, j) = \int_a^b G(x, \tau) \left[f(\tau, y(\tau), y'(\tau)) - \left(1 - \frac{j}{s}\right) f_*(\tau) \right] d\tau + \phi_j(x), \quad (23)$$

where $\phi_j(x)$ is the solution of the BVP $y'' = 0$, (18)_{2,3}, and $G(x, \tau)$ is *Green's function* defined in (22). With simple but tedious computation it is easy to prove that

$$\phi_j(x) = \frac{j}{s} \phi(x) - \left(1 - \frac{j}{s}\right) (Ax - B), \quad j = 0, \dots, s-1, \quad (24)$$

where $\phi(x)$ is the solution of the BVP $y'' = 0$, (1)_{2,3}, and

$$A = \frac{l_b v_a^* - l_a v_b^*}{l_b m_a + l_a [(b-a)l_b + m_b]}, \quad B = \frac{(bl_b + m_b) v_a^* - (al_a - m_a) v_b^*}{l_b m_a + l_a [(b-a)l_b + m_b]},$$

$$v_a^* = l_a y_*(a) - m_a y'_*(a), \quad v_b^* = l_b y_*(b) + m_b y'_*(b).$$

By using (23), (21) and (24) the following relation holds

$$|y(x, j) - y(x)| \leq \left(1 - \frac{j}{s}\right) (NM^0 + L + H), \quad (25)$$

where

$$N = \max_{x, \tau \in [a, b]} |G(x, \tau)(b-a)|, \quad M^0 = \max_{x \in [a, b]} |f_*(x)|,$$

$$L = \max_{x \in [a, b]} |\phi(x)|, \quad H = \max_{x \in [a, b]} |Ax - B|.$$

Remark 6. For the Dirichlet BVP (15) the relation (17) becomes

$$f(x, y_*(x), y'_*(x)) \neq 0, \quad \text{and/or} \quad \begin{cases} y_*(a) - v_a \neq 0, \\ y_*(b) - v_b \neq 0, \end{cases} \quad (26)$$

and (18) writes

$$A_j : \begin{cases} y'' = f(x, y, y') - \left(1 - \frac{j}{s}\right) f_*(x), & x \in [a, b], \\ y(a) - v_a = \left(1 - \frac{j}{s}\right) g_{*1}, \\ y(b) - v_b = \left(1 - \frac{j}{s}\right) g_{*2}, \end{cases} \quad (27)$$

where $g_{*1} = y_*(a) - v_a$, $g_{*2} = y_*(b) - v_b$. The Green function (22) writes

$$G(x, \tau) = \begin{cases} (b-x)(\tau-a)/(b-a), & a \leq \tau \leq x \leq b, \\ (b-\tau)(x-a)/(b-a), & a \leq x \leq \tau \leq b, \end{cases}$$

and, if we put $A = (y_*(a) - y_*(b))/(b-a)$, $B = (by_*(a) - ay_*(b))/(b-a)$ in (24), the relation (25) holds with $N = (b-a)^2/8$. Moreover, the Proposition 5 becomes

Proposition 7. Let $f \in C([a, b] \times \mathfrak{R}^2, \mathfrak{R})$ and for $(x, y_1, z_1), (x, y_2, z_2) \in [a, b] \times \mathfrak{R}^2$, we have

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \leq k_1 |y_1 - y_2| + k_2 |z_1 - z_2|,$$

where k_1, k_2 are positive constants such that

$$k_1 \frac{(b-a)^2}{8} + k_2 \frac{(b-a)}{2} < 1.$$

Then, the BVP (15) has one and only one solution.

Remark 8. If we assume that $f \in C([a, b] \times \mathfrak{R}^2, \mathfrak{R})$ is bounded in $[a, b] \times \mathfrak{R}^2$, then every BVP (15) has a solution (see [10] p. 9).

Remark 9. Usually, it is simple to verify the condition (17). In most cases it is sufficient to choose $y_*(x) = \text{const}$. For example, in [9] the numerical scheme (18) is implemented under the hypothesis that condition (17) is satisfied by the zero function.

4. The proposed scheme for BVP with nonlinear boundary conditions

In this section, we consider the following fully nonlinear BVP

$$\begin{cases} y'' = f(x, y, y'), & x \in [a, b], \\ \mathbf{G}((y(a), y(b)), (y'(a), y'(b))) = 0, \end{cases} \quad (28)$$

where $\mathbf{G} = (g_1, g_2)$, $g_i \in C([a, b] \times \mathfrak{R}^2, \mathfrak{R})$ are nonlinear functions, as well as its discrete approximation

$$\begin{cases} \mathcal{D}^2 y_k = f(x_k, y_k, \mathcal{D}y_k), & k = 1, \dots, n-1 \\ \mathbf{G}((y_0, y_n), (\mathcal{D}y_0, \mathcal{D}y_1)) = 0, \end{cases} \quad (29)$$

where the meaning of the adopted symbols is self-evident. As in the previous section, we consider the following family of BVPs A_j , $j = 0, \dots, s$,

$$A_j : \begin{cases} y'' = f(x, y, y') - \left(1 - \frac{j}{s}\right) f_*(x), & x \in [a, b], \\ g_1((y(a), y(b)), (y'(a), y'(b))) = \left(1 - \frac{j}{s}\right) \tilde{g}_{*1}, \\ g_2((y(a), y(b)), (y'(a), y'(b))) = \left(1 - \frac{j}{s}\right) \tilde{g}_{*2}, \end{cases} \quad (30)$$

where $f_*(x) = f(x, y_*(x), y'_*(x))$, $\tilde{g}_{*i} = g_i((y_*(a), y_*(b)), (y'_*(a), y'_*(b)))$, $i = 1, 2$, and $y_* = y_*(x)$ is such that

$$f(x, y_*(x), y'_*(x)) \neq 0, \quad \text{and/or} \quad \begin{cases} g_1((y_*(a), y_*(b)), (y'_*(a), y'_*(b))) \neq 0, \\ g_2((y_*(a), y_*(b)), (y'_*(a), y'_*(b))) \neq 0. \end{cases} \quad (31)$$

Once again, the problem A_0 admits only the exact solution $y_*(x)$, whereas the problem A_s coincides with the original problem (28). To each A_j in (30) the following finite system S_j is associated

$$S_j : \begin{cases} \mathcal{D}^2 y_k = f(x_k, y_k, \mathcal{D}y_k) - \left(1 - \frac{j}{s}\right) f_*(x_k), & k = 1, \dots, n-1 \\ g_1((y_0, y_n), (\mathcal{D}y_0, \mathcal{D}y_n)) = \left(1 - \frac{j}{s}\right) g_{*1}, \\ g_2((y_0, y_n), (\mathcal{D}y_0, \mathcal{D}y_n)) = \left(1 - \frac{j}{s}\right) g_{*2}, \end{cases} \quad (32)$$

which could be solved by using Newton-like methods based on starting values which are known from the step $j = s-1$. The considerations reported in Remark 9 hold for the condition (31). For the problems (28), the scheme (30) gives only a suggestion about the starting points of the Newton method. Indeed, the only theorems which ensure the convergence of the finite difference method and which do not refer to the linearization of (28) close to an isolated solution, are published in [4]. However, it is very difficult to verify the hypotheses on which these convergence and existence theorems for the problem (28) are based [14]. In the next section we will show some applications of the proposed method for fully nonlinear BVPs for which we know an analytical solution or an existence and uniqueness theorem.

5. The numerical simulations

In this section we propose few numerical simulations implemented by the notebooks **NBoundaryD.nb** and **NBoundaryM.nb** written by Mathematica, for solving the problems (15), (1) and (28), respectively. The use of Mathematica is suggested by the need of implementing the numerical scheme proposed in this paper, deputing to the routine **FindRoot** the task to numerically solve the finite difference systems (27), (19) and (32) with the most appropriate iterative method (Newton, Brent, secant, etc). Moreover, in these programs we express the first and the second derivatives in any BVPs (15), (1) and (28) by the formulae

$$\begin{aligned}\mathcal{D}y_k &= \frac{y_{k+1} - y_{k-1}}{2h} + O(h^2), \\ \mathcal{D}^2y_k &= \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + O(h^2).\end{aligned}\quad (33)$$

In order to keep the accuracy order $O(h^2)$, we write the finite difference schemes (10) and (29) in the form

$$\begin{cases} \mathcal{D}^2y_k = f(x_k, y_k, \mathcal{D}y_k), & k = 0, \dots, n \\ l_a y_0 - m_a \frac{y_0 - y_{-1}}{2h} = v_a, \\ l_b y_n + m_b \frac{y_{n+1} - y_n}{2h} = v_b, \end{cases}\quad (34)$$

$$\begin{cases} \mathcal{D}^2y_k = f(x_k, y_k, \mathcal{D}y_k), & k = 0, \dots, n \\ \mathbf{G}\left((y_0, y_n), \left(\frac{y_0 - y_{-1}}{2h}, \frac{y_{n+1} - y_n}{2h}\right)\right) = 0, \end{cases}\quad (35)$$

where $x_{-1} = a - h$, $x_{n+1} = b + h$.

Simulation 1. We consider the following Dirichlet BVP

$$\begin{cases} y'' = -\cos y \sin y' + 2y + \cos(1 - x^2) \sin(2x) - 2(x^2 - 1) + 2, & x \in [-1, 1] \\ y(-1) = 0, \quad y(1) = 0, \end{cases}\quad (36)$$

which admits the unique solution $y = x^2 - 1$. Since $f_y = 2 + \sin y \sin y'$, $f_{y'} = -\cos y \cos y'$, the hypotheses of [Theorem 1](#) are satisfied and the finite difference method converges for all $h \leq 2$, or equivalently for $n \geq 1$. Moreover, by using the computing notebook **NBoundaryD.nb** for $n = 5$, we have the following table of the approximations of the analytic solution

Starting solution	Max Absolute Errors for $s = 1$	Max Absolute Errors for $s = 2$
$y_*(x) = 0$	10^{-25}	10^{-28}
$y_*(x) = -2$	10^{-24}	10^{-27}
$y_*(x) = -x$	10^{-16}	10^{-27}
$y_*(x) = \sin 2x$	10^{-16}	10^{-25}
$y_*(x) = x^2$	10^{-25}	10^{-26}

(37)

Simulation 2. In this simulation the limits and the capacities of [Theorems 1](#) and [4](#) are discussed. The function $y = x(4x^2 - 1)$ is the only solution of the following BVP with mixed linear boundary conditions (see [Proposition 5](#))

$$\begin{cases} y'' = -\sin y' - \cos y + \cos x(4x^2 - 1) - \sin(1 - 12x^2) + 24x, & x \in [-1/2, 1/2] \\ y(-1/2) - y'(-1/2) = -2, \quad y(1/2) + y'(1/2) = 2. \end{cases}\quad (38)$$

Since $f_y = \sin y$, $f_{y'} = -\cos y'$, the hypotheses of [Theorem 4](#) are satisfied for $K = 1$, $m = 1/2$, and for any $\lambda : -1/2 + e/2 \leq \lambda < 1$ the relation (16) is verified. Then, the finite difference methods converge for all step h to

the unique solution that they can be computed with an iterative Newton-like method by starting with arbitrary initial guess. By using the notebook **NBoundaryM.nb**, we experimentally find the results of [Theorem 4](#), as it is shown in the following table

Starting solution	$y_*(x) = 0$	$y_*(x) = -x$
Max-Min Abs. Errors $n = 5$	$6 \times 10^{-2} - 7 \times 10^{-3}$	$6 \times 10^{-2} - 7 \times 10^{-3}$
Max-Min Abs. Errors $n = 30$	$2 \times 10^{-3} - 3 \times 10^{-5}$	$2 \times 10^{-3} - 3 \times 10^{-5}$
Max-Min Abs. Errors $n = 70$	$3 \times 10^{-4} - 4 \times 10^{-6}$	$3 \times 10^{-4} - 4 \times 10^{-6}$

(39)

We see that the approximate solution does not change in a significant way on increasing the number $s > 1$ of iterations.

If we substitute the boundary conditions (38)₂ with the following one

$$y'(-1/2) = 2, \quad y(1/2) + y'(1/2) = 2. \quad (40)$$

then there not exists any value of $\lambda \in (0, 1)$ such that (16) is valid, although the hypotheses of the [Proposition 5](#) are satisfied. **NBoundaryM.nb** shows that the finite difference method supplies results close to those of Table (39), or better still it gives results with a lower minimum absolute error (see Table (41)). Finally, the independence of the initial approximation in **FindRoot** still persists.

Starting solution	$y_*(x) = 0$	$y_*(x) = -x$
Max-Min Abs. Errors $n = 5$	$1.5 \times 10^{-1} - 4 \times 10^{-3}$	$1.5 \times 10^{-1} - 4 \times 10^{-3}$
Max-Min Abs. Errors $n = 30$	$4 \times 10^{-3} - 7 \times 10^{-5}$	$4 \times 10^{-3} - 7 \times 10^{-5}$
Max-Min Abs. Errors $n = 70$	$7 \times 10^{-4} - 1.2 \times 10^{-6}$	$7 \times 10^{-4} - 1.2 \times 10^{-6}$

(41)

It is not possible to apply the [Theorem 1](#) to the Dirichlet problem for the Eq. (38)₁ with the following boundary conditions

$$y(-1/2) = 0, \quad y(1/2) = 0.$$

However, the right-hand side in (38)₁ is bounded in $[a, b] \times \mathfrak{R}^2$ so that the problem has at least a solution which the notebook **NBoundaryD.nb** gives with high accuracy (see Table (42))

Starting solution	$y_*(x) = 0$	$y_*(x) = -x$
Max-Min Abs. Errors $n = 5$	$1.5 \times 10^{-2} - 4 \times 10^{-25}$	$1.5 \times 10^{-2} - 4 \times 10^{-25}$
Max-Min Abs. Errors $n = 30$	$4 \times 10^{-4} - 1.6 \times 10^{-29}$	$4 \times 10^{-4} - 1.6 \times 10^{-29}$
Max-Min Abs. Errors $n = 70$	$7.3 \times 10^{-5} - 5 \times 10^{-31}$	$7.3 \times 10^{-5} - 5 \times 10^{-31}$

(42)

Simulation 3. We consider the following problem which arises in the study of finite deflections of an elastic string under a transverse load

$$\begin{cases} y'' = -(1 + a^2(y')^2), & x \in [0, 1] \\ y(0) = 0, & y(1) = 0, \end{cases} \quad (43)$$

which admits the unique solution

$$y(x) = \ln \left(\frac{\cos a(x - 1/2)}{\cos a/2} \right) / a^2.$$

For $a = 1/7$, in [26] it is proved that the discrete problem, associated to (43), has a solution y_π satisfying $0 \leq y_\pi(x_k) \leq 4 - x_k^2$, $k = 0, \dots, n$ for h sufficiently small. By using the notebook **NBoundaryD.nb** for

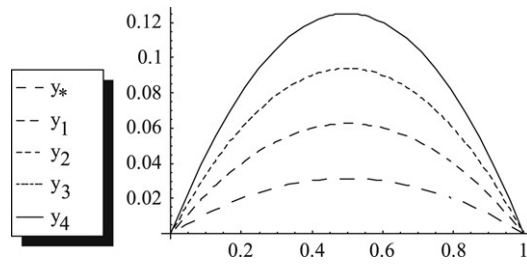
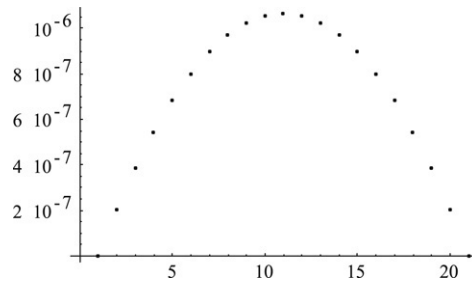
Fig. 1a. Graphics of $y = y(x, j)$.

Fig. 1b. Absolute error in the knots.

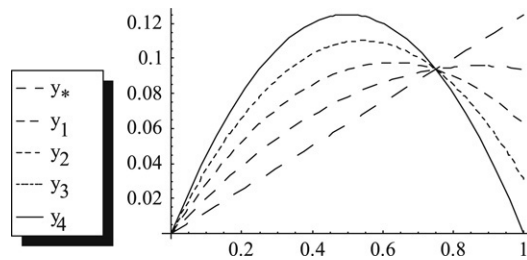
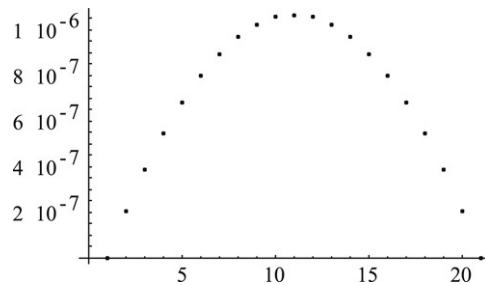
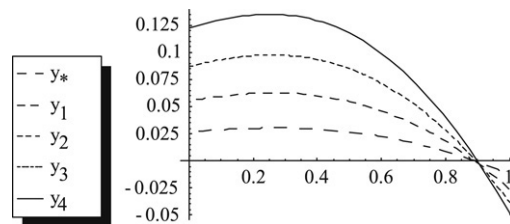
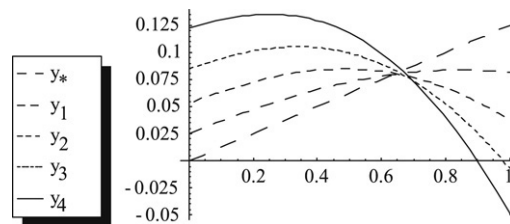
Fig. 2a. Graphics of $y = y(x, j)$.

Fig. 2b. Absolute error in the knots.

$n = 20, s = 4, y_*(x) = 0$, we have the following results (see Figs. 1a and 1b). By running again the notebook **NBoundaryD.nb** for $y_*(x) = x/8$ we get the results in Figs. 2a and 2b.

Fig. 3a. Graphics of $y(x, j)$.Fig. 3b. Graphics of $y(x, j)$.

Simulation 4. In [14] and [4], the authors prove that both the following fully nonlinear BVP and its discrete problem admit a solution

$$\begin{cases} y'' = -\sin x - 2 \cos(x - y^2) \sin y - (y')^5, & x \in [0, 1] \\ y(0) - y'(0) + [y^2(1) - (y'(1))^2] / 10 = 0, \\ y(0) + y'(0) + 6y(1) + y'(1) + \sin[y(0) - y'(1)] = 0. \end{cases} \quad (44)$$

Moreover, the solutions $y(x)$ and $y_\pi(x_k)$ of the above problems are such that

$$\begin{aligned} -\pi/2 &\leq y(x), \quad y_\pi(x_k) \leq \pi/2, \\ -2 &\leq y'(x), \quad y'_\pi(x_k) \leq 2, \quad k = 0, \dots, n. \end{aligned}$$

By resorting to the program **NBoundaryM.nb**, for $n = 40$, $s = 4$, we obtain the following results (Figs. 3a and 3b) where in the first case we chose $y_*(x) = 0$ and in the second one $y_*(x) = x/8$.

Moreover, comparing the numerical results we have obtained in the above computations, we can observe that the difference between them is of the order 10^{-18} .

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