Numerical regularization of a real inversion formula based on the Laplace transform's eigenfunction expansion of the inverse function

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2007 Inverse Problems 23 713
(http://iopscience.iop.org/0266-5611/23/2/015)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 192.167.11.69
This content was downloaded on 23/11/2016 at 15:06

Please note that terms and conditions apply.

You may also be interested in:

Regularization of a Fourier series method for the Laplace transform inversion with real data
Luisa D'Amore and Almerico Murli

Extended-domain-eigenfunction method (EDEM): a study of ill posedness and regularization
J Aarão, S J Miklavcic and D A Ward

A numerical solution of a Cauchy problem for an elliptic equation by Krylov subspaces
Lars Eldén and Valeria Simoncini

A smoothing spline that approximates Laplace transform functions only known on measurements on the real axis
L D'Amore, R Campagna, A Galletti et al.

Regularization by fractional filter methods and data smoothing
E Klann and R Ramlau

Regularization with randomized SVD for large-scale discrete inverse problems
Hua Xiang and Jun Zou

Integral equation models for image restoration
Yao Lu, Lixin Shen and Yuesheng Xu
Numerical regularization of a real inversion formula based on the Laplace transform’s eigenfunction expansion of the inverse function

A Murli, S Cuomo, L D’Amore and A Galletti

University of Naples, Federico II, Naples, Italy

Received 10 October 2006, in final form 23 January 2007
Published 8 March 2007
Online at stacks.iop.org/IP/23/713

Abstract
We describe the numerical approximation of the inverse Laplace function based on the Laplace transform’s eigenfunction expansion of the inverse function, in a real case. The error analysis allows us to introduce a regularization technique involving computable upper bounds of amplification factors of local errors introduced by the computational process. A regularized solution is defined as one which is obtained within the maximum attainable accuracy. Moreover the regularization parameter, that in this case coincides with the truncation parameter of the eigenfunction expansion, is dynamically computed by the algorithm itself in such a way that it provides the minimum of the global error bound.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Let
\[ \mathcal{L} : f(t) \rightarrow F(s) = \int_0^\infty e^{-st} f(t) \, dt \] (1)
be the Laplace transform integral operator. We deal with the numerical inversion of \( F(s) \) under the assumption that \( F \) is a given function computable on a real axis only. For instance, this problem may arise in the solution of differential equation problems by means of the Laplace transformation, in electric-circuit theory, in linear-control theory or in specific applications such as x-ray diffractometry.

This problem is ill posed and heavily ill conditioned. Since Tikhonov’s work on regularization methods the general theory for solving ill-posed problems developed systematically [5]. Briefly, regularization methods for computing an approximation of \( f = \mathcal{L}^{-1}[F] \) are based on the construction of a so-called regularized solution \( f_\alpha = R_\alpha[F] \), where \( f_\alpha \) is such that
\[ \lim_{\alpha \to 0} f_\alpha = f. \]
The parameter \( \alpha \) is called a regularization parameter and \( R_\alpha \) is the regularization operator. Regularized approximations \( f_\alpha \) belong to the so-called admissible set \( S \) that guarantees that \( R_\alpha \) is well defined [3]. \( R_\alpha \) is typically defined taking into account \textit{a priori} information regarding the desired solution, such as smoothing properties, asymptotic behaviour, etc, and it aims to mitigate the ill posedness of the original problem.

One of the most important problems in connection with regularization is the choice of the regularization parameter that balances the quantity of regularization introduced on the solution with respect to its reliability. Different techniques have been proposed in order to compute ‘optimal’ values of the regularization parameter (L-curve, Morozov’s discrepancy principle, cross-validation, etc), each using some kind of information \textit{a priori} that characterizes the operator \( R_\alpha \). Although such methods are well defined and quite satisfactory theoretically, computing the ‘optimal’ value of the regularization parameter in a reliable way is perhaps not as easy as it might seem. In the present paper, we introduce the numerical regularization operator (see definition 1.6) and we refer to the numerical regularization as the computation of an approximation of the inverse function \( f \) by requiring that the overall errors introduced during the numerical computation of \( f_\alpha \) have to be the minimum possible. The main feature of the proposed numerical approach is the dynamic computation of the regularization parameter that in this case coincides with the truncation parameter of the eigenfunction expansion. The regularization parameter is obtained by minimizing the errors. Moreover, this approach allows us to compute the solution with the best attainable accuracy and to estimate this. With this aim, a detailed error analysis is required, and section 2 is devoted to the errors’ analysis and their estimates. In section 3 we introduce the regularization technique that involves \textit{a posteriori} computable bounds and we discuss computation of the optimal value of the regularization parameter. Finally, in section 4, the main computational issues underlying the proposed algorithm are described and in section 5 we give some numerical experiments. Conclusions are summarized in section 6.

1.1. The inversion formula

The starting point of this work is the eigenfunction expansion of the inverse Laplace function, proposed in [6]. In [6] the authors derived analytical expressions for the eigenfunctions and eigenvalues of Laplace transform and similar Fredholm integral equations of the first kind. Moreover, the authors discussed the ill conditioning of the inverse problem and the difficulties associated with its solution. Provided that the functions \( f, F \in L^1_\infty (\mathbb{R}) \), where

\[
L^1_\infty (\mathbb{R}) = \left\{ h : \mathbb{R} \to \mathbb{R} : \int_0^\infty |h(x)| \frac{1}{\sqrt{x}} \, dx < \infty \right\},
\]

the inversion formula is

\[
f(t) = \int_0^\infty \left( \frac{F(\cdot)}{\lambda^+(\omega)} \right) \psi^+(\omega, t) \, d\omega + \int_0^\infty \left( \frac{F(\cdot)}{\lambda^-(\omega)} \right) \psi^-(\omega, t) \, d\omega,
\]

where \( \psi^\pm \) and \( \lambda^\pm \) are the eigenfunctions and the eigenvalues of the Laplace integral transform, respectively, and

\[
\langle F(\cdot), \psi^\pm(\cdot) \rangle = \int_0^\infty F(x) \cdot \psi^\pm(x) \, dx
\]

denotes the inner product between the functions \( F \) and \( \psi^\pm \). The eigenfunction \( \psi^\pm \) are given by [6]

\[
\psi^\pm(\omega, t) = \cos(\rho/2 - \omega \log t) \frac{1}{\sqrt{\pi t}}
\]
Numerical regularization of a real inversion formula based on the Laplace transform’s eigenfunction

Figure 1. Inverse function $f_1$ and its computed approximation. $f_1(t) = t e^{-\omega t}$. Left: $\omega = 4.5$. Centre: $\omega = 8.2$. Right: $\omega = 12.0$.

and

$$\psi^- (\omega, t) = \sin(\rho/2 - \omega \log t) \frac{1}{\sqrt{\pi t}},$$

where

$$\rho = \frac{s_\omega}{c_\omega}$$

and

$$s_\omega = \int_0^\infty t^{-1/2} \exp(-t) \sin(\omega \log t) \, dt, \quad c_\omega = \int_0^\infty t^{-1/2} \exp(-t) \cos(\omega \log t) \, dt.$$

The eigenvalues $\lambda^\pm(\omega)$ are [6]

$$\lambda^\pm(\omega) = \pm \sqrt{\frac{\pi}{\cosh(\pi \omega)}}.$$

Let

$$f_\omega(t) = \int_0^\infty \frac{\langle F(.), \psi^+(\omega, .) \rangle}{\lambda^+(\omega)} \psi^+(\omega, t) \, d\omega + \int_0^\infty \frac{\langle F(.), \psi^-(\omega, .) \rangle}{\lambda^-(\omega)} \psi^-(\omega, t) \, d\omega,$$

then

$$f(t) = \lim_{\omega \to \infty} f_\omega(t).$$

Although, as $\omega$ grows, we could expect $f_\omega$ to be a better approximation of $f$, intrinsic instabilities occurring during the numerical computation of $f_\omega$ and due to the ill posedness of the real inversion of the Laplace transform make the approximations not ‘good’. On the other hand, smaller values of $\omega$ may give better results.

Following [6], we show some results obtained for different values of $\omega$. We consider the functions

- $f_3(t) = t^N e^{-\omega t} \quad (N, p > 0)$
- $f_2(t) = \max(k - t, 0), \quad (k > 0)$
- $f_3(t) = \chi_{[0, k]}(k > 0) = \begin{cases} 1 & 0 \leq t < k \\ 0 & t \geq k > 0 \end{cases}$

As shown by the plots in figures 1–3, as $\omega$ grows, $f_\omega$ does not always provide a better approximation of $f$. Of course, if $\omega$ is too small, results cannot be accurate. As has been said before, this phenomenon is due to the ill posedness of the real inversion problem. It seems that there exists an ‘optimal’ value of $\omega$ corresponding to the maximum attainable accuracy for $f_\omega$. Moreover, as we will see later, the parameter $\omega$ acts like the regularization parameter that
characterizes the numerical regularized solution. In the following, we will describe how the ‘optimal’ value of $\omega$ and the corresponding numerical regularized solution can be obtained.

Before delving into details of our computation of this optimal value, which we present in section 2, we first review the main points here. In the following, we denote the function space $L^1(\mathbb{R}^+)$ as $S_{app}$ in order to highlight that this space function is the set of applicability of the inversion formula.

**Definition 1.1.** Let

$$S_{app} = \left\{ u : \text{Re} \rightarrow \text{Re} \mid \int_0^{+\infty} |u(t)| t^{-1/2} \, dt < +\infty \right\}$$

(3)

denote the set of real functions integrable with respect to $w(t) = t^{-1/2}$ on the real positive semi-axis.

If $F, f \in S_{app}$ formula (2) is well defined. Then, numerical approximation of $f(t)$ in (2) can be performed through the following steps.

**Step 1.** Let $\bar{\omega} > 0$ be given. Consider

$$\mathcal{T}_{\bar{\omega}} : F \mapsto f_{\bar{\omega}}(t).$$

(4)

**Step 2.** Let $Q^h$ be a quadrature rule on $[0, \bar{\omega}]$, where $h$ is the maximum step size. Consider

$$\mathcal{D}_h : f_{\bar{\omega}}(t) \mapsto Q^h(f_{\bar{\omega}})$$

(5)

and

$$f^h_{\bar{\omega}}(t) = Q^h(f_{\bar{\omega}}(t)) = \mathcal{D}_h(\mathcal{T}_{\bar{\omega}}(F)).$$

(6)
Step 3. Let $u$ be the machine epsilon. Consider
\[ C_u : f_{\omega}^h(t) \mapsto f_{\omega}^{h,u}(t) = f_{\omega}^h(t)(1 + \eta u) \quad (\eta = \text{const.}) . \] (7)

Starting from $F$, following steps 1–3, we get $f_{\omega}^{h,u}(t)$, the numerical approximation of $f(t)$. Observe that $h$ and $u$ are fixed once the quadrature rule and the floating point arithmetic system have been given.

Let us now give some definitions that will be used in the rest of the paper.

Definition 1.2. The difference
\[ \text{TE}_{\omega}(t) = f(t) - f_{\omega}(t) \] is the truncation error (TE) introduced in step 1.

Definition 1.3. The difference
\[ \text{DE}_{\omega}^h(t) = T_{\omega}(F) - D_h(T_{\omega}(F)) = f_{\omega}(t) - f_{\omega}^h(t) \] (9)
is the discretization error (DE) introduced in step 2.

Definition 1.4. The difference
\[ \text{CE}_{\omega}(t) = D_h(T_{\omega}(F)) - C_u(D_h(T_{\omega}(F))) = f_{\omega}^h(t) - f_{\omega}^{h,u}(t) \] (10)
is the condition error (CE) introduced in step 3.

Definition 1.5. The difference
\[ \text{GE}_{\omega}(t) = f(t) - f_{\omega}^{h,u}(t) \]
is the global error (GE) introduced on $f_{\omega}^{h,u}(t)$.

Using the global error function, we now introduce the numerical regularization operator.

Definition 1.6. Let
\[ \Phi(\tilde{\omega}, t) = \| \sqrt{t} \text{GE}_{\omega}(t) \|_\infty = \sup \sqrt{t} | f(t) - f_{\omega}^{h,u}(t) | \]
be the numerical regularization operator. The function
\[ f_{\omega_{\text{opt}}}(t) = \arg\min_{\omega > 0} \Phi(\tilde{\omega}, t) \]
is referred to as the numerical regularized solution corresponding to the optimal regularization parameter $\omega_{\text{opt}}$.

Note that
\[ f_{\omega_{\text{opt}}}(t) \to f, \quad (h, u \to 0, \omega_{\text{opt}} \to \infty). \]

The problem is the computation of $f_{\omega_{\text{opt}}}$ and of parameter $\omega_{\text{opt}}$. For notational simplicity, in the rest of the paper, we refer to $f_{\omega_{\text{opt}}}(t)$ as $f_{\omega}(t)$.

The following section is devoted to derivations of upper bounds of the errors and their propagation during steps 1–3.
2. Truncation error

We discuss how to estimate the truncation error which occurs at step 1.

**Lemma 2.1.** Let

\[ c^+(\omega) = \langle F(\cdot), \psi^+(\omega, \cdot) \rangle, \quad c^-(\omega) = \langle F(\cdot), \psi^-(\omega, \cdot) \rangle \]

be the eigenfunction projections of \( F \) and

\[ N(\omega) = \left( \frac{c^+(\omega)}{\lambda^+(\omega)} + \frac{c^-(\omega)}{\lambda^-(\omega)} \right)^2 \]

(11)

denote the modulus of the eigenfunction projections. If

\[ \int_{\tilde{\omega}}^{\infty} N(\omega) \, d\omega < \infty, \]

(12)

then

\[ |TE_{\tilde{\omega}}(t)| \leq \frac{2}{\sqrt{\pi t}} \int_{\tilde{\omega}}^{\infty} N(\omega) \, d\omega. \]

(13)

**Proof.** By observing that

\[ |\psi^\pm(\omega, t)| \leq \frac{1}{\sqrt{\pi t}}, \quad \left| \frac{c^\pm(\omega)}{\lambda^\pm(\omega)} \right| \leq N(\omega), \]

(14)

we have

\[ |TE_{\tilde{\omega}}(t)| \leq \int_{\tilde{\omega}}^{\infty} (N(\omega) |\psi^+(\omega, t)| + N(\omega) |\psi^-(\omega, t)|) \, d\omega \leq \frac{2}{\sqrt{\pi t}} \int_{\tilde{\omega}}^{\infty} N(\omega) \, d\omega. \]

Condition (12) guarantees that integrals in (2) exist and are finite. Observe that it acts like the Picard condition, characterizing the existence of the singular value expansion of solutions of Fredholm integral equations of the first kind [3].

Consider \( f_1, f_2 \) and \( f_3 \), as introduced in the list in section 1.1. Looking at \( N(\omega) \), plotted in figure 4 on a semi-logarithmic scale, we observe that as \( \omega \) grows, in the case of \( f_1 \), \( N_1(\omega) \) seems to be of an exponential order, while \( N_2(\omega) \) and \( N_3(\omega) \), corresponding to \( f_2 \) and \( f_3 \), have lower decreasing asymptotic orders. We found experimentally that

\[ N_1(\omega) \leq 12.1 e^{-1.25\omega}, \quad N_2(\omega) \leq 0.267\omega^{-1.896}, \quad N_3(\omega) \leq 0.528\omega^{-0.9596}. \]

The asymptotic behaviour of the truncation error strongly affects the accuracy of numerical computations. As already observed in [6], the accuracy of the inverse function depends on how fast the eigenvalues decay to zero (or equivalently towards a noise level). In particular, we actually prove that the (ill) conditioning of the discrete problem depends on the asymptotic order of eigenvalues: only if the truncation error rapidly goes to zero, can the amplification of inherent instabilities be controlled. Theorem 2.2 distinguishes between those Laplace functions for which the truncation error has an exponential decrease and the others that decrease with polynomial order.

**Theorem 2.2.** (Truncation error bound) Let \( C_2 > 0 \) and \( C_1 > 0 \) be fixed. Let

\[ TB(\tilde{\omega}) := \begin{cases} \frac{2}{\sqrt{\pi}} \frac{C_1}{C_2} e^{-C_2\omega} & \text{if } N(\omega) \leq C_1 e^{-C_2\omega} \\ \frac{2}{\sqrt{\pi}} \frac{C_1}{C_2} \omega^{-C_2} & \text{if } N(\omega) \leq C_1 \omega^{-(C_2+1)} \end{cases} \]

(15)
then
\[ |TE_\omega(t)| \leq \frac{1}{\sqrt{t}} TB(\bar{\omega}). \] (16)

**Proof.** This follows by (13). □

The previous result suggests that in order to obtain an upper bound of the truncation error, it is sufficient to determine \( C_1 \) and \( C_2 \) and to select the asymptotic behaviour of \( TB \). How to choose \( TB(\bar{\omega}) \) and to compute \( C_1 \) and \( C_2 \) will be explained in section 4.

### 2.1. Discretization error

Let us assume that the quadrature rule \( Q^h \), introduced at step 2 for discretizing the inversion formula, has algebraic precision degree \( d > 0 \).

**Theorem 2.3.** (Discretization error estimate) Let
\[ G(\omega, t) = \frac{c^+(\omega)}{\lambda^+(\omega)} \psi^+(\omega, t) + \frac{c^-(-\omega)}{\lambda^-(-\omega)} \psi^-(\omega, t). \] (17)

If \( G(\cdot, t) \in C^{k+1}[0, \bar{\omega}], \forall t > 0 \) and if there exists \( H(t), t > 0 \) such that
\[ |G^{(k+1)}(\omega, t)| \leq H(t), \quad \omega \in [0, \bar{\omega}], \quad t > 0, \] (18)
then the discretization error of the quadrature rule can be bounded as follows:
\[ |DE_h(t)| \leq c_{k,d} H(t) \bar{\omega} h^{k+1}, \] (19)
where \( c_{k,d} = 2^{3k+1} \frac{e^{\bar{\omega} h^{k+1}}}{d^{(d-k)(k+1)}} \) and \( d > k \).

**Proof.** Let \( \tau_j \) be the length of the subinterval \([t_{j-1}, t_j]\) of \( Q^h \). Then,
\[ |DE_h(t)| \leq c_{k,d} H(t) \sum_{j=1}^{n} \tau_j h^{k+1} \leq c_{k,d} H(t) \sum_{j=1}^{n} t_j h^{k+1} = c_{k,d} H(t) \bar{\omega} h^{k+1}. \] □
From (19), in particular it follows that
\[
\lim_{h \to 0} |T^h_0(F) - D^h_0(F)| = 0 \quad \forall t > 0.
\] (20)

2.2. Condition error

The analysis of the propagation of the condition error during numerical computations is of the greatest practical importance in the design of algorithms. This is particularly significant in the case of ill-posed problems. Indeed, we expect unbounded amplifications of the errors, and a detailed investigation of numerical computations allows one to understand how to control and mitigate the instabilities. Concerning formula (1), and its numerical approximation described by steps 1–3, this means that we need to focus essentially on steps 2 and 3. In particular, we refer to the computation of the integrand function \( G \) in (17) which depends on the eigenfunction projections \( c^\pm \) and on the eigenvalues \( \lambda^\pm \). Let \( \tilde{F}(s) \) denote the perturbed value of \( F(s) \):
\[
\tilde{F}(s) = F(s)(1 + \delta(s)), \quad s \geq 0.
\] (21)

We analyse the propagation of \( \delta(s) \) at steps 2 and 3. Let
\[
\tilde{c}^\pm(\omega) = \langle \psi^\pm(\omega, \cdot), \tilde{F}(\cdot) \rangle
\] (22)
denote the perturbed value of the eigenfunction projections, and
\[
\Theta^\pm(\omega) = c^\pm(\omega) - \tilde{c}^\pm(\omega)
\] (23)
denotes the absolute error on the eigenfunction projections. The following lemma describes the propagation of \( \delta(s) \) on \( \Theta^\pm \).

**Lemma 2.4.** Let \( \tilde{F}(s), \delta(s), \tilde{c}^\pm(\omega) \) and \( \Theta^\pm(\omega) \) be as in (21), (22) and (23). If
\[
|\delta(s)| \leq K, \quad \forall s > 0,
\] (24)
then
\[
|\Theta^\pm(\omega)| \leq C_F K, \quad \forall \omega \geq 0,
\] (25)
where
\[
C_F = \int_0^{+\infty} |F(s)||s\|^{-1/2} \; ds.
\] (26)

**Proof.** We have
\[
|\Theta^\pm(\omega)| = |\tilde{c}^\pm(\omega) - c^\pm(\omega)| = |\langle \psi^\pm(\omega, \cdot), \tilde{F}(\cdot) \rangle - \langle \psi^\pm(\omega, \cdot), F(\cdot) \rangle|
\]
\[
= |\langle \psi^\pm(\omega, \cdot), F(\cdot)\delta(\cdot) \rangle| = \left| \int_0^{+\infty} F(s)\delta(s)\psi^\pm(\omega, s) \; ds \right|
\]
\[
\leq K \int_0^{+\infty} |F(s)||\psi^\pm(\omega, s)| \; ds.
\]

Finally, by (14), the result follows. \( \square \)

We now discuss the propagation of \( \Theta^\pm \) on
\[
\tilde{G}(\omega, t) = \frac{\tilde{c}^+(\omega)}{\lambda^+(\omega)} \psi^+(\omega, t) + \frac{\tilde{c}^-(\omega)}{\lambda^-(\omega)} \psi^-(\omega, t), \quad t, \omega \in \mathbb{R}^+.
\] (27)

1 If \( F \) does not change sign, then \( C_F = |c^+(0)|. \)
Let
\[ \beta(\omega, t) = \tilde{G}(\omega, t) - G(\omega, t), \quad \omega > 0, \]
(28)
denote the absolute error in \( G \). Then

**Lemma 2.5.** Let \( \tilde{c}_k^{\pm}(\omega), \Theta_k^{\pm}(\omega), \tilde{G}(\omega, t) \) and \( \beta(\omega, t) \) be as in (22), (23), (27) and (28), respectively. If \( \Theta \geq 0 \) is such that
\[ |\Theta(\omega)| \leq \Theta, \quad \forall \ 0 \leq \omega \leq \hat{\omega}, \]
(29)
then
\[ |\beta(\omega, t)| \leq \frac{1}{\pi} \sqrt{\frac{2}{t}} \left( 1 + e^{\pi \hat{\omega}/2} \right), \quad 0 \leq \omega \leq \hat{\omega}, \quad t > 0. \]
(30)

**Proof.** By (14)
\[ |\beta(\omega, t)| = |\tilde{G}(\omega, t) - G(\omega, t)| \leq \frac{|\Theta^{\pm}(\omega)|}{|\lambda(\omega)|} |\psi^{\pm}(\omega, t)| + \frac{|\Theta(\omega)|}{|\lambda^{-}(\omega)|} |\psi^{-}(\omega, t)| \]
\[ \leq 2 \Theta \max_{\omega \leq \hat{\omega}} |\psi^{\pm}(\omega, t)| \max_{\omega \leq \hat{\omega}} \left| \frac{1}{\lambda^{\pm}(\omega)} \right| = 2 \Theta \frac{1}{\sqrt{\pi t}} \max_{\omega \leq \hat{\omega}} \left| \frac{1}{\lambda^{\pm}(\omega)} \right|. \]
(31)
Observing that
\[ \left| \frac{1}{\lambda^{\pm}(\omega)} \right| = \sqrt{e^{2\pi \pm \omega} + e^{-2\pi \pm \omega}} < \frac{1 + e^{\pm \omega}}{2\pi}, \]
then
\[ \max_{\omega \leq \hat{\omega}} \left| \frac{1}{\lambda^{\pm}(\omega)} \right| \leq \frac{1 + e^{\mp \omega}}{2\pi}, \]
(32)
and
\[ |\beta(\omega, t)| \leq \frac{1}{\pi} \sqrt{\frac{2}{t}} \left( 1 + e^{\mp \omega/2} \right), \]
(33)
and the proof is complete. \( \Box \)

The following lemma relates \( \beta \), the error propagated on \( G \), to the condition error of the quadrature rule \( Q^h \).

**Lemma 2.6.** Let \( \tilde{G}(\omega, t), \beta(\omega, t), f_{\omega}(t) \) and the condition error \( CE_{\omega}(t) \) be as in (27), (28), (7) and (10), respectively. If
\[ |\beta(\omega, t)| \leq h(t), \quad 0 \leq \omega \leq \hat{\omega}, \quad t > 0, \]
(34)
then the following inequality holds:
\[ |CE_{\omega}(t)| \leq h(t) v(Q^h), \quad 0 \leq \omega \leq \hat{\omega}, \quad t > 0, \]
(35)
where \( v(Q^h) \) is the absolute condition number of the quadrature rule \( Q^h \).

**Proof.** Applying (34), it follows that
\[ |CE_{\omega}(t)| = |f_{\omega}^h(t) - f_{\omega}(t)| = |Q^h(G(\cdot, t)) - Q^h(\tilde{G}(\cdot, t))| \]
\[ \leq v(Q^h) \max_{0 \leq \omega \leq \hat{\omega}} |\tilde{G}(\omega, t) - G(\omega, t)| = v(Q^h) \max_{0 \leq \omega \leq \hat{\omega}} |\beta(\omega, t)|. \]
(36)
Theorem 2.7. (Condition error estimate) Let $\tilde{F}(s)$ be as in (21). If

$$CB(\tilde{\omega}) := \frac{\sqrt{2}}{\pi} \nu(Qh) C_F K (1 + e^{\pi \tilde{\omega}/2}),$$

(37)

where $C_F$ is as in (26) and $\nu(Qh)$ is the absolute condition number of $Qh$, and

$$|\delta(s)| \leq K, \quad \forall s > 0,$$

(38)

then

$$|CE(\tilde{\omega})| \leq \frac{1}{\sqrt{t}} CB(\tilde{\omega}).$$

(39)

Proof. By using lemma 2.5 with $\Theta = C_F K$ and lemma 2.6 with $h(t) = \frac{1}{\pi} \frac{\sqrt{2}}{\sqrt{t}} (1 + e^{\pi \tilde{\omega}/2})$, the proof is complete.

The upper bound, CB, of $CE(\tilde{\omega})$ in (39) reveals amplification of local errors, $\delta(s)$, and $\beta(\omega, t)$. Observe that CB has at least an exponential growth with asymptotic order $O(e^{\pi \tilde{\omega}/2})$ that depends on the asymptotic behaviour of the eigenvalues $\lambda ^ \pm$. This fact confirms that for sufficiently large values of $\tilde{\omega}$, we may get the computed approximation $f_\omega$ completely wrong. For instance, at $\tilde{\omega} = 12.0$, it is

$$e^{\pi \tilde{\omega}/2} = e^{\pi 12/2} = 1.5 \ldots \times 10^8.$$

From (16), the estimate of the truncation error (see theorem 2.2) could be of decreasing exponential order. We expect that there exists a value of $\tilde{\omega}$ at which the exponential growth factor of the condition error estimate can be balanced by TB. This suggests that we compute a numerical regularized solution $f_\omega$ as the one corresponding to the choice of $\tilde{\omega}$ that gives a trade-off between TB and CB errors. The following section is devoted to the characterization of the optimal value of $\tilde{\omega}$ and therefore to the computation of $f_\omega$.

3. Numerical regularization

We discuss how to determine $f_{\text{opt}}(t)$ as given in definition 1.6. To this end we need a computable estimate of $GE(\tilde{\omega})$.

Theorem 3.1. (Global error upper bound) Under the assumptions of theorems 2.2 and 2.7, let

$$\Psi(\tilde{\omega}) := CB(\tilde{\omega}) + TB(\tilde{\omega}),$$

(40)

then

$$|GE(\tilde{\omega})| \leq \frac{\Psi(\tilde{\omega})}{\sqrt{t}} + O(h^{k+1}) \quad \forall t, \omega.$$ 

(41)

Proof. $\forall t, \omega$ we have

$$|GE(\tilde{\omega})| \leq |TE(\tilde{\omega})| + |DE^k(\tilde{\omega})| + |CE(\tilde{\omega})| \leq [TB(\tilde{\omega}) + CB(\tilde{\omega})]/\sqrt{t} + O(h^{k+1})$$

$$= \frac{\Psi(\tilde{\omega})}{\sqrt{t}} + O(h^{k+1}).$$

This means that up to an approximation of $O(h^{k+1})$, $\Psi$ can be regarded as an estimate of $GE$. Observe that the Laplace transform is analytic on its half plane of convergence; then $k$ can be considered to be sufficiently large to neglect $O(h^{k+1})$. 
**Definition 3.2.** Let
\[
\tilde{\omega}_{\text{opt}} = \arg\min \Psi(\bar{\omega})
\]
be the optimal value of the regularization parameter \(\bar{\omega}\) to which corresponds the least global error estimate. Moreover, we refer to \(\epsilon^* = \Psi(\tilde{\omega}_{\text{opt}})\) as the best attainable accuracy.

From (41), neglecting \(O(h^{k+1})\) it follows that
\[
\left| G_{E_{\text{opt}}} (t) \right| \leq \Psi(\tilde{\omega}_{\text{opt}}) \sqrt{t},
\]
then
\[
\Phi[\tilde{\omega}_{\text{opt}}, t] = \| \sqrt{t} G_{E_{\text{opt}}} (t) \|_{\infty} \leq \Psi(\tilde{\omega}_{\text{opt}}),
\]
and \(f_{\tilde{\omega}_{\text{opt}}}\) can be considered as a reliable approximation of \(f_{\text{app}}\), the numerical regularized solution as given in definition 1.6. It is worth noting that, as \(t \to 0\), then \(G_{E_{\text{opt}}}\) grows and we are not able to provide an accurate solution, while as \(t \to \infty\), the global error goes to zero as well as \(f\).

\(\Psi(\tilde{\omega})\) is a convex function of \(\tilde{\omega}\) (see, for instance, the plot in figure 5), and this property guarantees the existence of the regularization parameter \(\tilde{\omega}_{\text{opt}}\). By means of \(\epsilon^*\), we define the set of admissible solutions.

**Definition 3.3.** The set
\[
N_{\text{app}} = \{ g \in S_{\text{app}} \text{ s.t. } \epsilon^* < C_f \}
\]
is defined as the set of numerical applicability of (2).
Observe that the upper bound of $\epsilon^*$ depends on $F$ and, in particular, on the order of magnitude of $F$ and of its inverse function. This means that (43) states that formula (2) is numerically applicable if its numerical approximation gives the best attainable accuracy whose order of magnitude scales according to that of the Laplace transform and its inverse. The following theorem gives a sufficient condition on $F$ that guarantees that it belongs to $N_{\text{app}}$.

**Theorem 3.4.** Let $F, f \in S_{\text{app}}$ and $F \geq 0$. If

$$N(\omega) \leq C_1 e^{-C_2\omega},$$

where

$$C_2 > \frac{2\pi}{\pi^2 - 4} = 1.07 \ldots,$$

and $\nu(Q^h) \cdot K < \frac{2\sqrt{2}}{\pi^2}$, then we have

$$\epsilon^* < C_F,$$

and $F$ belongs to $N_{\text{app}}$.

**Proof.** If $F \geq 0$, then, from (15) and taking into account the expression for the eigenvalues $\lambda(\omega)$, we have

$$C_F = |e^*(0)| = N(0)\lambda^*(0) \leq C_1\lambda^*(0) \leq C_1\sqrt{\pi}.$$ 

Moreover, we have

$$\Psi(\overline{\omega}) = \sqrt{\frac{2}{\pi}} \left( \sqrt{\frac{\sqrt{2}}{C_2}} e^{-C_2\pi} + \frac{C_F}{\sqrt{\pi}} K \nu(Q^h) e^{\pi\overline{\omega}/2} \right)$$

$$\leq \sqrt{\frac{2}{\pi}} \left( \sqrt{\frac{\sqrt{2}}{C_2}} C_F e^{-C_2\pi} + \frac{C_2}{\sqrt{\pi}} \frac{2\sqrt{2}}{\pi} e^{\pi\overline{\omega}/2} \right) = \frac{2}{C_2} \frac{C_F}{\pi} \left( \frac{e^{-C_2\pi}}{\pi} + \frac{2C_2}{\pi} e^{\pi\omega/2} \right).$$

Then

$$\epsilon^* = \min_{\overline{\omega}} \Psi(\overline{\omega}) \leq \frac{2}{C_2} \frac{C_F}{\pi} \min_{\overline{\omega}} \left( e^{-C_2\pi} + \frac{2C_2}{\pi} e^{\pi\overline{\omega}/2} \right) = \frac{2}{C_2} \frac{C_F}{\pi} \frac{2C_2 + \pi}{\pi} = C_F \frac{4C_2 + 2\pi}{\pi^2 C_2} < C_F.$$ 

An example of a function belonging to $N_{\text{app}}$ is $F_1$ as introduced in the list in section 1.1. □

4. **Computational issues**

In this section we describe the algorithm that we have developed for computing the numerical regularized solutions $\tilde{f}_{\omega^*}$, $\tilde{f}_{\omega^*}$. The core of the algorithm is a dynamic computation of $\omega_{\text{opt}}$. This can be done at run time by using upper bounds $TB$ and $CB$. The following box summarizes the main steps needed to compute $\omega_{\text{opt}}$:

1. computation of $\Psi(\omega)$:
   (i) computation of $TB(\omega)$;
   (ii) computation of $CB(\omega)$;
2. computation of $\omega_{\text{opt}}$ minimizing $TB(\omega) + CB(\omega)$.

Algorithm 0: sketch of the main steps of the computation of $\omega_{\text{opt}}$. 
• At Step 1(i), \( T_B(\tilde{\omega}) \) is computed assuming

(a) \( N(\omega) = C_1 \, e^{-C_2 \omega} \) or (b) \( N(\omega) = C_1 \, \omega^{-(C_2 + 1)} \).

To this end, the algorithm tabulates \( N(\omega) \) on a finite set of points and extrapolates its asymptotic behaviour, by choosing between (a) and (b) using the least-square approximation.

In particular, given \( N(\omega_i) \) at \( \omega_i = i \times 0.1, i = 1, \ldots, m \),

(i) it computes the distance between the points \((\omega_i, N(\omega_i))\) and the function \( y_1(\omega) = C_1 \, e^{-C_2 \omega} \) using the 2-norm

\[
E^a(C_1, C_2) = \sum_{i=0}^{m} \left[ \log C_1 - C_2 \omega_i - \log N(\omega_i) \right]^2,
\]

and \((C_1^a, C_2^a) = \text{argmin} \, E^a(C_1, C_2)\) are obtained by solving a \( 2 \times 2 \) least-square approximation problem;

(ii) it computes the distance between the points \((\omega_i, N(\omega_i))\) and the function \( y_2(\omega) = C_1 \, \omega^{-(C_2 + 1)} \) using the 2-norm

\[
E^b(C_1, C_2) = \sum_{i=1}^{m} \left[ \log C_1 - (C_2 + 1) \log \omega_i - \log N(\omega_i) \right]^2,
\]

and \((C_1^b, C_2^b) = \text{argmin} \, E^b(C_1, C_2)\) are obtained by solving a \( 2 \times 2 \) least-square approximation problem.

If \( E^a(C_1^a, C_2^a) \leq E^b(C_1^b, C_2^b) \), then the algorithm chooses the approximation

(a) \( N(\omega) := C_1^a \, e^{-C_2^a \omega}, \)

otherwise, the approximation

(b) \( N(\omega) := C_1^b \, \omega^{-(C_2^b + 1)}. \)

Finally, following theorem 2.2, the algorithm sets

\[
T_B(\tilde{\omega}) := \begin{cases} 
\frac{2 \, C_1^a}{\sqrt{\pi} \, C_2^a} \, e^{-C_2^a \tilde{\omega}} & \text{if } E^a(C_1^a, C_2^a) \leq E^b(C_1^b, C_2^b) \\
\frac{2 \, C_1^b}{\sqrt{\pi} \, C_2^b} \, \tilde{\omega}^{-(C_2^b + 1)} & \text{otherwise}. 
\end{cases}
\]

Observe that if the computed value of \( C_2 \) is negative, theorem 2.2 cannot be applied and the algorithm does not work. Actually, in this case the fitting models that we have considered for describing the asymptotic behaviour of \( N(\omega) \) are nondecreasing functions. Then, condition (12) may not be satisfied and the inversion formula (2) may not be well defined. In other words, through the sign of the constant \( C_2 \) the algorithm recognizes if the inversion formula (2) may be applied or not.

• Concerning 1(ii), we have

\[
\text{CB}(\tilde{\omega}) = \frac{\sqrt{2}}{\pi} \tilde{\omega} \Theta(1 + e^{\pi \tilde{\omega}/2}),
\]

where we assume that

(i) the condition number of the quadrature rule is \( \nu(Q_\tilde{\omega}) = \tilde{\omega} \) (this is true for the compound trapezoidal rule and for the Gauss-based quadrature rules),

(ii) from (25) and (29), \( \Theta \) is an estimate of \( C_F \cdot K \).

In the following box, we detail Step 1 related to computation of \( \Psi(\bar{\omega}). \)
Regarding Step 2, we tabulate $\Psi(\tilde{\omega})$ with stepsize $h = 0.1$ up to $\Psi(\tilde{\omega}_0) \leq \Psi(\tilde{\omega}_{i+1})$. Then, we assume

$$\tilde{\omega}_\text{opt} := \tilde{\omega}_i + O(h).$$

This means that we get the approximation of $\tilde{\omega}_\text{opt}$ correct up to the first decimal digit. Details are described by algorithm 1 in the following box.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>minimize $E^a(C_1, C_2)$</td>
</tr>
<tr>
<td>2.</td>
<td>minimize $E^b(C_1, C_2)$</td>
</tr>
<tr>
<td>3.</td>
<td>if $(E^a(C_1^a, C_2^a) \leq E^b(C_1^b, C_2^b))$</td>
</tr>
<tr>
<td>4.</td>
<td>evaluate $TB(\tilde{\omega}) := \frac{2}{\sqrt{\pi}} C_1^a e^{-C_2^a \tilde{\omega}}$</td>
</tr>
<tr>
<td>5.</td>
<td>else</td>
</tr>
<tr>
<td>6.</td>
<td>evaluate $TB(\tilde{\omega}) := \frac{2}{\sqrt{\pi}} C_1^b e^{-C_2^b \tilde{\omega}}$</td>
</tr>
<tr>
<td>7.</td>
<td>endif</td>
</tr>
<tr>
<td>8.</td>
<td>evaluate $CB(\tilde{\omega}) := \frac{2}{\sqrt{\pi}} \tilde{\omega} C_\Delta e^{\pi \tilde{\omega}/2}$</td>
</tr>
<tr>
<td>9.</td>
<td>$\Psi(\tilde{\omega}) := CB(\tilde{\omega}) + TB(\tilde{\omega})$</td>
</tr>
</tbody>
</table>

Algorithm 1: $\Psi(\tilde{\omega})$ computational scheme.

5. Numerical experiments

In this section we report the numerical results obtained on test functions used in figures 1–3. These experiments are aimed at explaining and predicting the performance of the inversion formula on these functions. In particular, the experiments are intended to show the reliability of the proposed numerical algorithm. This means that we aim to show the capability of the algorithm to predict its behaviour with respect to the propagation of the errors. Moreover, we show the numerical applicability of the algorithm, that is, its ability to provide the predicted accuracy.

We consider as test functions the same that we have introduced in figures 1–3 because they are representative of three different situations. We rewrite them in the following box:

<table>
<thead>
<tr>
<th>Test</th>
<th>Function $F_1(s)$</th>
<th>Function $f_1(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$F_1(s) = \frac{1}{(4\pi)^2}$</td>
<td>$f_1(t) = t e^{-4t}$</td>
</tr>
<tr>
<td>2</td>
<td>$F_2(s) = \frac{e^{-5s}-1}{s^2}$</td>
<td>$f_2(t) = \max(k - t, 0)$</td>
</tr>
<tr>
<td>3</td>
<td>$F_3(s) = \frac{1-e^{-s}}{s}$</td>
<td>$f_3(t) = \chi_{[0,1]}$</td>
</tr>
</tbody>
</table>

Test functions as introduced in figures 1–3.

Regarding $f_1$, $TB$ is of exponential order. We expect that $f_1$ can be computed accurately. Concerning $f_2$, $TB$ is of polynomial order and in this case the best attainable accuracy cannot be as much as for $f_1$. We expect a propagation of the condition error on the computed
Numerical regularization of a real inversion formula based on the Laplace transform’s eigenfunction approximation. Finally, \( f_3 \) belongs to \( S_{app} \) but the algorithm is not able to predict and to mitigate the error amplification. In other words, \( f_3 \) does not belong to \( N_{app} \).

For the numerical integration of the inversion formula we choose the automatic adaptive quadrature rules based on the Clenshaw–Curtis scheme, provided by the QUADPACK software package [4]. In particular, we use the routine QAWF for computing the outermost integrals over the bounded integration interval \( [0, \bar{\omega}] \) and the routine QAWO for computing the innermost integrals, over the unbounded integration interval \( [0, +\infty[ \), which occur in the computation of the eigenfunction projections. It is worth noting that, with the aim of computing the inverse function with the maximum attainable accuracy, these routines were chosen because in the presence of oscillating functions, they give the result with the best accuracy. Moreover, the software returns the estimate of the roundoff and of the discretization error on the computed value of the eigenfunction projections. This is assumed to be the value of the constant \( \Theta \) in the expression of \( CB \).

Numerical results have been obtained using the double precision floating point arithmetic system.

5.1. Test function 1

We get
\[
TB(\bar{\omega}) = \frac{2}{\sqrt{\pi}} \frac{1.344 \times 10^{-1}}{1.175 \times 10^{0}} e^{1.11 \frac{10^{0}}{\bar{\omega}}}
\]
and
\[
CB(\bar{\omega}) = \frac{\sqrt{\pi}}{5} \times 10^{-8} \bar{\omega}(1 + e^{\pi \bar{\omega}/2}),
\]
where \( \Theta = 5 \times 10^{-8} \) is the accuracy obtained on the eigenfunction projections by the quadrature software.

(1) In figure 6 we plot \( \Psi(\bar{\omega}) \) and its minimum corresponding to \( \bar{\omega}_{opt} = 5 \). The maximum attainable accuracy is \( \epsilon^* = 6.5012 \times 10^{-4} \). \( C_F = 1.333 \), then \( \epsilon^* < C_F \) and \( F_1 \in N_{app} \).

This means that we expect at least three correct significant digits, as confirmed by looking at the global error plotted in figure 7 and at the computed result in figure 1.

(2) In figure 7, we compare the global error \( GE_5(t) \) and its estimate \( \Psi(5)/\sqrt{t} \) at 40 values of \( t \) uniformly distributed on \( [0, 4.0] \).

5.2. Test function 2

We get
\[
TB(\bar{\omega}) = \frac{2}{\sqrt{\pi}} \frac{3.8543 \times 10^{0}}{7.0880 \times 10^{-1}} \bar{\omega}^{-7.0880 \times 10^{-1}}
\]
and
\[
CB(\bar{\omega}) = \frac{\sqrt{\pi}}{5} \times 10^{-8} \bar{\omega}(1 + e^{\pi \bar{\omega}/2}),
\]
where \( \Theta = 5 \times 10^{-8} \) is the accuracy obtained on the eigenfunction projections by the quadrature software. In figures 8 and 9,

(1) we plot \( \Psi(\bar{\omega}) \) and its minimum corresponding to \( \bar{\omega}_{opt} = 7.6 \), the best attainable accuracy is \( \epsilon^* = 3.9 \times 10^{-1} \) and \( \epsilon^* < C_F = 0.6 \), then \( F_2 \in N_{app} \). We expect only one correct digit, as confirmed by global error plotted in figure 9 and by the result in figure 2;

(2) we compare \( GE_{7.6}(t) \) and its estimate \( \Psi(7.6)/\sqrt{t} \) at 40 values of \( t \) uniformly distributed on \( [0.1, 10.0] \).
Figure 6. Test 1: plot of $\Psi(\tilde{\omega})$ (full line) as a function of $\tilde{\omega}$, reported on the x-axis. The two dashed lines represent the condition error $CB$ (the increasing line on the left side of the plot) and the truncation error $TB$ (the decreasing line on the right side of the plot), respectively. The small circle on the full line indicates the point corresponding to the value of $\tilde{\omega} = 5$ at which $\Psi(\tilde{\omega})$ is minimum.

Figure 7. Test 1: a comparison between the function $\Psi(5)/\sqrt{\tilde{t}}$ that is the estimate of the global error (full line) and the global error $GE_5(t)$ (dashed line) versus $t$.

5.3. Test function 3

We get $N_3(\omega) \leq 0.528\omega^{-0.9596}$, then $C_1 = 0.528, C_2 = -0.0404 < 0$. Theorem 2.2 cannot be applied. This means that the algorithm is not able to compute the optimal value of the regularization parameter as given by definition 3.2.
Numerical regularization of a real inversion formula based on the Laplace transform’s eigenfunction

Figure 8. Test 2: plot of the function $\Psi(\bar{\omega})$ (full line) versus $\bar{\omega}$ reported on the x-axis. The two dotted lines represent the estimate of the condition error (the increasing line on the left side of the plot) and the estimate of the truncation error (the decreasing line on the right side of the plot). The small circle on the full line indicates the point corresponding to the value of $\bar{\omega} = 7.6$ at which $\Psi(\bar{\omega})$ is minimum.

Figure 9. Test 2: comparison between the function $\Psi(7.6)/\sqrt{t}$ (full line) that is the estimate of the global error, and the global error $GE_{7.6}(t)$ (dashed line), as a function of $t$ reported on the x-axis.

In this example the inversion formula (2) can be theoretically used, that is $f_3 \in S_{app}$ but the accuracy of numerical computations cannot be predicted. Indeed, $f_3$ does not belong to $\mathcal{N}_{app}$. Table 1 reports the global error $GE_{\omega}$ corresponding to three values of $\bar{\omega} = 4.3, 8.5, 12.0$ and computed at 20 uniformly distributed values of $t$ at $[0.1, 2.0]$. As expected, the
Table 1. Global errors for $f_3$ where $\bar{\omega} = 4.3, 8.5, 12.0$.

<table>
<thead>
<tr>
<th>t</th>
<th>$GE_{4.3}(t)$</th>
<th>$GE_{8.5}(t)$</th>
<th>$GE_{12.0}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$0.0278 \times 10^0$</td>
<td>$0.731 \times 10^{-2}$</td>
<td>$0.9377 \times 10^2$</td>
</tr>
<tr>
<td>0.2</td>
<td>$0.0366 \times 10^0$</td>
<td>$0.177 \times 10^{-1}$</td>
<td>$0.1095 \times 10^3$</td>
</tr>
<tr>
<td>0.3</td>
<td>$0.0231 \times 10^0$</td>
<td>$0.193 \times 10^{-1}$</td>
<td>$0.4832 \times 10^2$</td>
</tr>
<tr>
<td>0.4</td>
<td>$0.0591 \times 10^0$</td>
<td>$0.141 \times 10^{-2}$</td>
<td>$0.2710 \times 10^2$</td>
</tr>
<tr>
<td>0.5</td>
<td>$0.0930 \times 10^0$</td>
<td>$0.242 \times 10^{-1}$</td>
<td>$0.7469 \times 10^2$</td>
</tr>
<tr>
<td>0.6</td>
<td>$0.0488 \times 10^0$</td>
<td>$0.425 \times 10^{-1}$</td>
<td>$0.9090 \times 10^2$</td>
</tr>
<tr>
<td>0.7</td>
<td>$0.0591 \times 10^0$</td>
<td>$0.180 \times 10^{-1}$</td>
<td>$0.1181 \times 10^2$</td>
</tr>
<tr>
<td>0.8</td>
<td>$0.02060 \times 10^0$</td>
<td>$0.809 \times 10^{-1}$</td>
<td>$0.8484 \times 10^2$</td>
</tr>
<tr>
<td>0.9</td>
<td>$0.3703 \times 10^0$</td>
<td>$0.2538 \times 10^{-0}$</td>
<td>$0.3298 \times 10^2$</td>
</tr>
<tr>
<td>1.0</td>
<td>$0.5365 \times 10^0$</td>
<td>$0.4742 \times 10^{-0}$</td>
<td>$0.5463 \times 10^2$</td>
</tr>
<tr>
<td>1.1</td>
<td>$0.3538 \times 10^0$</td>
<td>$0.3413 \times 10^{-0}$</td>
<td>$0.7112 \times 10^2$</td>
</tr>
<tr>
<td>1.2</td>
<td>$0.2551 \times 10^0$</td>
<td>$0.1715 \times 10^{-0}$</td>
<td>$0.1978 \times 10^2$</td>
</tr>
<tr>
<td>1.3</td>
<td>$0.1700 \times 10^0$</td>
<td>$0.354 \times 10^{-1}$</td>
<td>$0.3964 \times 10^2$</td>
</tr>
<tr>
<td>1.4</td>
<td>$0.0994 \times 10^0$</td>
<td>$0.591 \times 10^{-1}$</td>
<td>$0.6418 \times 10^2$</td>
</tr>
<tr>
<td>1.5</td>
<td>$0.0426 \times 10^0$</td>
<td>$0.1126 \times 10^{-0}$</td>
<td>$0.4771 \times 10^2$</td>
</tr>
<tr>
<td>1.6</td>
<td>$0.0015 \times 10^0$</td>
<td>$0.1311 \times 10^{-0}$</td>
<td>$0.8711 \times 10^1$</td>
</tr>
<tr>
<td>1.7</td>
<td>$0.0344 \times 10^0$</td>
<td>$0.1231 \times 10^{-0}$</td>
<td>$0.2936 \times 10^2$</td>
</tr>
<tr>
<td>1.8</td>
<td>$0.0577 \times 10^0$</td>
<td>$0.977 \times 10^{-1}$</td>
<td>$0.5072 \times 10^2$</td>
</tr>
<tr>
<td>1.9</td>
<td>$0.0730 \times 10^0$</td>
<td>$0.629 \times 10^{-1}$</td>
<td>$0.5081 \times 10^2$</td>
</tr>
<tr>
<td>2.0</td>
<td>$0.0816 \times 10^0$</td>
<td>$0.253 \times 10^{-1}$</td>
<td>$0.3391 \times 10^2$</td>
</tr>
</tbody>
</table>

computed values of $f_{\bar{\omega}}$, corresponding to these values of $\bar{\omega}$ and plotted in figure 3, give poor approximations.

6. Conclusions

We introduce the numerical regularization of the inversion formula of Laplace transform based on the eigenfunctions truncated expansion. Numerical regularized solutions are obtained by minimizing the errors introduced during numerical approximations. That means that they are obtained with the best attainable accuracy. The optimal value of the regularization parameter is computed dynamically, and it is given by minimizing the global error estimate. Finally, we describe the computational issues aiming to enable the algorithm to predict the best attainable accuracy of the computed solution. Numerical experiments show the reliability of the final algorithm on three test functions representing three different applications of the inversion formula.

Acknowledgments

We thank Professor M Bertero and Professor E R Pike for their suggestions on how to present this paper.

References

Numerical regularization of a real inversion formula based on the Laplace transform’s eigenfunction


