

CINEMATICA DIFFERENZIALE

- relazioni tra velocità dei giunti e velocità dell'organo terminale

Jacobiano geometrico

Singularità cinematiche

Analisi della ridondanza

Inversione della cinematica differenziale

Jacobiano analitico

Algoritmi per l'inversione cinematica

STATICA

- relazioni tra forze all'organo terminale e coppie ai giunti

JACOBIANO GEOMETRICO

$$\mathbf{T}(\mathbf{q}) = \begin{bmatrix} \mathbf{R}(\mathbf{q}) & \mathbf{p}(\mathbf{q}) \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- Obiettivo

$$\dot{\mathbf{p}}_e = \mathbf{J}_P(\mathbf{q})\dot{\mathbf{q}}$$

$$\boldsymbol{\omega}_e = \mathbf{J}_O(\mathbf{q})\dot{\mathbf{q}}$$

$$\mathbf{v}_e = \begin{bmatrix} \dot{\mathbf{p}} \\ \boldsymbol{\omega} \end{bmatrix} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

Derivata di una matrice di rotazione

$$\mathbf{R}(t)\mathbf{R}^T(t) = \mathbf{I}$$

$$\dot{\mathbf{R}}(t)\mathbf{R}^T(t) + \mathbf{R}(t)\dot{\mathbf{R}}^T(t) = \mathbf{O}$$

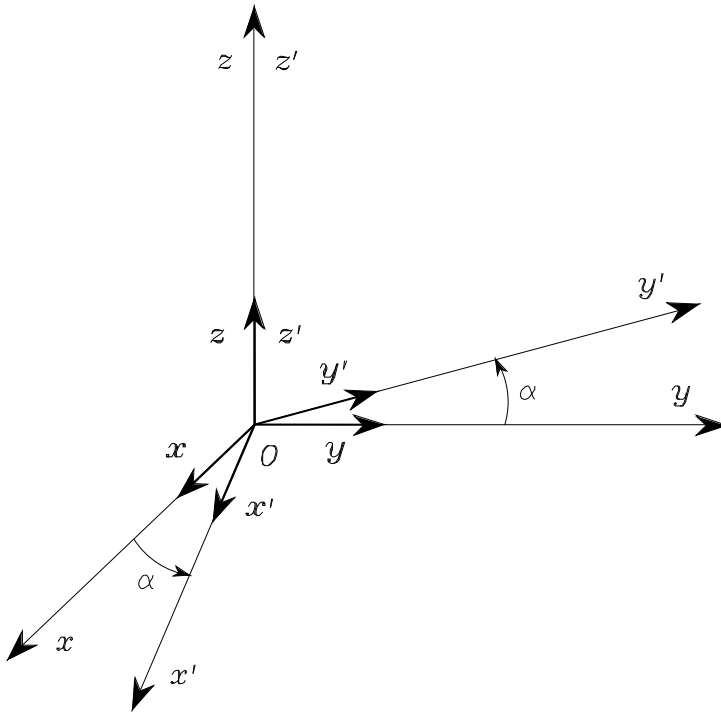
- Posto $\mathbf{S}(t) = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)$

$$\mathbf{S}(t) + \mathbf{S}^T(t) = \mathbf{O}$$

$$\dot{\mathbf{R}}(t) = \mathbf{S}(\boldsymbol{\omega}(t))\mathbf{R}(t)$$

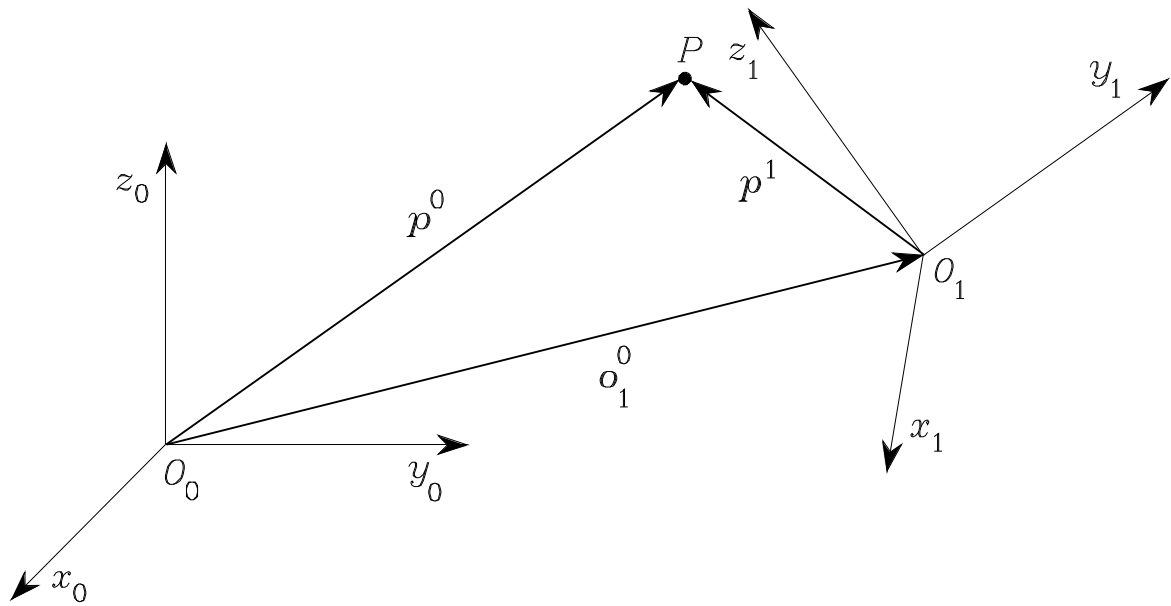
$$\mathbf{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- Esempio



$$\mathbf{R}_z(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}(t) &= \begin{bmatrix} -\dot{\alpha} \sin \alpha & -\dot{\alpha} \cos \alpha & 0 \\ \dot{\alpha} \cos \alpha & -\dot{\alpha} \sin \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\dot{\alpha} & 0 \\ \dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{S}(\boldsymbol{\omega}(t)) \end{aligned}$$

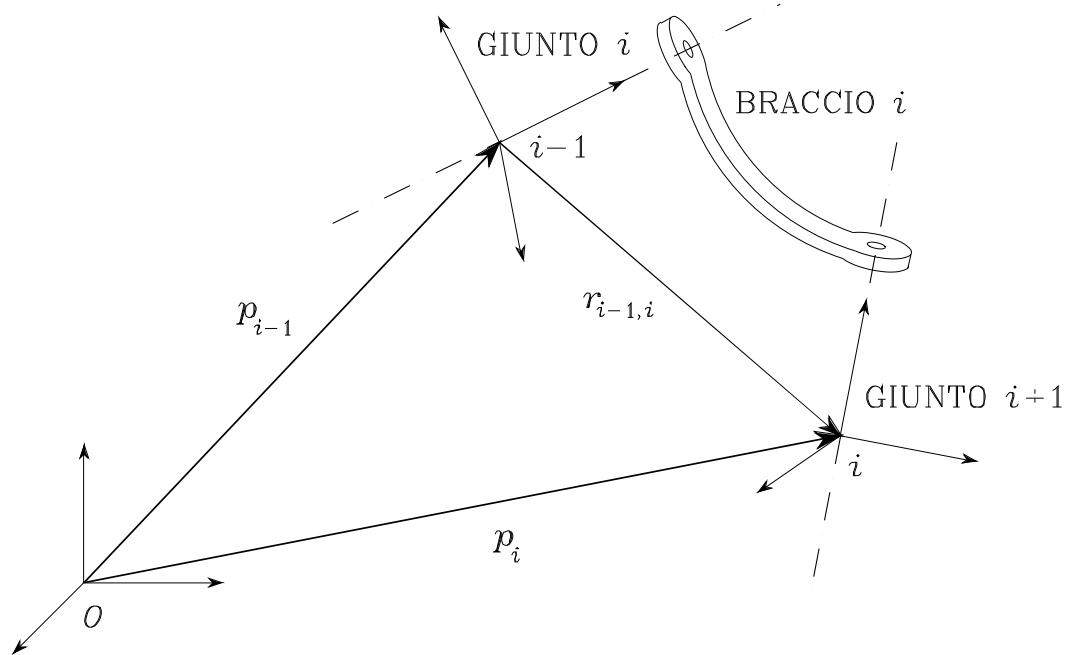


$$\mathbf{p}^0 = \mathbf{o}_1^0 + \mathbf{R}_1^0 \mathbf{p}^1$$

$$\begin{aligned} \dot{\mathbf{p}}^0 &= \dot{\mathbf{o}}_1^0 + \mathbf{R}_1^0 \dot{\mathbf{p}}^1 + \dot{\mathbf{R}}_1^0 \mathbf{p}^1 \\ &= \dot{\mathbf{o}}_1^0 + \mathbf{R}_1^0 \dot{\mathbf{p}}^1 + \mathbf{S}(\boldsymbol{\omega}_1^0) \mathbf{R}_1^0 \mathbf{p}^1 \\ &= \dot{\mathbf{o}}_1^0 + \mathbf{R}_1^0 \dot{\mathbf{p}}^1 + \boldsymbol{\omega}_1^0 \times \mathbf{r}_1^0 \end{aligned}$$

Velocità di un braccio

- Velocità lineare



$$p_i = p_{i-1} + R_{i-1} r_{i-1,i}^{i-1}$$

$$\begin{aligned} \dot{p}_i &= \dot{p}_{i-1} + R_{i-1} \dot{r}_{i-1,i}^{i-1} + \omega_{i-1} \times R_{i-1} r_{i-1,i}^{i-1} \\ &= \dot{p}_{i-1} + v_{i-1,i} + \omega_{i-1} \times r_{i-1,i} \end{aligned}$$

- Velocità angolare

$$\mathbf{R}_i = \mathbf{R}_{i-1} \mathbf{R}_i^{i-1}$$

$$\mathbf{S}(\boldsymbol{\omega}_i) \mathbf{R}_i = \mathbf{S}(\boldsymbol{\omega}_{i-1}) \mathbf{R}_i + \mathbf{R}_{i-1} \mathbf{S}(\boldsymbol{\omega}_{i-1,i}^{i-1}) \mathbf{R}_i^{i-1}$$

$$= \mathbf{S}(\boldsymbol{\omega}_{i-1}) \mathbf{R}_i + \mathbf{S}(\mathbf{R}_{i-1} \boldsymbol{\omega}_{i-1,i}^{i-1}) \mathbf{R}_i$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \mathbf{R}_{i-1} \boldsymbol{\omega}_{i-1,i}^{i-1}$$

$$= \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i}$$

- Giunto prismatico

$$\boldsymbol{\omega}_{i-1,i} = \mathbf{0}$$

$$\boldsymbol{v}_{i-1,i} = \dot{d}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \dot{d}_i \boldsymbol{z}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}$$

- Giunto rotoidale

$$\boldsymbol{\omega}_{i-1,i} = \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{v}_{i-1,i} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,i}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}$$

Calcolo dello Jacobiano

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{P1} & \dots & \mathbf{J}_{Pn} \\ \mathbf{J}_{O1} & & \mathbf{J}_{On} \end{bmatrix}$$

- Velocità angolare

- ★ giunto *i* prismatico

$$\dot{q}_i \mathbf{J}_{O_i} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{J}_{O_i} = \mathbf{0}$$

- ★ giunto *i* rotoidale

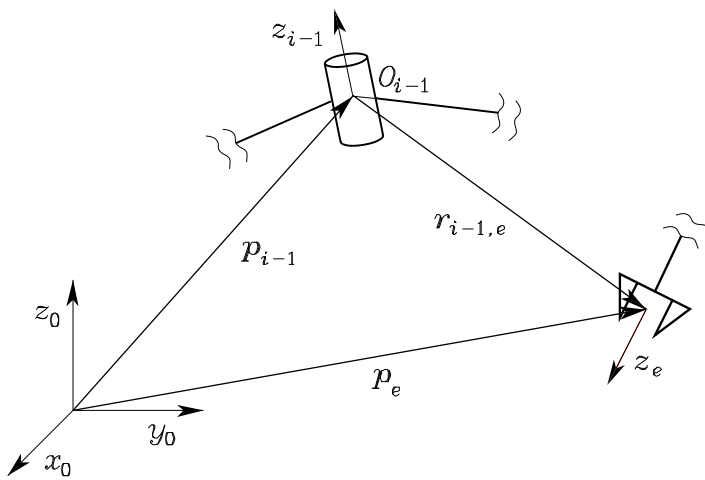
$$\dot{q}_i \mathbf{J}_{O_i} = \dot{\vartheta}_i \mathbf{z}_{i-1} \quad \Longrightarrow \quad \mathbf{J}_{O_i} = \mathbf{z}_{i-1}$$

- Velocità lineare

- ★ giunto *i* prismatico

$$\dot{q}_i \mathbf{J}_{Pi} = \dot{d}_i \mathbf{z}_{i-1} \quad \Longrightarrow \quad \mathbf{J}_{Pi} = \mathbf{z}_{i-1}$$

- ★ giunto *i* rotoidale



$$\begin{aligned} \dot{q}_i \mathbf{J}_{Pi} &= \boldsymbol{\omega}_{i-1,i} \times \mathbf{r}_{i-1,n} \\ &= \dot{\vartheta}_i \mathbf{z}_{i-1} \times (\mathbf{p}_e - \mathbf{p}_{i-1}) \end{aligned}$$

$$\Downarrow$$

$$\mathbf{J}_{Pi} = \mathbf{z}_{i-1} \times (\mathbf{p}_e - \mathbf{p}_{i-1})$$

- Colonna dello Jacobiano geometrico

$$\begin{bmatrix} \mathbf{J}^{Pi} \\ \mathbf{J}^{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} \mathbf{z}_{i-1} \times (\mathbf{p}_e - \mathbf{p}_{i-1}) \\ \mathbf{z}_{i-1} \end{bmatrix} \end{cases}$$

per un giunto *prismatico*

per un giunto *rotoidale*

$$\star \mathbf{z}_{i-1} = \mathbf{R}_1^0(q_1) \dots \mathbf{R}_{i-1}^{i-2}(q_{i-1}) \mathbf{z}_0$$

$$\star \tilde{\mathbf{p}}_e = \mathbf{A}_1^0(q_1) \dots \mathbf{A}_n^{n-1}(q_n) \tilde{\mathbf{p}}_0$$

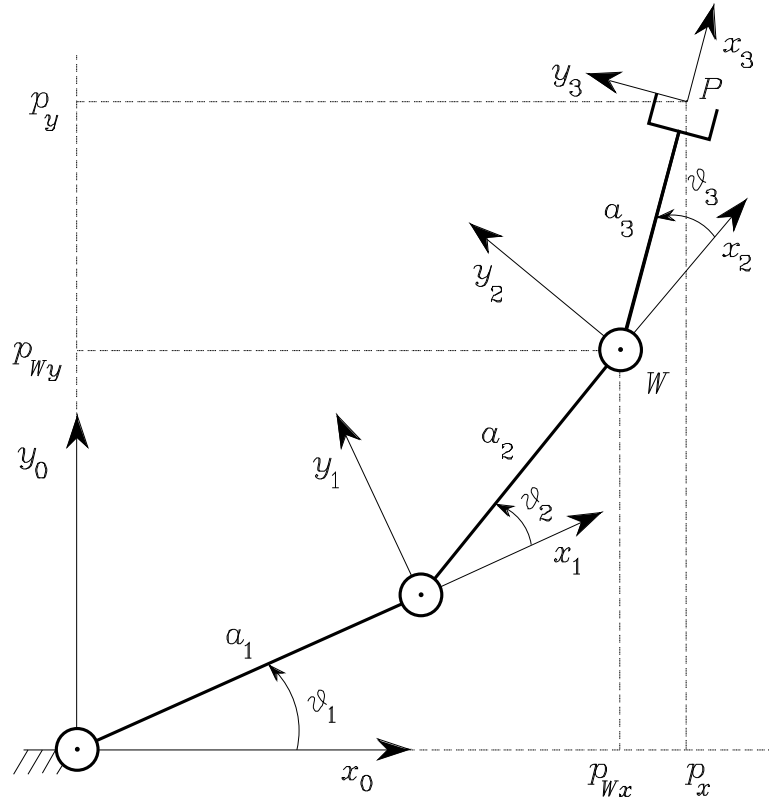
$$\star \tilde{\mathbf{p}}_{i-1} = \mathbf{A}_1^0(q_1) \dots \mathbf{A}_{i-1}^{i-2}(q_{i-1}) \tilde{\mathbf{p}}_0$$

- Rappresentazione in terna differente

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{p}}^t \\ \boldsymbol{\omega}^t \end{bmatrix} &= \begin{bmatrix} \mathbf{R}^t & \mathbf{O} \\ \mathbf{O} & \mathbf{R}^t \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \boldsymbol{\omega} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}^t & \mathbf{O} \\ \mathbf{O} & \mathbf{R}^t \end{bmatrix} \mathbf{J} \dot{\mathbf{q}} \end{aligned}$$

$$\mathbf{J}^t = \begin{bmatrix} \mathbf{R}^t & \mathbf{O} \\ \mathbf{O} & \mathbf{R}^t \end{bmatrix} \mathbf{J}$$

Manipolatore planare a tre bracci



$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_3 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_3 - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_3 - \mathbf{p}_2) \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 \end{bmatrix}$$

$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

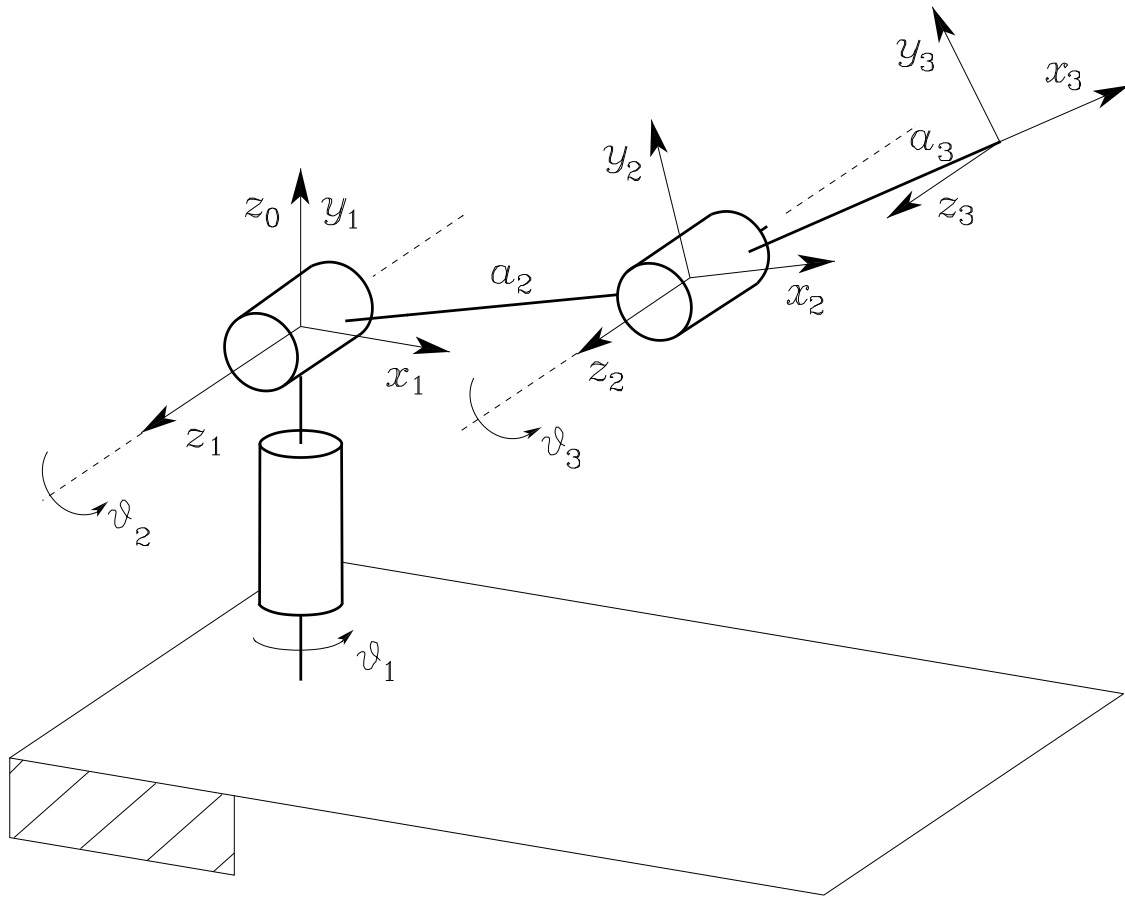
$$\mathbf{p}_3 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix}$$

$$\mathbf{z}_0 = \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{J}_P = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \end{bmatrix}$$

Manipolatore antropomorfo



$$\mathbf{J} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_3 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_3 - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_3 - \mathbf{p}_2) \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 \end{bmatrix}$$

$$\mathbf{p}_0 = \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} a_2 c_1 c_2 \\ a_2 s_1 c_2 \\ a_2 s_2 \end{bmatrix}$$

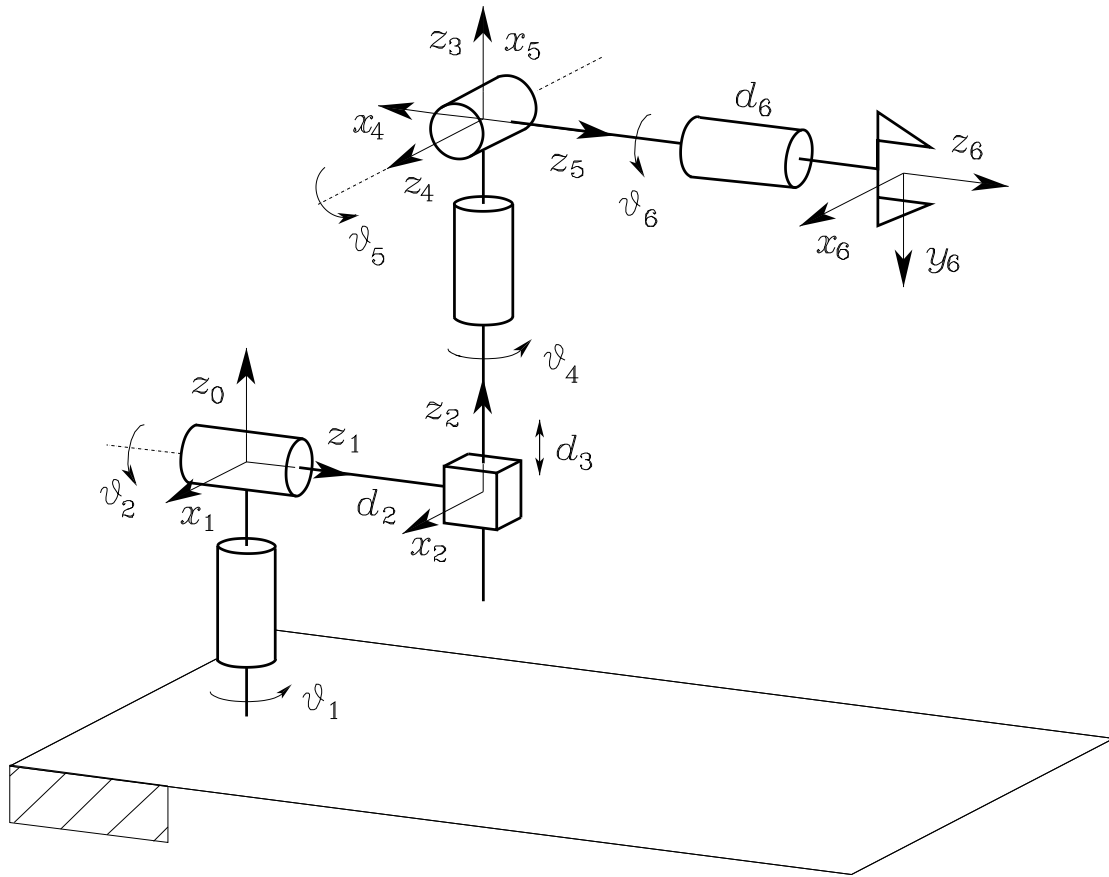
$$\mathbf{p}_3 = \begin{bmatrix} c_1(a_2 c_2 + a_3 c_{23}) \\ s_1(a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{bmatrix}$$

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J}_P = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix}$$

Manipolatore di Stanford



$$\mathbf{J} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_6 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_6 - \mathbf{p}_1) & \mathbf{z}_2 \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{0} \\ \mathbf{z}_3 \times (\mathbf{p}_6 - \mathbf{p}_3) & \mathbf{z}_4 \times (\mathbf{p}_6 - \mathbf{p}_4) & \mathbf{z}_5 \times (\mathbf{p}_6 - \mathbf{p}_5) \\ \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{bmatrix}$$

$$\mathbf{p}_0 = \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \mathbf{p}_4 = \mathbf{p}_5 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$\mathbf{p}_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6 (c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 + c_1 d_2 + d_6 (c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6 (c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \mathbf{z}_2 = \mathbf{z}_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$\mathbf{z}_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \quad \mathbf{z}_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

SINGOLARITÀ CINEMATICHE

$$v_e = J(q)\dot{q}$$

- se J diminuisce di rango \implies *singolarità cinematiche*
 - (a) perdita di mobilità
 - (b) infinite soluzioni al problema cinematico inverso
 - (c) velocità elevate nello spazio dei giunti (nell'intorno di una singolarità)

- Classificazione
 - ★ Singolarità *ai confini dello spazio di lavoro raggiungibile*
 - ★ Singolarità *all'interno dello spazio di lavoro raggiungibile*

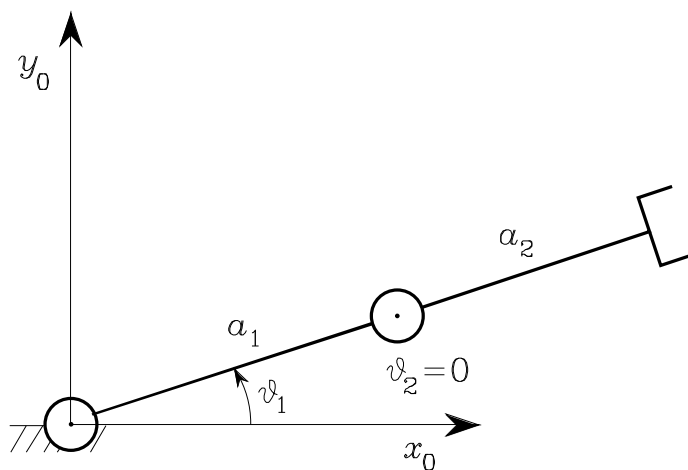
- Manipolatore planare a due bracci

$$\mathbf{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\det(\mathbf{J}) = a_1 a_2 s_2$$

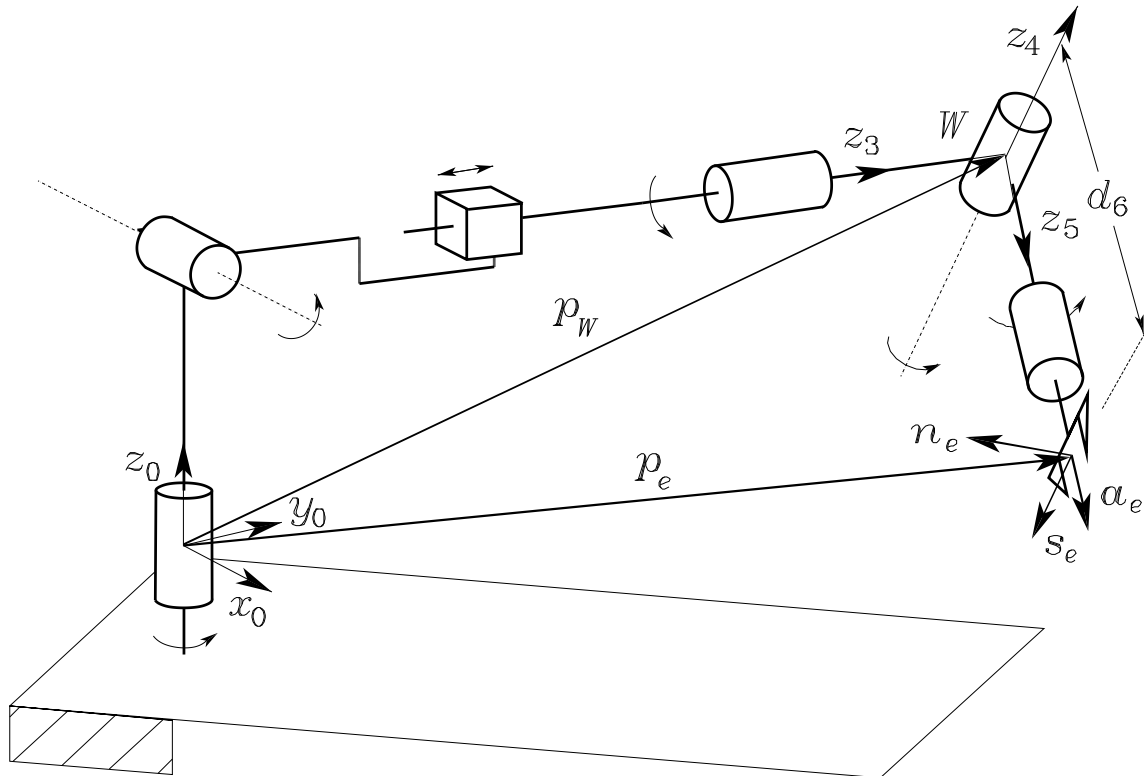
$$\Downarrow$$

$$\vartheta_2 = 0 \quad \vartheta_2 = \pi$$



★ $[-(a_1 + a_2)s_1 \quad (a_1 + a_2)c_1]^T$ parallelo a $[-a_2s_1 \quad a_2c_1]^T$
 (componenti di velocità dell'organo terminale non indipendenti)

Disaccoppiamento di singolarità



- calcolo delle *singolarità della struttura portante*
- calcolo delle *singolarità del polso*

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}$$

$$\mathbf{J}_{12} = \left[\mathbf{z}_3 \times (\mathbf{p}_e - \mathbf{p}_3) \quad \mathbf{z}_4 \times (\mathbf{p}_e - \mathbf{p}_4) \quad \mathbf{z}_5 \times (\mathbf{p}_e - \mathbf{p}_5) \right]$$

$$\mathbf{J}_{22} = \left[\mathbf{z}_3 \quad \mathbf{z}_4 \quad \mathbf{z}_5 \right]$$

- $\mathbf{p}_e = \mathbf{p}_W \implies \mathbf{p}_W - \mathbf{p}_i$ paralleli a $\mathbf{z}_i, i = 3, 4, 5$

$$\mathbf{J}_{12} = \left[\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \right]$$

$$\det(\mathbf{J}) = \det(\mathbf{J}_{11})\det(\mathbf{J}_{22})$$

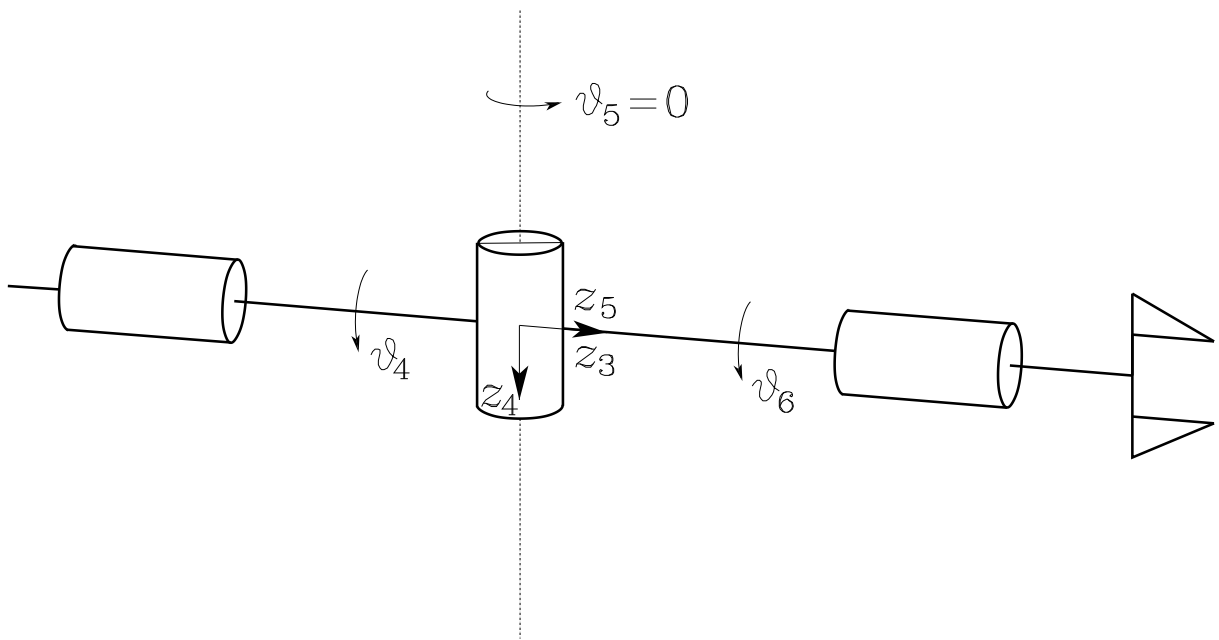
$$\det(\mathbf{J}_{11}) = 0$$

$$\det(\mathbf{J}_{22}) = 0$$

Singularità di polso

- z_3 parallelo a z_5

$$\vartheta_5 = 0 \quad \vartheta_5 = \pi$$



- ★ rotazioni uguali e opposte di ϑ_4 e ϑ_6 non producono alcuna rotazione dell'organo terminale

Singularità di struttura portante

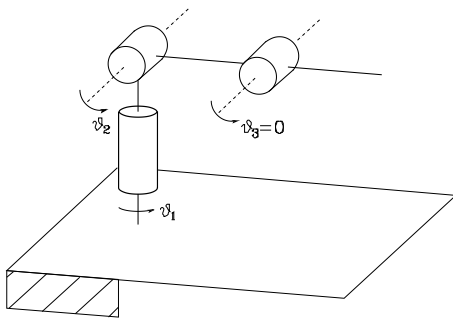
- Manipolatore antropomorfo

$$\det(\mathbf{J}_P) = -a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

$$s_3 = 0 \quad a_2 c_2 + a_3 c_{23} = 0$$

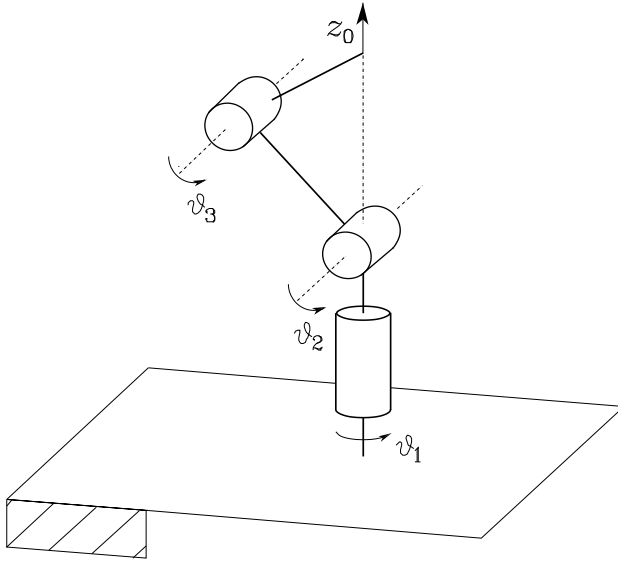
- ★ Singularità di *gomito*

$$\vartheta_3 = 0 \quad \vartheta_3 = \pi$$



★ Singularità di *spalla*

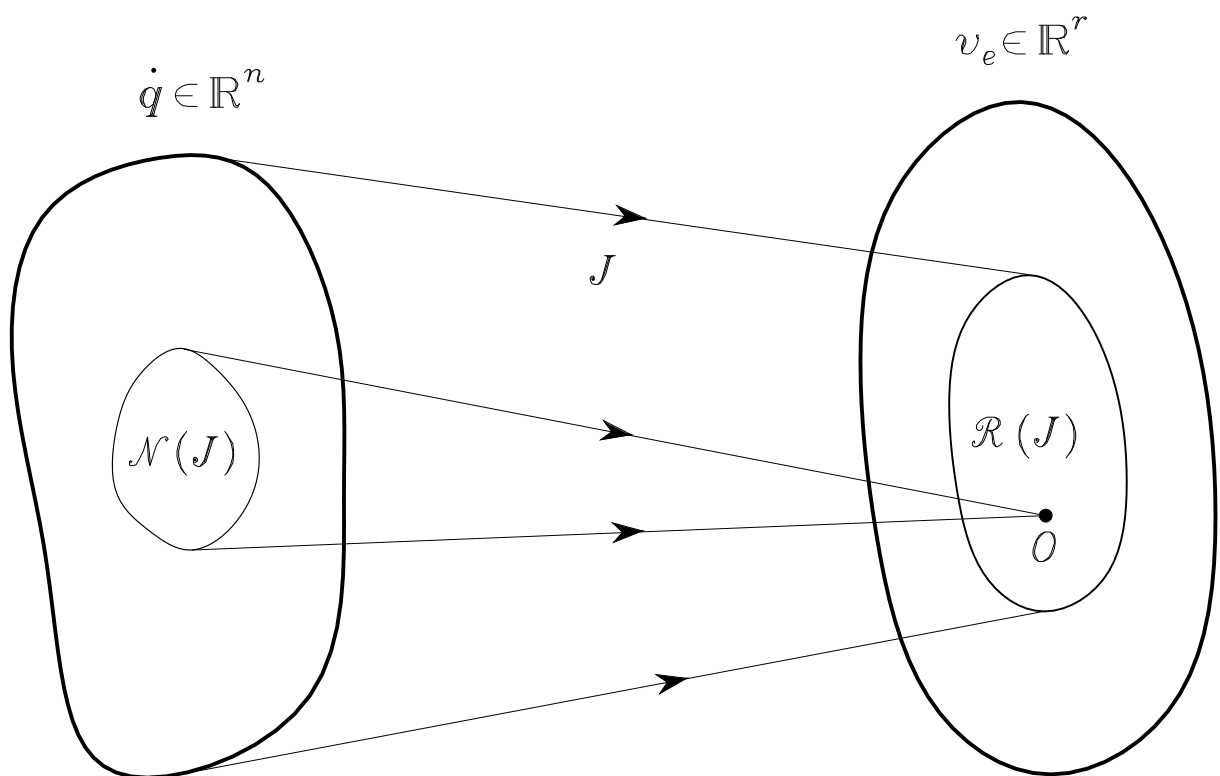
$$p_x = p_y = 0$$



ANALISI DELLA RIDONDANZA

- Cinematica differenziale

$$\mathbf{v}_e = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$



★ se $\rho(\mathbf{J}) = r$

$$\dim(\mathcal{R}(\mathbf{J})) = r \quad \dim(\mathcal{N}(\mathbf{J})) = n - r$$

★ in generale

$$\dim(\mathcal{R}(\mathbf{J})) + \dim(\mathcal{N}(\mathbf{J})) = n$$

- Se $\mathcal{N}(\mathbf{J}) \neq \emptyset$

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^* + \mathbf{P}\dot{\mathbf{q}}_a$$

ove

$$\mathcal{R}(\mathbf{P}) \equiv \mathcal{N}(\mathbf{J})$$

★ verifica:

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}}^* + \mathbf{J}\mathbf{P}\dot{\mathbf{q}}_a = \mathbf{J}\dot{\mathbf{q}}^* = \mathbf{v}_e$$

- $\dot{\mathbf{q}}_a$ genera *moti interni* della struttura

INVERSIONE DELLA CINEMATICA DIFFERENZIALE

- Equazione cinematica non lineare
- Equazione cinematica differenziale lineare nelle velocità
- Data $\mathbf{v}(t)$ + condizioni iniziali $\implies (\mathbf{q}(t), \dot{\mathbf{q}}(t))$

★ se $n = r$

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})\mathbf{v}_e$$

$$\mathbf{q}(t) = \int_0^t \dot{\mathbf{q}}(\varsigma) d\varsigma + \mathbf{q}(0)$$

★ regola di integrazione numerica (Eulero)

$$\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t$$

Manipolatori ridondanti

- Per una data configurazione q , trovare le soluzioni \dot{q} che soddisfino

$$v_e = J\dot{q}$$

e che minimizzino

$$g(\dot{q}) = \frac{1}{2}\dot{q}^T W \dot{q}$$

★ *metodo dei moltiplicatori di Lagrange*

$$g(\dot{q}, \lambda) = \frac{1}{2}\dot{q}^T W \dot{q} + \lambda^T (v_e - J\dot{q})$$

$$\left(\frac{\partial g}{\partial \dot{q}} \right)^T = 0 \quad \left(\frac{\partial g}{\partial \lambda} \right)^T = 0$$

★ *soluzione ottima*

$$\dot{q} = W^{-1} J^T (J W^{-1} J^T)^{-1} v_e$$

★ *se $W = I$*

$$\dot{q} = J^\dagger v_e$$

ove

$$J^\dagger = J^T (J J^T)^{-1}$$

è la *pseudo-inversa destra* di J

- Utilizzo della ridondanza

$$g'(\dot{\mathbf{q}}) = \frac{1}{2}(\dot{\mathbf{q}}^T - \dot{\mathbf{q}}_a^T)(\dot{\mathbf{q}} - \dot{\mathbf{q}}_a)$$

★ come sopra ...

$$g'(\dot{\mathbf{q}}, \boldsymbol{\lambda}) = \frac{1}{2}(\dot{\mathbf{q}}^T - \dot{\mathbf{q}}_a^T)(\dot{\mathbf{q}} - \dot{\mathbf{q}}_a) + \boldsymbol{\lambda}^T(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}})$$

★ soluzione ottima

$$\dot{\mathbf{q}} = \mathbf{J}^\dagger \mathbf{v}_e + (\mathbf{I} - \mathbf{J}^\dagger \mathbf{J})\dot{\mathbf{q}}_0$$

- Caratterizzazione dei moti interni

$$\dot{\mathbf{q}}_0 = k_0 \left(\frac{\partial w(\mathbf{q})}{\partial \mathbf{q}} \right)^T$$

- ★ *misura di manipolabilità*

$$w(\mathbf{q}) = \sqrt{\det(\mathbf{J}(\mathbf{q})\mathbf{J}^T(\mathbf{q}))}$$

- ★ *distanza dai fine-corsa dei giunti*

$$w(\mathbf{q}) = -\frac{1}{2n} \sum_{i=1}^n \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right)^2$$

- ★ *distanza da un ostacolo*

$$w(\mathbf{q}) = \min_{\mathbf{p}, \mathbf{o}} \|\mathbf{p}(\mathbf{q}) - \mathbf{o}\|$$

Singularità cinematiche

- Le soluzioni precedenti valgono solo se \mathbf{J} è di rango pieno
- Se \mathbf{J} non è di rango pieno (singularità)
 - ★ se $\mathbf{v}_e \in \mathcal{R}(\mathbf{J}) \implies$ soluzione $\dot{\mathbf{q}}$ estraendo tutte le equazioni linearmente indipendenti (traiettoria “fisicamente” eseguibile)
 - ★ se $\mathbf{v}_e \notin \mathcal{R}(\mathbf{J}) \implies$ il sistema non è risolvibile (traiettoria non eseguibile)
- Inversione nell’intorno di singularità
 - ★ $\det(\mathbf{J})$ piccolo $\implies \dot{\mathbf{q}}$ elevate
 - ★ *inversa a minimi quadrati smorzata*

$$\mathbf{J}^* = \mathbf{J}^T (\mathbf{J}\mathbf{J}^T + k^2 \mathbf{I})^{-1}$$

ove $\dot{\mathbf{q}}$ minimizza

$$g''(\dot{\mathbf{q}}) = \frac{1}{2}(\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}})^T (\mathbf{v}_e - \mathbf{J}\dot{\mathbf{q}}) + \frac{1}{2}k^2 \dot{\mathbf{q}}^T \dot{\mathbf{q}}$$

JACOBIANO ANALITICO

$$\mathbf{p}_e = \mathbf{p}_e(\mathbf{q})$$

$$\phi_e = \phi_e(\mathbf{q})$$

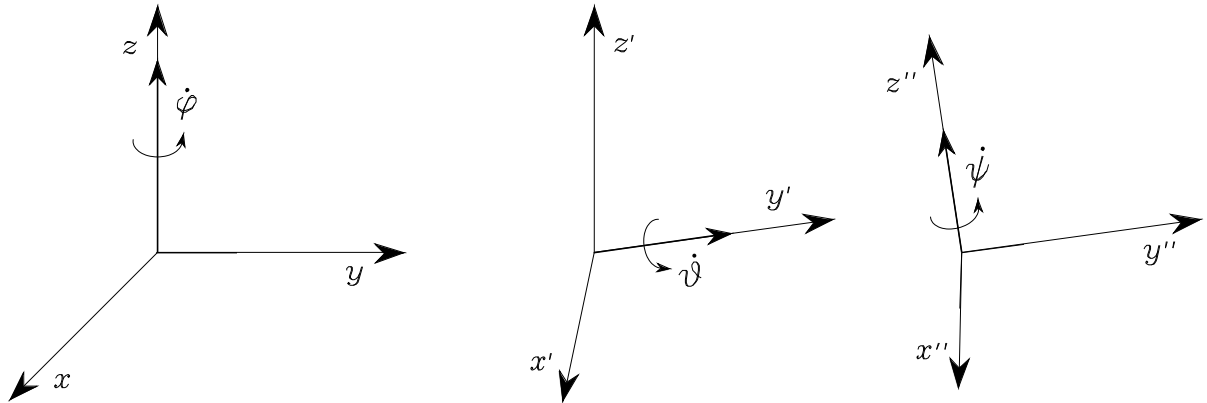
$$\dot{\mathbf{p}}_e = \frac{\partial \mathbf{p}_e}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_P(\mathbf{q}) \dot{\mathbf{q}}$$

$$\dot{\phi}_e = \frac{\partial \phi_e}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_\phi(\mathbf{q}) \dot{\mathbf{q}}$$

$$\begin{aligned} \dot{\mathbf{x}}_e &= \begin{bmatrix} \dot{\mathbf{p}}_e \\ \dot{\phi}_e \end{bmatrix} = \begin{bmatrix} \mathbf{J}_P(\mathbf{q}) \\ \mathbf{J}_\phi(\mathbf{q}) \end{bmatrix} \dot{\mathbf{q}} \\ &= \mathbf{J}_A(\mathbf{q}) \dot{\mathbf{q}} \end{aligned}$$

$$\mathbf{J}_A(\mathbf{q}) = \frac{\partial \mathbf{k}(\mathbf{q})}{\partial \mathbf{q}}$$

- Velocità di rotazione in terna corrente di angoli di Eulero ZYZ

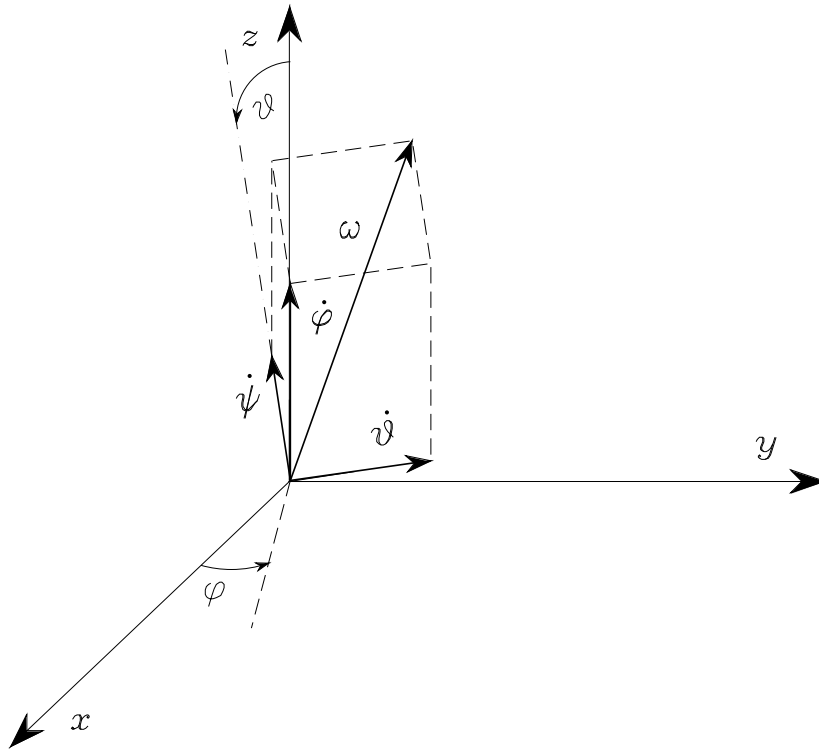


★ per effetto di $\dot{\varphi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\varphi} [0 \quad 0 \quad 1]^T$

★ per effetto di $\dot{\vartheta}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\vartheta} [-s_\varphi \quad c_\varphi \quad 0]^T$

★ per effetto di $\dot{\psi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\psi} [c_\varphi s_\vartheta \quad s_\varphi s_\vartheta \quad c_\vartheta]^T$

- Composizione di velocità di rotazione elementari



$$\omega_e = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix} \dot{\phi} = \mathbf{T}(\phi_e) \dot{\phi}_e$$

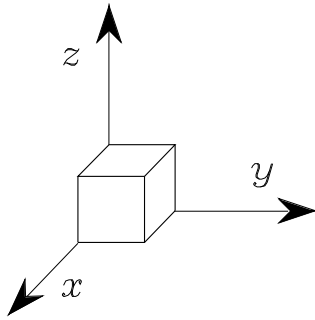
- Significato fisico di ω

$$\omega = [\pi/2 \quad 0 \quad 0]^T \quad 0 \leq t \leq 1$$

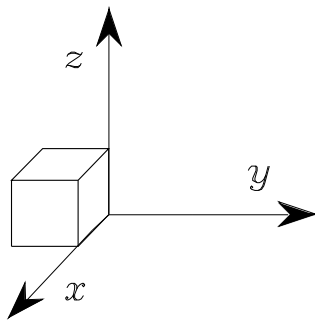
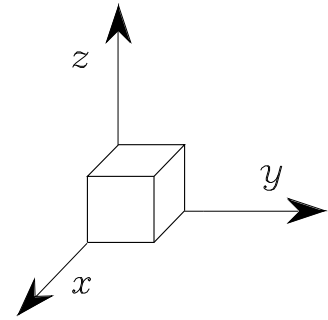
$$\omega = [0 \quad \pi/2 \quad 0]^T \quad 1 < t \leq 2$$

$$\omega = [0 \quad \pi/2 \quad 0]^T \quad 0 \leq t \leq 1$$

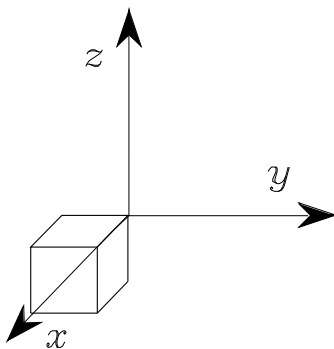
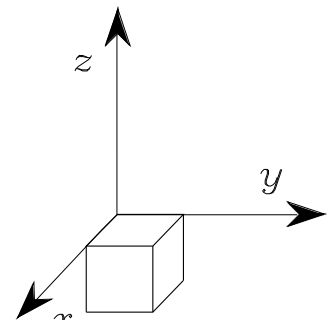
$$\omega = [\pi/2 \quad 0 \quad 0]^T \quad 1 < t \leq 2$$



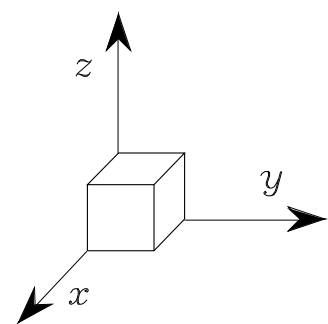
$t = 0$



$t = 1$



$t = 2$



$$\int_0^2 \omega dt = [\pi/2 \quad \pi/2 \quad 0]^T$$

Relazione tra Jacobiano analitico e Jacobiano geometrico

$$\mathbf{v}_e = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{T}(\phi_e) \end{bmatrix} \dot{\mathbf{x}} = \mathbf{T}_A(\phi_e) \dot{\mathbf{x}}$$

$$\mathbf{J} = \mathbf{T}_A(\phi_e) \mathbf{J}_A$$

- Jacobiano geometrico
 - ★ grandezze di significato fisico
- Jacobiano analitico
 - ★ grandezze differenziali di variabili nello spazio operativo

ALGORITMI PER L'INVERSIONE CINEMATICA

- Inversione cinematica

$$\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \mathbf{J}^{-1}(\mathbf{q}(t_k))\mathbf{v}_e(t_k)\Delta t$$

★ fenomeni di *deriva* della soluzione

- Soluzione algoritmica

★ *errore nello spazio operativo*

$$\mathbf{e} = \mathbf{x}_d - \mathbf{x}_e$$

$$\begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{x}}_d - \dot{\mathbf{x}}_e \\ &= \dot{\mathbf{x}}_d - \mathbf{J}_A(\mathbf{q})\dot{\mathbf{q}}\end{aligned}$$

★ trovare $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{e})$: $\mathbf{e} \rightarrow \mathbf{0}$

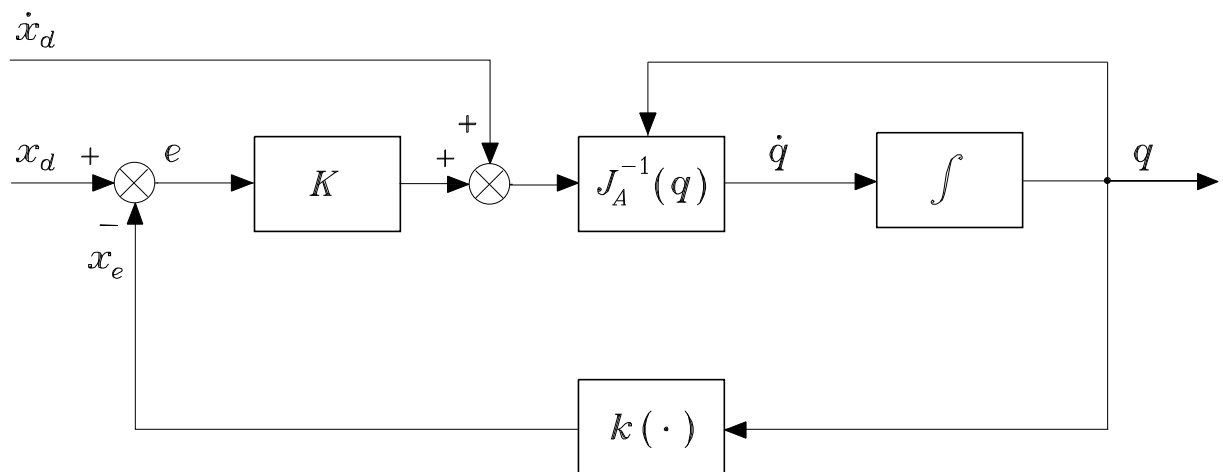
(Pseudo-)inversa dello Jacobiano

- Linearizzazione della dinamica di errore

$$\dot{q} = J_A^{-1}(q)(\dot{x}_d + Ke)$$

⇓

$$\dot{e} + Ke = 0$$



★ Per un *manipolatore ridondante*

$$\dot{q} = J_A^\dagger(\dot{x}_d + Ke) + (I - J_A^\dagger J_A)\dot{q}_0$$

Trasposta dello Jacobiano

- $\dot{q} = \dot{q}(e)$ senza linearizzare la dinamica di errore
- Metodo di Lyapunov

$$V(e) = \frac{1}{2} e^T K e$$

ove

$$V(e) > 0 \quad \forall e \neq \mathbf{0} \quad V(\mathbf{0}) = 0$$

$$\begin{aligned} \dot{V}(e) &= e^T K \dot{x}_d - e^T K \dot{x}_e \\ &= e^T K \dot{x}_d - e^T K J_A(q) \dot{q} \end{aligned}$$

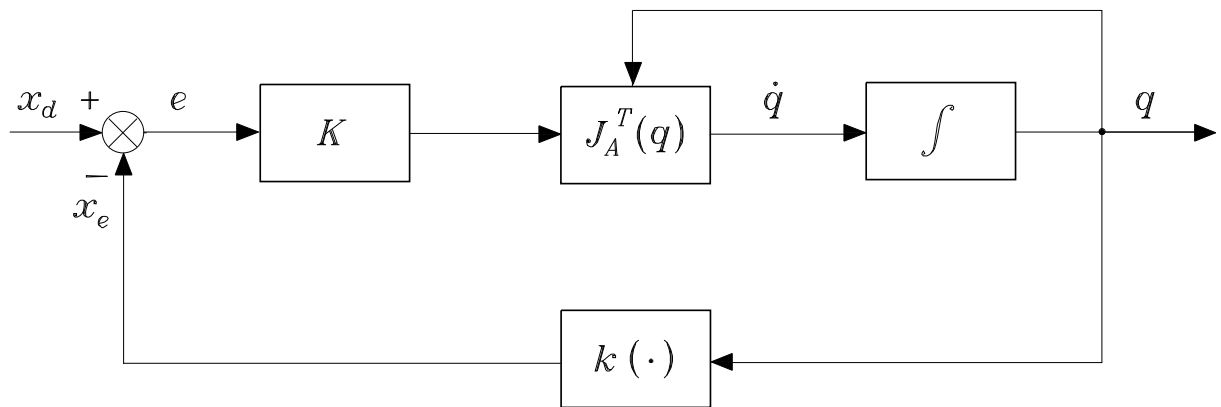
★ la scelta

$$\dot{q} = J_A^T(q) K e$$

comporta che

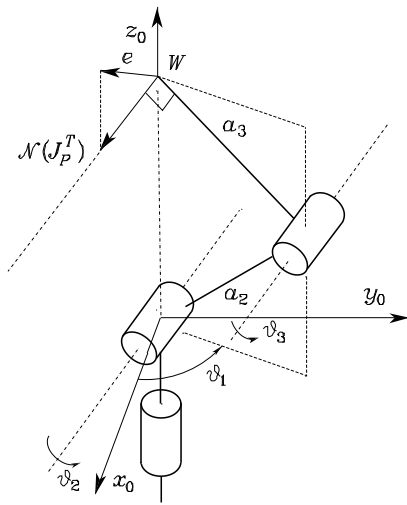
$$\dot{V}(e) = e^T K \dot{x}_d - e^T K J_A(q) J_A^T(q) K e$$

- ★ se $\dot{x}_d = \mathbf{0} \implies \dot{V} < 0$ con $V > 0$ (*asintotica stabilità*)
- ★ se $\mathcal{N}(J_A^T) \neq \emptyset \implies \dot{V} = 0$ se $K e \in \mathcal{N}(J_A^T)$
 $\dot{q} = \mathbf{0}$ con $e \neq \mathbf{0}$ (stallo?)



- Se $\dot{x}_d \neq 0$
 - ★ $e(t)$ limitato (conviene aumentare la norma di K)
 - ★ $e(\infty) \rightarrow 0$

- Esempio



$$\mathbf{J}_P^T = \begin{bmatrix} 0 & 0 & 0 \\ -c_1(a_2 s_2 + a_3 s_{23}) & -s_1(a_2 s_2 + a_3 s_{23}) & 0 \\ -a_3 c_1 s_{23} & -a_3 s_1 s_{23} & a_3 c_{23} \end{bmatrix}$$

★ nullo di \mathbf{J}_P^T

$$\frac{\nu_y}{\nu_x} = -\frac{1}{\tan \vartheta_1} \quad \nu_z = 0$$

Errore di orientamento

- Errore di posizione

$$\mathbf{e}_P = \mathbf{p}_d - \mathbf{p}_e(\mathbf{q})$$

$$\dot{\mathbf{e}}_P = \dot{\mathbf{p}}_d - \dot{\mathbf{p}}_e$$

- Angoli di Eulero

$$\mathbf{e}_O = \boldsymbol{\phi}_d - \boldsymbol{\phi}_e(\mathbf{q})$$

$$\dot{\mathbf{e}}_O = \dot{\boldsymbol{\phi}}_d - \dot{\boldsymbol{\phi}}_e$$

$$\dot{\mathbf{q}} = \mathbf{J}_A^{-1}(\mathbf{q}) \begin{bmatrix} \dot{\mathbf{p}}_d + \mathbf{K}_P \mathbf{e}_P \\ \dot{\boldsymbol{\phi}}_d + \mathbf{K}_O \mathbf{e}_O \end{bmatrix}$$

★ agevole per assegnare l'andamento temporale $\boldsymbol{\phi}_d(t)$

★ richiede comunque il passaggio attraverso $\mathbf{R}_e = \begin{bmatrix} \mathbf{n}_e & \mathbf{s}_e & \mathbf{a}_e \end{bmatrix}$

- Manipolatore con polso sferico

★ calcolare $\mathbf{q}_P \implies \mathbf{R}_W$

★ calcolare $\mathbf{R}_W^T \mathbf{R}_d \implies \mathbf{q}_O$ (angoli di Eulero ZYZ)

- Asse/angolo

$$\mathbf{R}(\vartheta, \mathbf{r}) = \mathbf{R}_d \mathbf{R}_e^T$$

- ★ errore di orientamento

$$\begin{aligned} \mathbf{e}_O &= \mathbf{r} \sin \vartheta \\ &= \frac{1}{2} (\mathbf{n}_e \times \mathbf{n}_d + \mathbf{s}_e \times \mathbf{s}_d + \mathbf{a}_e \times \mathbf{a}_d) \end{aligned}$$

$$\dot{\mathbf{e}}_O = \mathbf{L}^T \boldsymbol{\omega}_d - \mathbf{L} \boldsymbol{\omega}_e$$

ove

$$\mathbf{L} = -\frac{1}{2} (\mathbf{S}(\mathbf{n}_d) \mathbf{S}(\mathbf{n}_e) + \mathbf{S}(\mathbf{s}_d) \mathbf{S}(\mathbf{s}_e) + \mathbf{S}(\mathbf{a}_d) \mathbf{S}(\mathbf{a}_e))$$

$$\begin{aligned} \dot{\mathbf{e}} &= \begin{bmatrix} \dot{\mathbf{e}}_P \\ \dot{\mathbf{e}}_O \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{p}}_d - \mathbf{J}_P(\mathbf{q}) \dot{\mathbf{q}} \\ \mathbf{L}^T \boldsymbol{\omega}_d - \mathbf{L} \mathbf{J}_O(\mathbf{q}) \dot{\mathbf{q}} \end{bmatrix} \\ &= \begin{bmatrix} \dot{\mathbf{p}}_d \\ \mathbf{L}^T \boldsymbol{\omega}_d \end{bmatrix} - \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{L} \end{bmatrix} \mathbf{J} \dot{\mathbf{q}} \end{aligned}$$

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q}) \begin{bmatrix} \dot{\mathbf{p}}_d + \mathbf{K}_P \mathbf{e}_P \\ \mathbf{L}^{-1} (\mathbf{L}^T \boldsymbol{\omega}_d + \mathbf{K}_O \mathbf{e}_O) \end{bmatrix}$$

- Quaternione unitario

$$\Delta Q = Q_d * Q_e^{-1}$$

- ★ errore di orientamento

$$e_O = \Delta \epsilon = \eta_e(\mathbf{q})\epsilon_d - \eta_d\epsilon_e(\mathbf{q}) - \mathbf{S}(\epsilon_d)\epsilon_e(\mathbf{q})$$

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q}) \begin{bmatrix} \dot{\mathbf{p}}_d + \mathbf{K}_P \mathbf{e}_P \\ \boldsymbol{\omega}_d + \mathbf{K}_O \mathbf{e}_O \end{bmatrix}$$

$$\boldsymbol{\omega}_d - \boldsymbol{\omega}_e + \mathbf{K}_O \mathbf{e}_O = \mathbf{0}$$

- ★ propagazione del quaternione

$$\dot{\eta} = -\frac{1}{2}\boldsymbol{\epsilon}^T \boldsymbol{\omega}$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\eta \mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon})) \boldsymbol{\omega}$$

- ★ studio della stabilità

$$V = (\eta_d - \eta_e)^2 + (\boldsymbol{\epsilon}_d - \boldsymbol{\epsilon}_e)^T (\boldsymbol{\epsilon}_d - \boldsymbol{\epsilon}_e)$$

$$\dot{V} = -\mathbf{e}_O^T \mathbf{K}_O \mathbf{e}_O$$

- Algoritmi del secondo ordine

- ★ derivata della cinematica differenziale

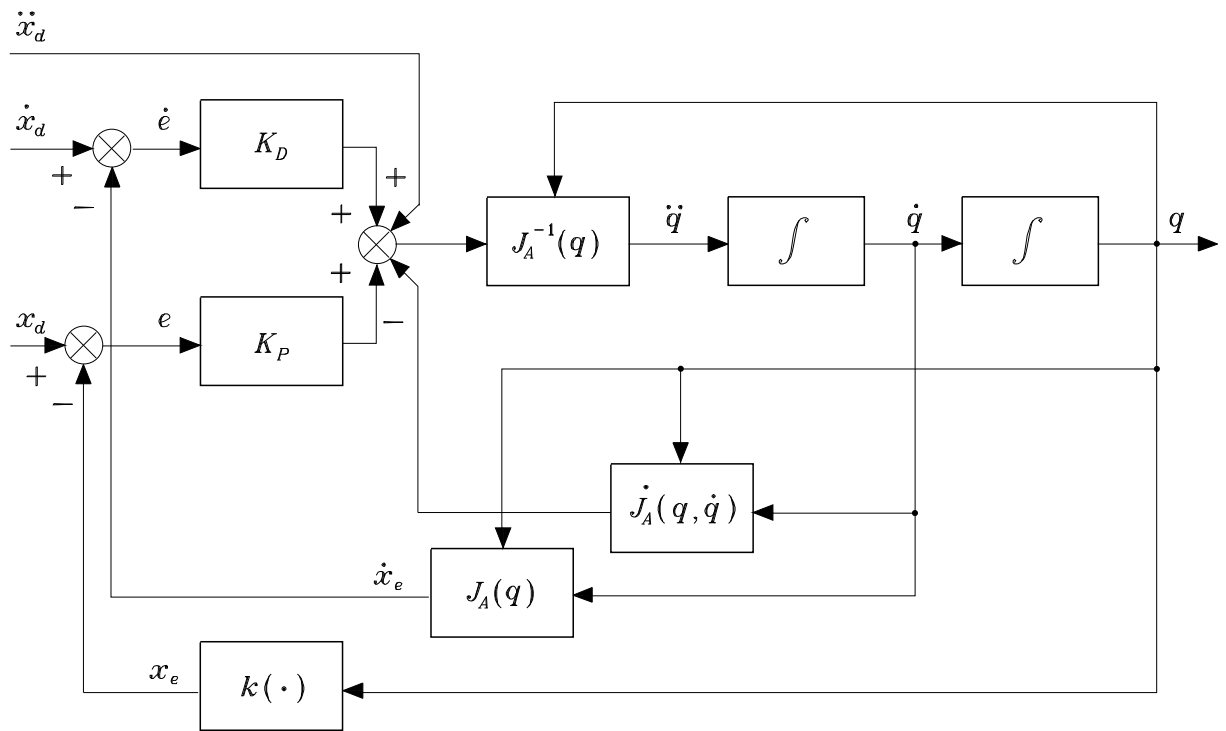
$$\ddot{\mathbf{x}}_e = \mathbf{J}_A(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}_A(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$$

- ★ soluzione in accelerazione ai giunti

$$\ddot{\mathbf{q}} = \mathbf{J}_A^{-1}(\mathbf{q}) \left(\ddot{\mathbf{x}}_d + \mathbf{K}_D \dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} - \dot{\mathbf{J}}_A(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \right)$$

⇓

$$\ddot{\mathbf{e}} + \mathbf{K}_D \dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} = \mathbf{0}$$



Confronto tra gli algoritmi per l'inversione cinematica

- Manipolatore planare a tre bracci

$$\mathbf{x}_e = \mathbf{k}(\mathbf{q})$$

$$\begin{bmatrix} p_x \\ p_y \\ \phi_e \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

$$\star a_1 = a_2 = a_3 = 0.5 \text{ m}$$

$$\mathbf{J}_A = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\star \mathbf{q}_i = [\pi \quad -\pi/2 \quad -\pi/2]^T \text{ rad}$$

↓

$$\star \mathbf{p}_{di} = [0 \quad 0.5]^T \text{ m} \quad \phi_{di} = 0 \text{ rad}$$

★ traiettoria desiderata

$$\mathbf{p}_d(t) = \begin{bmatrix} 0.25(1 - \cos \pi t) \\ 0.25(2 + \sin \pi t) \end{bmatrix} \quad 0 \leq t \leq 4$$

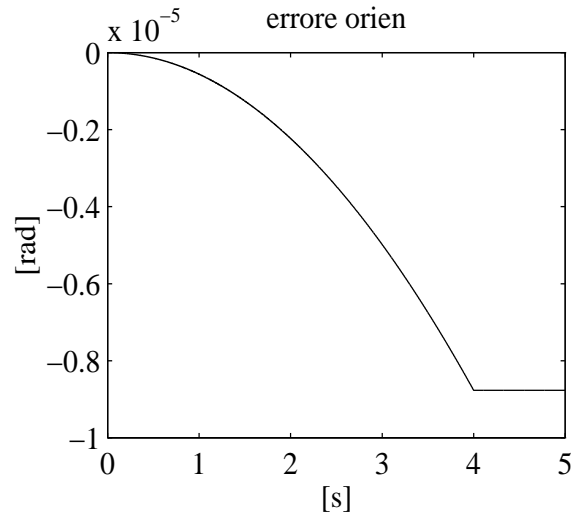
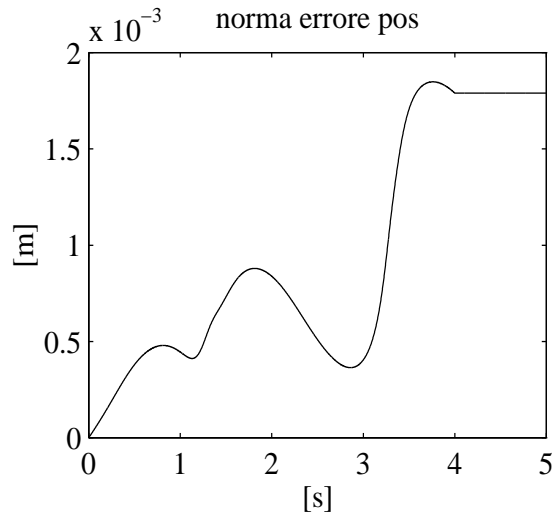
$$\phi_d(t) = \sin \frac{\pi}{24} t \quad 0 \leq t \leq 4$$

- Simulazione in MATLAB con integrazione numerica di Eulero

$$\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k) \Delta t$$

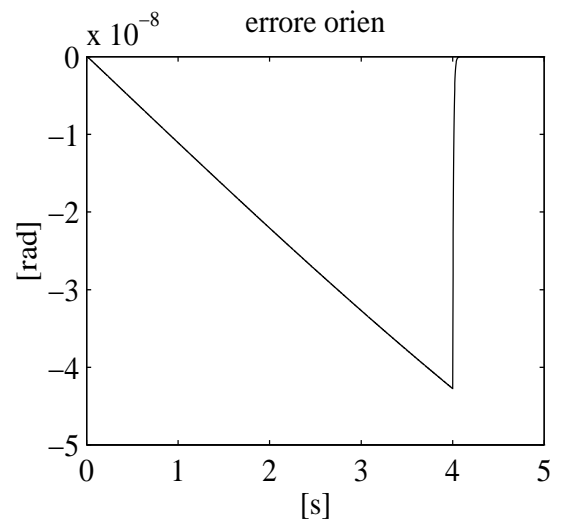
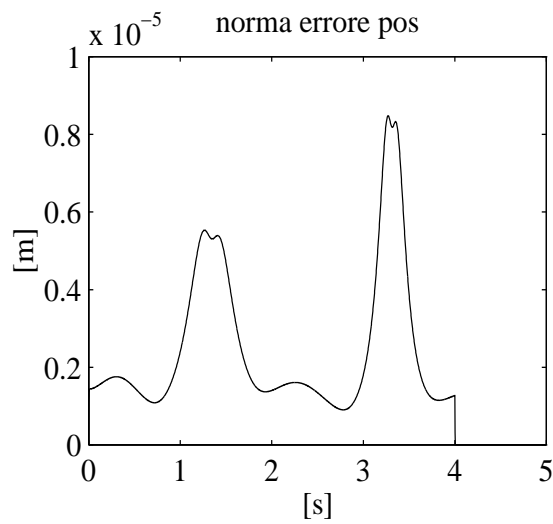
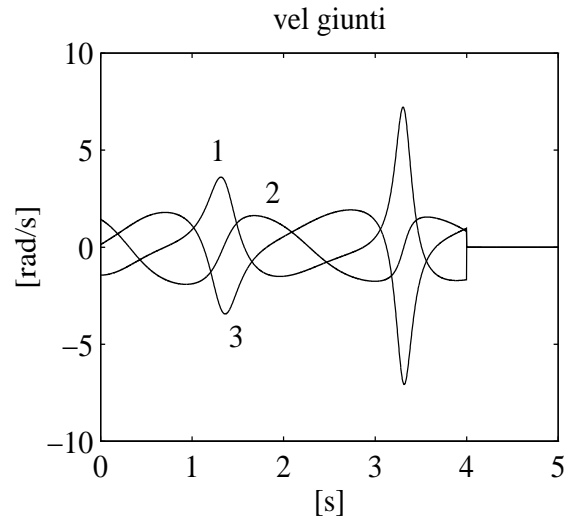
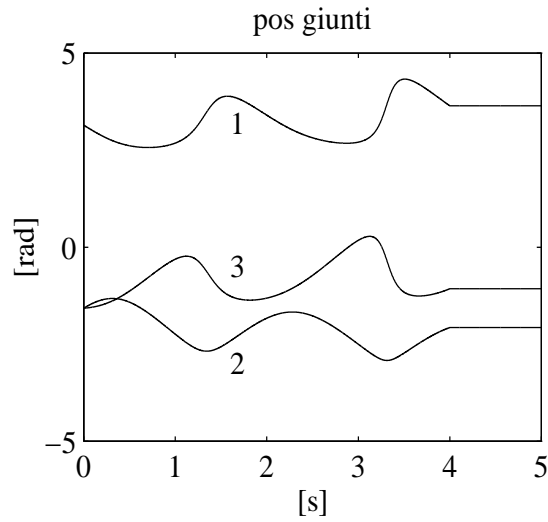
e $\Delta t = 1 \text{ ms}$

$$\bullet \dot{\mathbf{q}} = \mathbf{J}_A^{-1}(\mathbf{q}) \dot{\mathbf{x}}_e$$



$$\bullet \dot{\mathbf{q}} = \mathbf{J}_A^{-1}(\mathbf{q})(\dot{\mathbf{x}}_d + \mathbf{K}\mathbf{e})$$

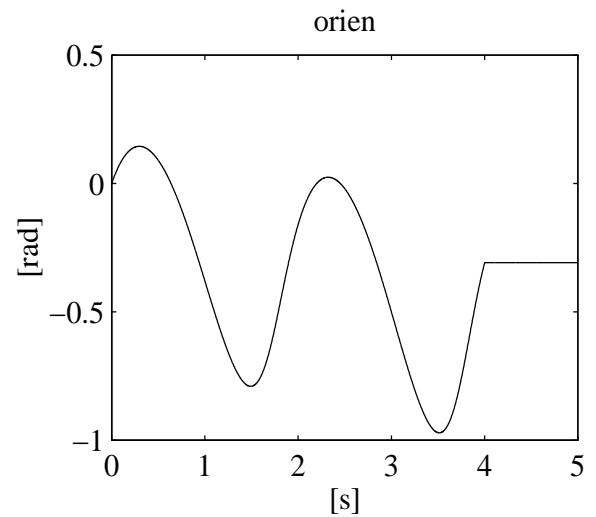
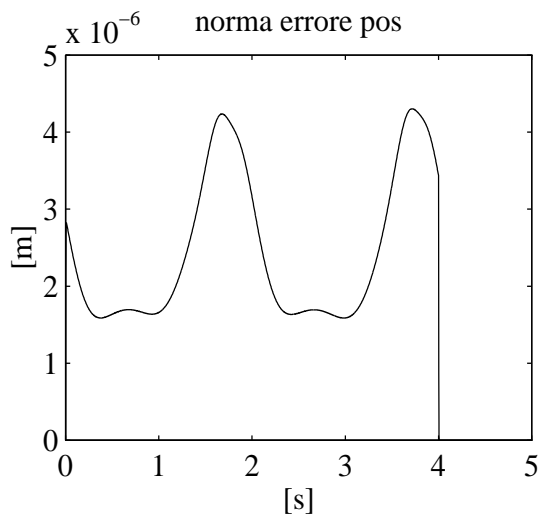
$$\mathbf{K} = \text{diag}\{500, 500, 100\}$$



- ϕ libero ($r = 2, n = 3$)

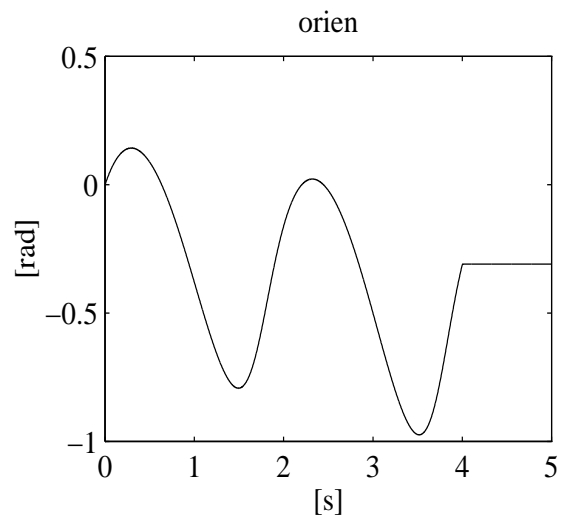
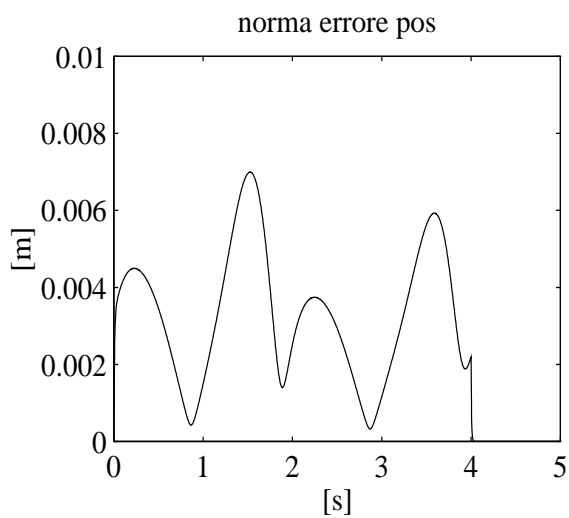
- $\dot{\mathbf{q}} = \mathbf{J}_P^\dagger(\dot{\mathbf{p}}_d + \mathbf{K}_P \mathbf{e}_P)$

$$\mathbf{K}_P = \text{diag}\{500, 500\}$$



- $\dot{\mathbf{q}} = \mathbf{J}_P^T(\mathbf{q})\mathbf{K}_P \mathbf{e}_P$

$$\mathbf{K}_P = \text{diag}\{500, 500\}$$



$$\bullet \dot{\mathbf{q}} = \mathbf{J}_P^\dagger(\dot{\mathbf{p}}_d + \mathbf{K}_P \mathbf{e}_P) + (\mathbf{I} - \mathbf{J}_P^\dagger \mathbf{J}_P) \dot{\mathbf{q}}_0$$

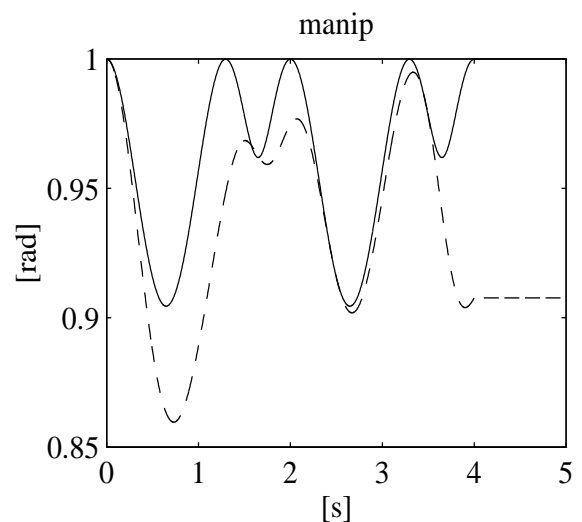
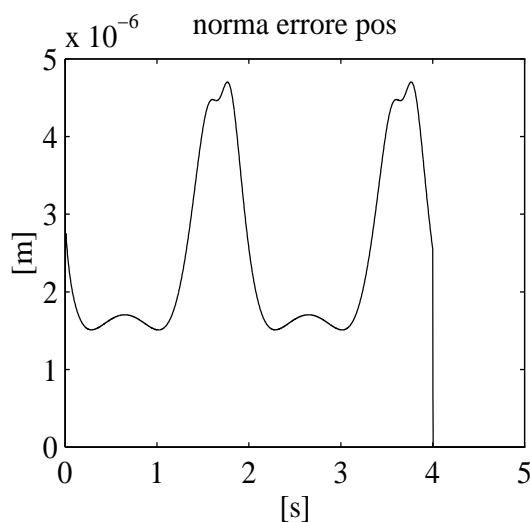
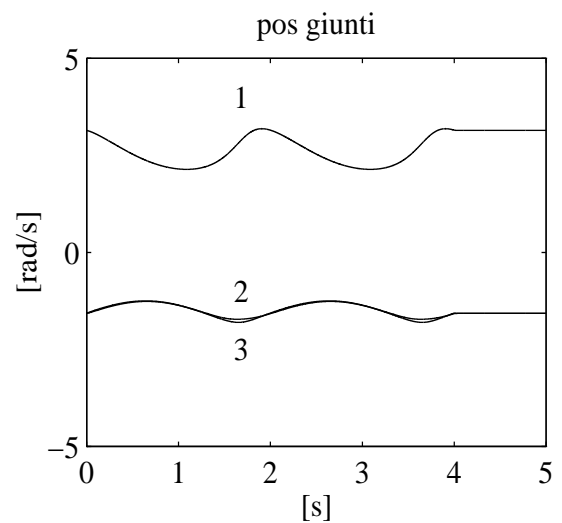
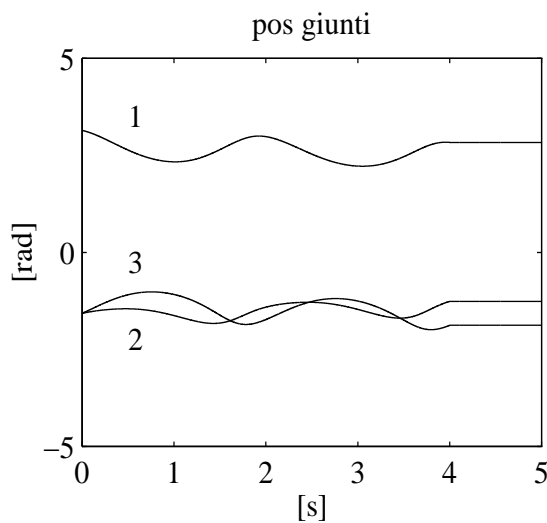
$$\mathbf{K}_P = \text{diag}\{500, 500\}$$

★ misura di manipolabilità

$$w(\vartheta_2, \vartheta_3) = \frac{1}{2}(s_2^2 + s_3^2)$$

$$\star \dot{\mathbf{q}}_0 = k_0 \left(\frac{\partial w(\mathbf{q})}{\partial \mathbf{q}} \right)^T$$

$$k_0 = 50$$

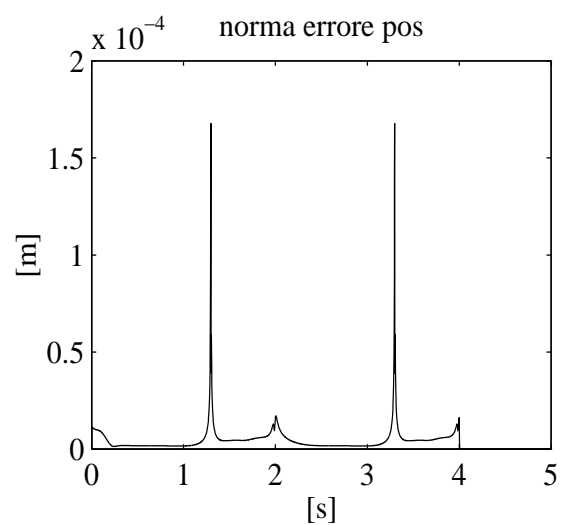
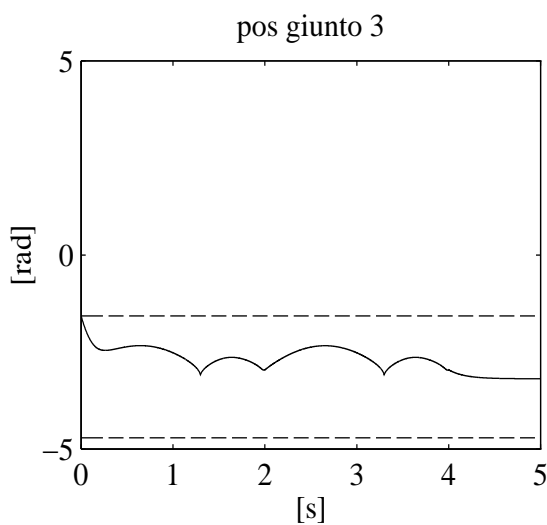
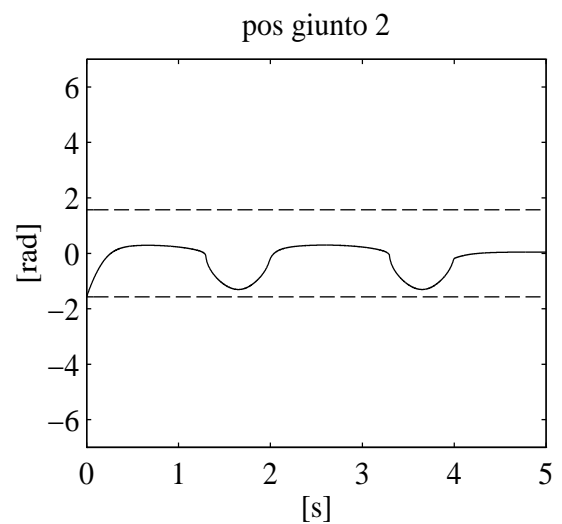
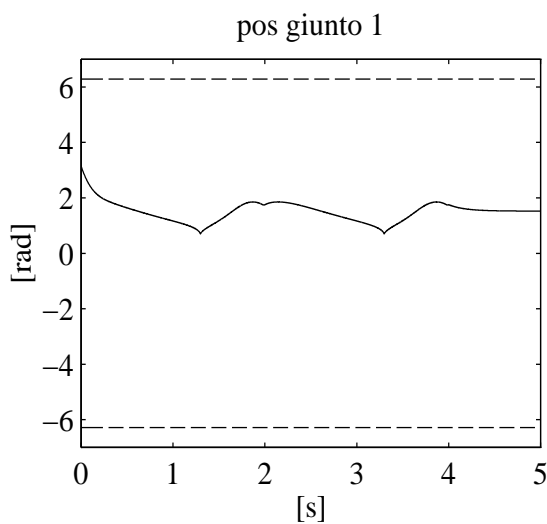


★ distanza dai fine-corsa dei giunti

$$w(\mathbf{q}) = -\frac{1}{6} \sum_{i=1}^3 \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right)^2$$

$$-2\pi \leq q_1 \leq 2\pi \quad -\pi/2 \leq q_2 \leq \pi/2 \quad -3\pi/2 \leq q_3 \leq -\pi/2$$

$$\star \dot{\mathbf{q}}_0 = k_0 \left(\frac{\partial w(\mathbf{q})}{\partial \mathbf{q}} \right)^T \quad k_0 = 250$$



STATICA

- Relazione tra forze e momenti (*forze*) γ_e all'organo terminale e forze e/o coppie (*coppie*) τ ai giunti con il manipolatore in configurazione di equilibrio

- ★ lavoro elementare compiuto dalle coppie

$$dW_\tau = \boldsymbol{\tau}^T d\mathbf{q}$$

- ★ lavoro elementare compiuto dalle forze

$$\begin{aligned}dW_\gamma &= \mathbf{f}_e^T d\mathbf{p}_e + \boldsymbol{\mu}_e^T \boldsymbol{\omega}_e dt \\ &= \mathbf{f}_e^T \mathbf{J}_P(\mathbf{q}) d\mathbf{q} + \boldsymbol{\mu}_e^T \mathbf{J}_O(\mathbf{q}) d\mathbf{q} \\ &= \boldsymbol{\gamma}_e^T \mathbf{J}(\mathbf{q}) d\mathbf{q}\end{aligned}$$

- ★ spostamenti elementari \equiv spostamenti virtuali

$$\begin{aligned}\delta W_\tau &= \boldsymbol{\tau}^T \delta \mathbf{q} \\ \delta W_\gamma &= \boldsymbol{\gamma}_e^T \mathbf{J}(\mathbf{q}) \delta \mathbf{q}\end{aligned}$$

- Principio dei lavori virtuali

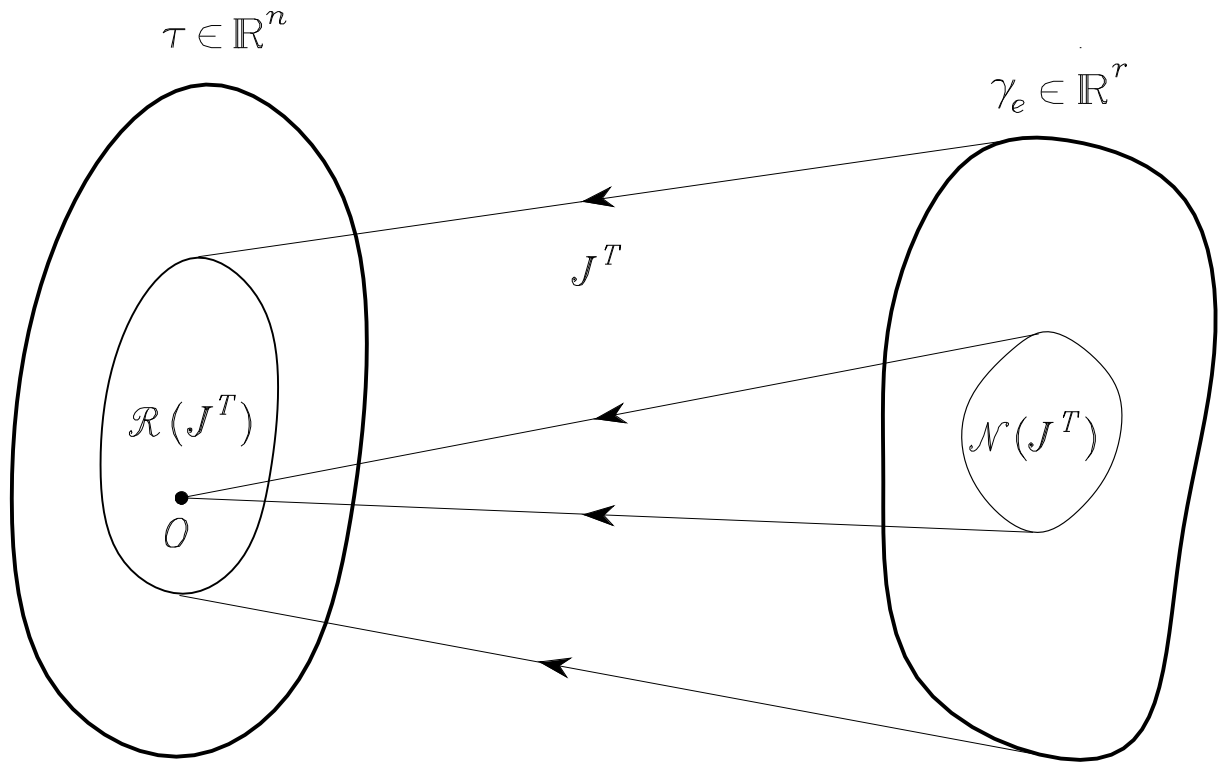
★ il manipolatore è in *equilibrio statico* se e solo se

$$\delta W_{\tau} = \delta W_{\gamma} \quad \forall \delta \mathbf{q}$$

⇓

$$\boldsymbol{\tau} = \mathbf{J}^T(\mathbf{q})\boldsymbol{\gamma}_e$$

Dualità cineto-statica

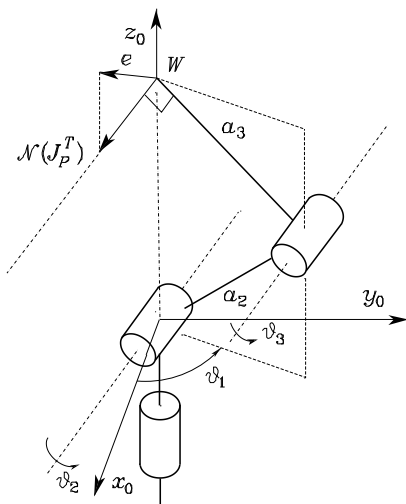


$$\mathcal{N}(\mathbf{J}) \equiv \mathcal{R}^\perp(\mathbf{J}^T)$$

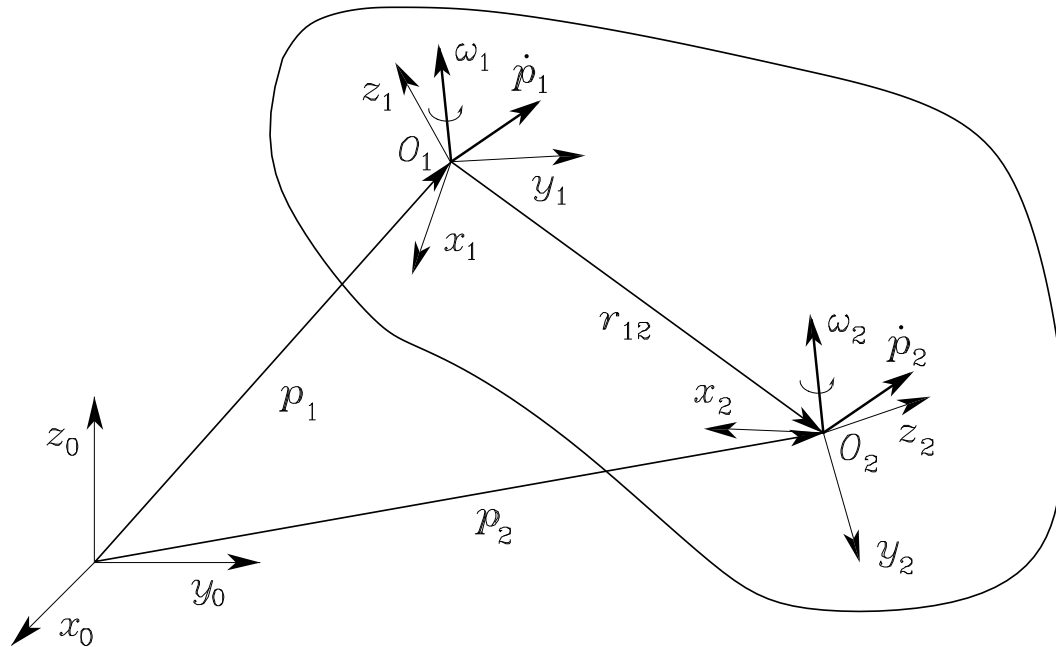
$$\mathcal{R}(\mathbf{J}) \equiv \mathcal{N}^\perp(\mathbf{J}^T)$$

- forze $\gamma_e \in \mathcal{N}(\mathbf{J}^T)$ interamente assorbite dalla struttura meccanica

- Interpretazione fisica dello schema con la trasposta dello Jacobiano
 - ★ dinamica ideale $\tau = \dot{q}$
 - ★ forza elastica $\mathbf{K}e$ che tira l'organo terminale verso la posa desiderata nello spazio operativo
 - ★ ha effetto solo se $\mathbf{K}e \notin \mathcal{N}(\mathbf{J}^T)$



Trasformazione di velocità e forze



$$\begin{bmatrix} \dot{p}_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{S}(r_{12}) \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \omega_1 \end{bmatrix}$$

$$r_{12} = \mathbf{R}_1 r_{12}^1$$

$$\begin{aligned} \dot{p}_1 &= \mathbf{R}_1 \dot{p}_1^1 & \dot{p}_2 &= \mathbf{R}_2 \dot{p}_2^2 = \mathbf{R}_1 \mathbf{R}_2^1 \dot{p}_2^2 \\ \omega_1 &= \mathbf{R}_1 \omega_1^1 & \omega_2 &= \mathbf{R}_2 \omega_2^2 = \mathbf{R}_1 \mathbf{R}_2^1 \omega_2^2 \end{aligned}$$

$$\begin{bmatrix} \dot{\mathbf{p}}_2^2 \\ \boldsymbol{\omega}_2^2 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^2 & -\mathbf{R}_1^2 \mathbf{S}(\mathbf{r}_{12}^1) \\ \mathbf{O} & \mathbf{R}_1^2 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1^1 \\ \boldsymbol{\omega}_1^1 \end{bmatrix}$$

$$\mathbf{v}_2^2 = \mathbf{J}_1^2 \mathbf{v}_1^1$$

★ in virtù della dualità cineto-statica:

$$\boldsymbol{\gamma}_1^1 = \mathbf{J}_1^{2T} \boldsymbol{\gamma}_2^2$$

$$\begin{bmatrix} \mathbf{f}_1^1 \\ \boldsymbol{\mu}_1^1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_2^1 & \mathbf{O} \\ \mathbf{S}(\mathbf{r}_{12}^1) \mathbf{R}_2^1 & \mathbf{R}_2^1 \end{bmatrix} \begin{bmatrix} \mathbf{f}_2^2 \\ \boldsymbol{\mu}_2^2 \end{bmatrix}$$

Catena chiusa

- Struttura ad albero equivalente a catena aperta

$$\mathbf{q}_o = \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_u \end{bmatrix}$$

- Risolvendo i vincoli: $\mathbf{q}_u = \mathbf{q}_u(\mathbf{q}_a)$

$$\dot{\mathbf{q}}_o = \boldsymbol{\Upsilon} \dot{\mathbf{q}}_a$$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \mathbf{I} \\ \frac{\partial \mathbf{q}_u}{\partial \mathbf{q}_a} \end{bmatrix}$$

★ per la dualità cineto-statica

$$\boldsymbol{\tau}_a = \boldsymbol{\Upsilon}^T \boldsymbol{\tau}_o$$

- Manipolatore a parallelogramma

$$\mathbf{q}_a = \begin{bmatrix} \vartheta_{1'} \\ \vartheta_{1''} \end{bmatrix} \quad \mathbf{q}_u = \begin{bmatrix} \vartheta_{2'} \\ \vartheta_{3'} \end{bmatrix}$$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{\tau}_a = \begin{bmatrix} \tau_{1'} - \tau_{2'} + \tau_{3'} \\ \tau_{1''} + \tau_{2'} - \tau_{3'} \end{bmatrix} \quad \boldsymbol{\tau}_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ELLISSOIDI DI MANIPOLABILITÀ

- *Ellissoide di manipolabilità in velocità*

- ★ insieme delle velocità ai giunti a norma costante

$$\dot{\mathbf{q}}^T \dot{\mathbf{q}} = 1$$

- ★ manipolatore ridondante

$$\dot{\mathbf{q}} = \mathbf{J}^\dagger(\mathbf{q}) \mathbf{v}_e$$

⇓

$$\mathbf{v}_e^T (\mathbf{J}(\mathbf{q}) \mathbf{J}^T(\mathbf{q}))^{-1} \mathbf{v}_e = 1$$

- Assi

- ★ autovettori \mathbf{u}_i di $\mathbf{J} \mathbf{J}^T \implies$ direzioni

- ★ valori singolari $\sigma_i = \sqrt{\lambda_i(\mathbf{J} \mathbf{J}^T)} \implies$ dimensioni

- Volume

- ★ proporzionale a

$$w(\mathbf{q}) = \sqrt{\det(\mathbf{J}(\mathbf{q}) \mathbf{J}^T(\mathbf{q}))}$$

Manipolatore planare a due bracci

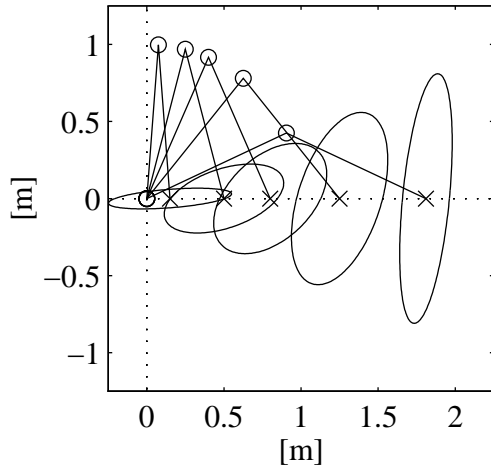
- Misura di manipolabilità

$$w = |\det(\mathbf{J})| = a_1 a_2 |s_2|$$

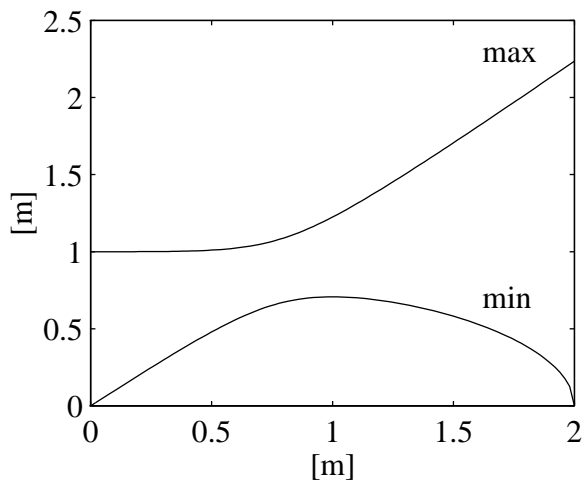
★ max per $\vartheta_2 = \pm\pi/2$

★ max per $a_1 = a_2$ (a parità di estensione $a_1 + a_2$)

- Ellissi di manipolabilità in velocità



- Valori singolari



- *Ellissoide di manipolabilità in forza*

- ★ insieme delle coppie ai giunti a norma costante

$$\boldsymbol{\tau}^T \boldsymbol{\tau} = 1$$

⇓

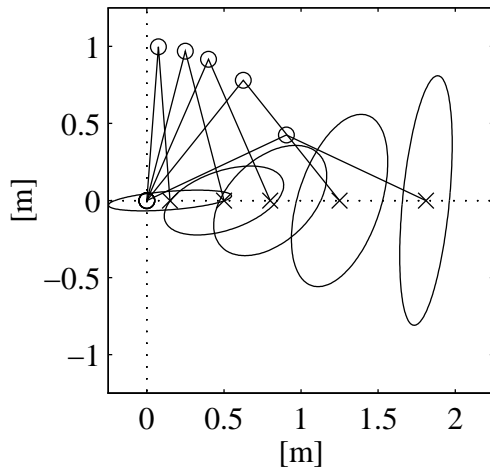
$$\boldsymbol{\gamma}_e^T (\mathbf{J}(\mathbf{q})\mathbf{J}^T(\mathbf{q}))\boldsymbol{\gamma}_e = 1$$

- Dualità cineto-statica

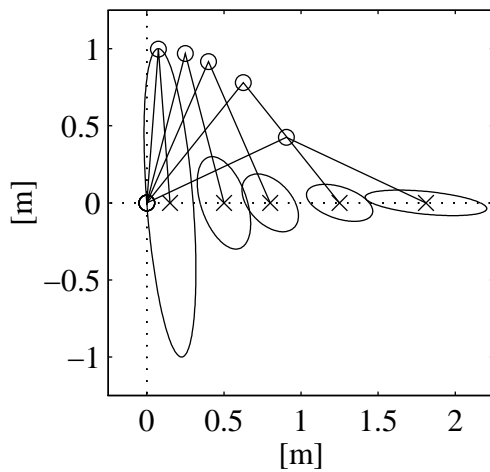
- ★ una direzione lungo la quale si ha elevata manipolabilità in velocità è una direzione lungo la quale si ha scarsa manipolabilità in forza, e viceversa

Manipolatore planare a due bracci

- Ellissi di manipolabilità in velocità



- Ellissi di manipolabilità in forza



- Manipolatore \equiv *trasformatore meccanico* di velocità e forze dallo spazio dei giunti allo spazio operativo
 - ★ rapporto di trasformazione lungo una direzione per l'ellissoide in forza

$$\alpha(\mathbf{q}) = \left(\mathbf{u}^T \mathbf{J}(\mathbf{q}) \mathbf{J}^T(\mathbf{q}) \mathbf{u} \right)^{-1/2}$$

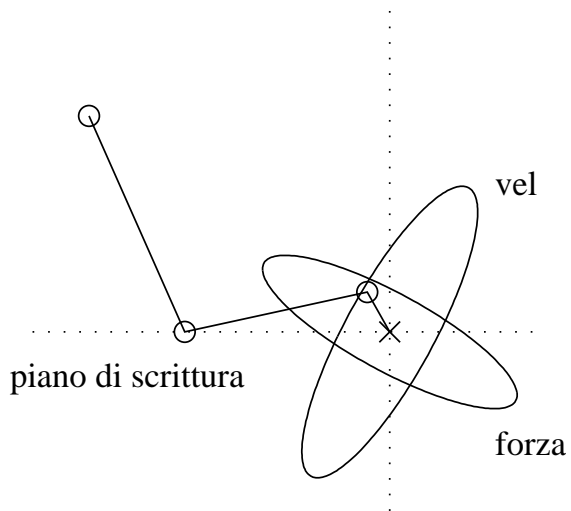
- ★ rapporto di trasformazione lungo una direzione per l'ellissoide in velocità

$$\beta(\mathbf{q}) = \left(\mathbf{u}^T (\mathbf{J}(\mathbf{q}) \mathbf{J}^T(\mathbf{q}))^{-1} \mathbf{u} \right)^{-1/2}$$

- ★ utilizzazione di gradi di mobilità ridondanti

- Compatibilità della struttura ad eseguire un compito assegnato lungo una direzione

★ compito di scrittura su superficie orizzontale



★ compito di lancio di un peso in direzione orizzontale

