DINAMICA

- equazioni del moto del manipolatore in funzione delle forze e momenti agenti su di esso

Formulazione di Lagrange

Proprietà notevoli del modello dinamico

Identificazione dei parametri dinamici

Formulazione di Newton–Eulero

Dinamica diretta e dinamica inversa

Modello dinamico nello spazio operativo

Ellissoide di manipolabilità dinamica

Scalatura dinamica di traiettorie

FORMULAZIONE DI LAGRANGE

- Lagrangiana $\equiv$ (energia cinetica) $-$ (energia potenziale)

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

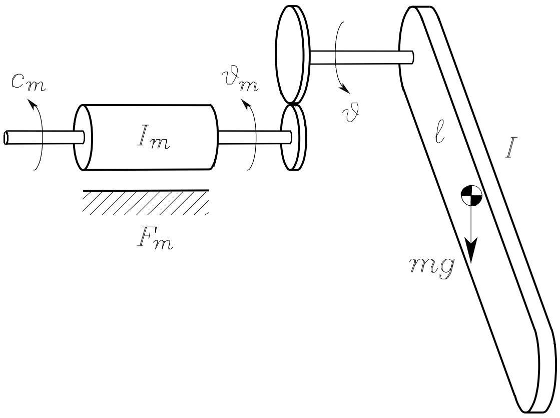
- Equazioni di Lagrange

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\lambda}_i} - \frac{\partial \mathcal{L}}{\partial \lambda_i} = \xi_i \qquad i = 1, \ldots, n$$

$\star$ *coordinate generalizzate*

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \boldsymbol{q}$$

- Esempio



⋆ energia cinetica

$$\mathcal{T} = \frac{1}{2} I \dot{\vartheta}^2 + \frac{1}{2} I_m k_r^2 \dot{\vartheta}^2$$

⋆ energia potenziale

$$\mathcal{U} = mg\ell(1 - \cos\vartheta)$$

⋆ lagrangiana

$$\mathcal{L} = \frac{1}{2} I \dot{\vartheta}^2 + \frac{1}{2} I_m k_r^2 \dot{\vartheta}^2 - mg\ell(1 - \cos\vartheta)$$

⋆ equazione del moto

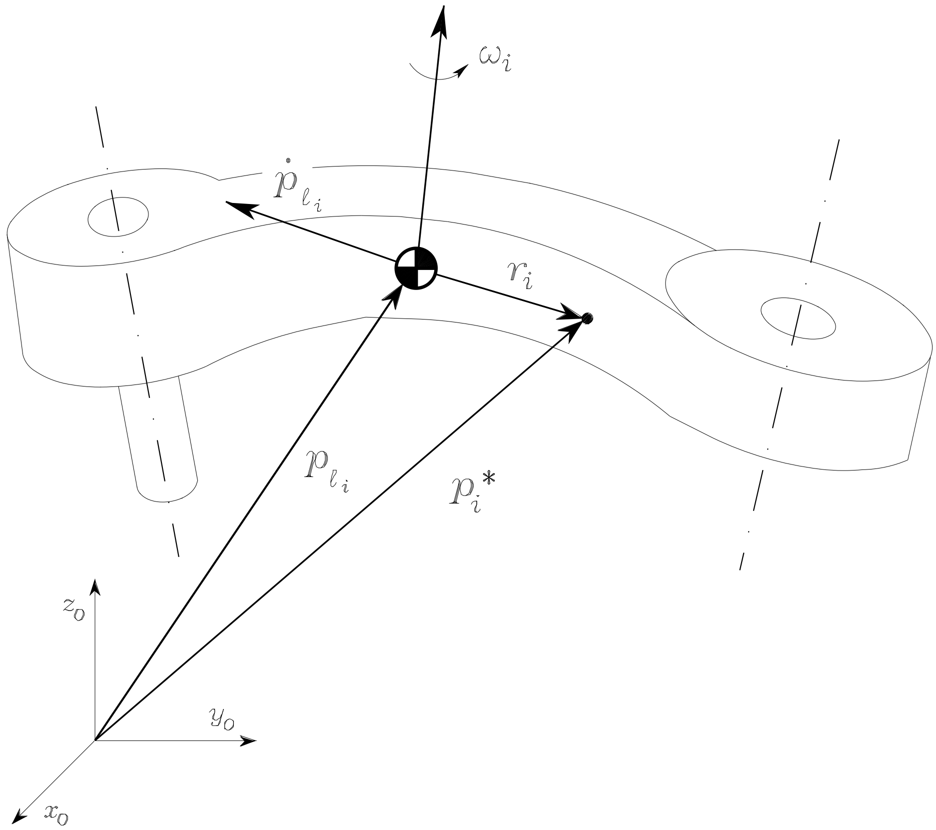
$$\begin{aligned}(I + I_m k_r^2)\ddot{\vartheta} + mg\ell \sin\vartheta &= \xi \\ &= \tau - F\dot{\vartheta} - F_m k_r \dot{\vartheta}\end{aligned}$$

Energia cinetica

- Contributo bracci + attuatori

$$\mathcal{T} = \sum_{i=1}^{n} (\mathcal{T}_{\ell_i} + \mathcal{T}_{m_i})$$

- Braccio i



$$\mathcal{T}_{\ell_i} = \frac{1}{2}\int_{V_{\ell_i}} \dot{\boldsymbol{p}}_i^{*T}\dot{\boldsymbol{p}}_i^* \rho dV$$

⋆ baricentro

$$\boldsymbol{p}_{\ell_i} = \frac{1}{m_{\ell_i}}\int_{V_{\ell_i}} \boldsymbol{p}_i^* \rho dV$$

⋆ velocità lineare della particella elementare

$$\dot{\boldsymbol{p}}_i^* = \dot{\boldsymbol{p}}_{\ell_i} + \boldsymbol{\omega}_i \times \boldsymbol{r}_i = \dot{\boldsymbol{p}}_{\ell_i} + \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i$$

- Traslazionale

$$\frac{1}{2}\int_{V_{\ell_i}} \dot{\boldsymbol{p}}_{\ell_i}^T \dot{\boldsymbol{p}}_{\ell_i} \rho dV = \frac{1}{2} m_{\ell_i} \dot{\boldsymbol{p}}_{\ell_i}^T \dot{\boldsymbol{p}}_{\ell_i}$$

- Mutuo

$$2\left(\frac{1}{2}\int_{V_{\ell_i}} \dot{\boldsymbol{p}}_{\ell_i}^T \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i \rho dV\right) =$$

$$2\left(\frac{1}{2}\dot{\boldsymbol{p}}_{\ell_i}^T \boldsymbol{S}(\boldsymbol{\omega}_i)\int_{V_{\ell_i}} (\boldsymbol{p}_i^* - \boldsymbol{p}_{\ell_i})\rho dV\right) = 0$$

- Rotazionale

$$\frac{1}{2}\int_{V_{\ell_i}} \boldsymbol{r}_i^T \boldsymbol{S}^T(\boldsymbol{\omega}_i)\boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i \rho dV = \frac{1}{2}\boldsymbol{\omega}_i^T \left(\int_{V_{\ell_i}} \boldsymbol{S}^T(\boldsymbol{r}_i)\boldsymbol{S}(\boldsymbol{r}_i)\rho dV\right)\boldsymbol{\omega}_i$$

$$= \frac{1}{2}\boldsymbol{\omega}_i^T \boldsymbol{I}_{\ell_i}\boldsymbol{\omega}_i$$

⋆ *tensore di inerzia*

$$\boldsymbol{I}_{\ell_i} = \begin{bmatrix} \int(r_{iy}^2 + r_{iz}^2)\rho dV & -\int r_{ix}r_{iy}\rho dV & -\int r_{ix}r_{iz}\rho dV \\ * & \int(r_{ix}^2 + r_{iz}^2)\rho dV & -\int r_{iy}r_{iz}\rho dV \\ * & * & \int(r_{ix}^2 + r_{iy}^2)\rho dV \end{bmatrix}$$

$$= \begin{bmatrix} I_{\ell_i xx} & -I_{\ell_i xy} & -I_{\ell_i xz} \\ * & I_{\ell_i yy} & -I_{\ell_i yz} \\ * & * & I_{\ell_i zz} \end{bmatrix}$$

- Energia cinetica del braccio i

$$\mathcal{T}_{\ell_i} = \frac{1}{2} m_{\ell_i} \dot{\boldsymbol{p}}_{\ell_i}^T \dot{\boldsymbol{p}}_{\ell_i} + \frac{1}{2} \boldsymbol{\omega}_i^T \boldsymbol{R}_i \boldsymbol{I}_{\ell_i}^i \boldsymbol{R}_i^T \boldsymbol{\omega}_i$$
$$= \frac{1}{2} m_{\ell_i} \dot{\boldsymbol{q}}^T \boldsymbol{J}_P^{(\ell_i)T} \boldsymbol{J}_P^{(\ell_i)} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{J}_O^{(\ell_i)T} \boldsymbol{R}_i \boldsymbol{I}_{\ell_i}^i \boldsymbol{R}_i^T \boldsymbol{J}_O^{(\ell_i)} \dot{\boldsymbol{q}}$$

$\star$ velocità lineare

$$\dot{\boldsymbol{p}}_{\ell_i} = \boldsymbol{j}_{P1}^{(\ell_i)} \dot{q}_1 + \ldots + \boldsymbol{j}_{Pi}^{(\ell_i)} \dot{q}_i = \boldsymbol{J}_P^{(\ell_i)} \dot{\boldsymbol{q}}$$

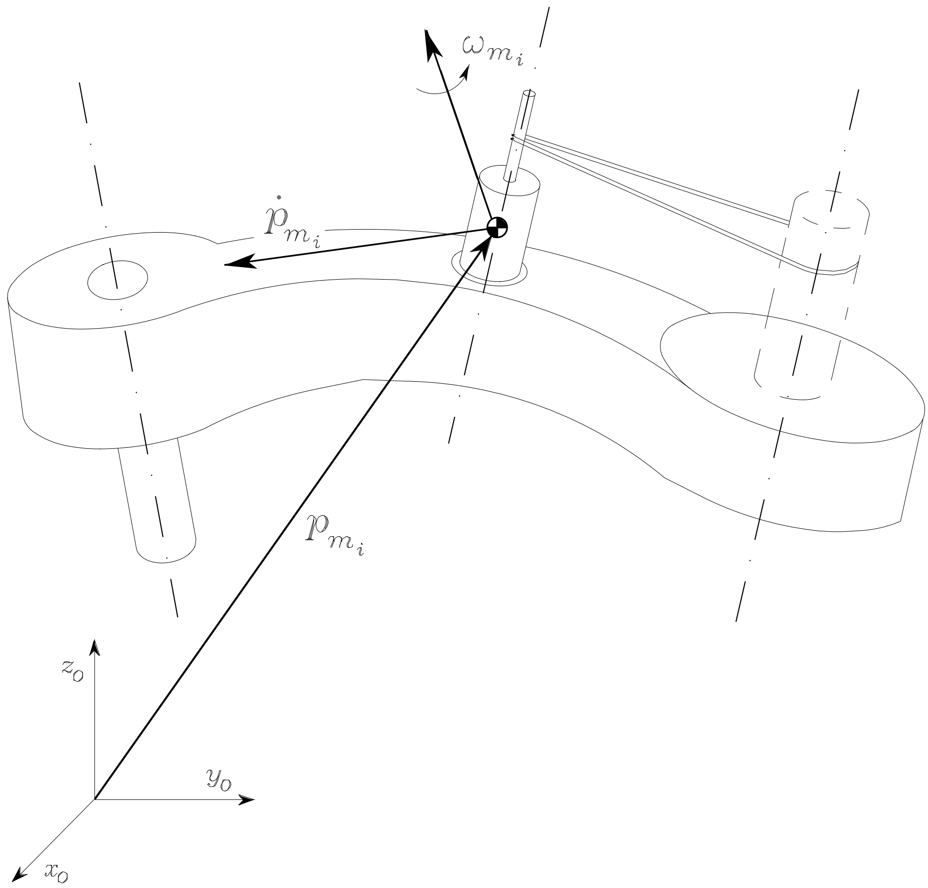
$$\boldsymbol{j}_{Pj}^{(\ell_i)} = \begin{cases} \boldsymbol{z}_{j-1} & \text{per un giunto } \textit{prismatico} \\ \boldsymbol{z}_{j-1} \times (\boldsymbol{p}_{\ell_i} - \boldsymbol{p}_{j-1}) & \text{per un giunto } \textit{rotoidale} \end{cases}$$

$\star$ velocità angolare

$$\boldsymbol{\omega}_i = \boldsymbol{j}_{O1}^{(\ell_i)} \dot{q}_1 + \ldots + \boldsymbol{j}_{Oi}^{(\ell_i)} \dot{q}_i = \boldsymbol{J}_O^{(\ell_i)} \dot{\boldsymbol{q}}$$

$$\boldsymbol{j}_{Oj}^{(\ell_i)} = \begin{cases} \boldsymbol{0} & \text{per un giunto } \textit{prismatico} \\ \boldsymbol{z}_{j-1} & \text{per un giunto } \textit{rotoidale} \end{cases}$$

- Motore i (elettrico rotante)



⋆ trasmissione rigida

$$\boldsymbol{\omega}_{m_i} = \boldsymbol{\omega}_{i-1} + k_{ri}\dot{q}_i \boldsymbol{z}_{m_i}$$

- Energia cinetica del rotore i

$$
\begin{aligned}
\mathcal{T}_{m_i} &= \frac{1}{2} m_{m_i} \dot{\boldsymbol{p}}_{m_i}^T \dot{\boldsymbol{p}}_{m_i} + \frac{1}{2} \boldsymbol{\omega}_{m_i}^T \boldsymbol{I}_{m_i} \boldsymbol{\omega}_{m_i} \\
&= \frac{1}{2} m_{m_i} \dot{\boldsymbol{q}}^T \boldsymbol{J}_P^{(m_i)T} \boldsymbol{J}_P^{(m_i)} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{J}_O^{(m_i)T} \boldsymbol{R}_{m_i} \boldsymbol{I}_{m_i}^{m_i} \boldsymbol{R}_{m_i}^T \boldsymbol{J}_O^{(m_i)} \dot{\boldsymbol{q}}
\end{aligned}
$$

⋆ velocità lineare

$$
\boldsymbol{J}_P^{(m_i)} = \begin{bmatrix} \boldsymbol{\jmath}_{P1}^{(m_i)} & \dots & \boldsymbol{\jmath}_{P,i-1}^{(m_i)} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}
$$

$$
\boldsymbol{\jmath}_{Pj}^{(m_i)} = \begin{cases} \boldsymbol{z}_{j-1} & \text{per un giunto } \textit{prismatico} \\ \boldsymbol{z}_{j-1} \times (\boldsymbol{p}_{m_i} - \boldsymbol{p}_{j-1}) & \text{per un giunto } \textit{rotoidale} \end{cases}
$$

⋆ velocità angolare

$$
\boldsymbol{J}_O^{(m_i)} = \begin{bmatrix} \boldsymbol{\jmath}_{O1}^{(m_i)} & \dots & \boldsymbol{\jmath}_{O,i-1}^{(m_i)} & \boldsymbol{\jmath}_{Oi}^{(m_i)} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}
$$

$$
\boldsymbol{\jmath}_{Oj}^{(m_i)} = \begin{cases} \boldsymbol{\jmath}_{Oj}^{(\ell_i)} & j = 1, \dots, i-1 \\ k_{ri} \boldsymbol{z}_{m_i} & j = i \end{cases}
$$

- Energia cinetica totale

$$\mathcal{T} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- *Matrice di inerzia*

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{q}) = \sum_{i=1}^{n}\Big(& m_{\ell_i}\boldsymbol{J}_P^{(\ell_i)T}\boldsymbol{J}_P^{(\ell_i)} + \boldsymbol{J}_O^{(\ell_i)T}\boldsymbol{R}_i\boldsymbol{I}_{\ell_i}^{i}\boldsymbol{R}_i^T\boldsymbol{J}_O^{(\ell_i)} \\ & + m_{m_i}\boldsymbol{J}_P^{(m_i)T}\boldsymbol{J}_P^{(m_i)} + \boldsymbol{J}_O^{(m_i)T}\boldsymbol{R}_{m_i}\boldsymbol{I}_{m_i}^{m_i}\boldsymbol{R}_{m_i}^T\boldsymbol{J}_O^{(m_i)}\Big)\end{aligned}$$

  - ⋆ *simmetrica*
  - ⋆ *definita positiva*
  - ⋆ *dipendente dalla configurazione* (in generale)

Energia potenziale

$$\mathcal{U} = \sum_{i=1}^{n} (\mathcal{U}_{\ell_i} + \mathcal{U}_{m_i})$$

- Braccio i

$$\mathcal{U}_{\ell_i} = -\int_{V_{\ell_i}} \boldsymbol{g}_0^T \boldsymbol{p}_i^* \rho dV = -m_{\ell_i} \boldsymbol{g}_0^T \boldsymbol{p}_{\ell_i}$$

- Rotore i

$$\mathcal{U}_{m_i} = -m_{m_i} \boldsymbol{g}_0^T \boldsymbol{p}_{m_i}$$

- Energia potenziale totale

$$\mathcal{U} = -\sum_{i=1}^{n} (m_{\ell_i} \boldsymbol{g}_0^T \boldsymbol{p}_{\ell_i} + m_{m_i} \boldsymbol{g}_0^T \boldsymbol{p}_{m_i})$$

Equazioni del moto

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \mathcal{T}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j + \sum_{i=1}^{n}\left(m_{\ell_i}\boldsymbol{g}_0^T\boldsymbol{p}_{\ell_i}(\boldsymbol{q}) + m_{m_i}\boldsymbol{g}_0^T\boldsymbol{p}_{m_i}(\boldsymbol{q})\right)
\end{aligned}
$$

- Equazioni di Lagrange

$$
\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\xi}
$$

$$
\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T + \left(\frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T
$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) &= \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^{n} \frac{db_{ij}(\boldsymbol{q})}{dt}\dot{q}_j \\ &= \sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{ij}(\boldsymbol{q})}{\partial q_k}\dot{q}_k\dot{q}_j\end{aligned}$$

$$\frac{\partial \mathcal{T}}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{jk}(\boldsymbol{q})}{\partial q_i}\dot{q}_k\dot{q}_j$$

$$\begin{aligned}\frac{\partial \mathcal{U}}{\partial q_i} &= -\sum_{j=1}^{n}\left(m_{\ell_j}\boldsymbol{g}_0^T\frac{\partial \boldsymbol{p}_{\ell_j}}{\partial q_i} + m_{m_j}\boldsymbol{g}_0^T\frac{\partial \boldsymbol{p}_{m_j}}{\partial q_i}\right) \\ &= -\sum_{j=1}^{n}\left(m_{\ell_j}\boldsymbol{g}_0^T\boldsymbol{\jmath}_{Pi}^{(\ell_j)}(\boldsymbol{q}) + m_{m_j}\boldsymbol{g}_0^T\boldsymbol{\jmath}_{Pi}^{(m_j)}(\boldsymbol{q})\right) \\ &= g_i(\boldsymbol{q})\end{aligned}$$

- Equazioni del moto

$$\sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(\boldsymbol{q})\dot{q}_k\dot{q}_j + g_i(\boldsymbol{q}) = \xi_i \qquad i = 1,\dots,n$$

ove

$$h_{ijk} = \frac{\partial b_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial b_{jk}}{\partial q_i}$$

⋆ *termini in accelerazione*

il coefficiente b_{ii} rappresenta il momento di inerzia visto all'asse del giunto i, nella configurazione corrente del manipolatore, quando gli altri giunti sono bloccati

il coefficiente b_{ij} tiene conto dell'effetto dell'accelerazione del giunto j sul giunto i

⋆ *termini quadratici in velocità*

il termine $h_{ijj}\dot{q}_j^2$ rappresenta l'effetto *centrifugo* indotto al giunto i dalla velocità del giunto j; si noti che $h_{iii} = 0$, poiché $\partial b_{ii}/\partial q_i = 0$

il termine $h_{ijk}\dot{q}_j\dot{q}_k$ rappresenta l'effetto di *Coriolis* indotto al giunto i dalle velocità dei giunti j e k

⋆ *termini dipendenti solo dalla configurazione*

il termine g_i rappresenta la coppia generata all'asse del giunto i nella configurazione corrente del manipolatore per effetto della gravità

- Forze non conservative
  - ⋆ coppie di attuazione $\boldsymbol{\tau}$
  - ⋆ coppie di attrito viscoso $-\boldsymbol{F}_v \dot{\boldsymbol{q}}$
  - ⋆ coppie di attrito statico $-\boldsymbol{f}_s(\boldsymbol{q}, \dot{\boldsymbol{q}}) \approx -\boldsymbol{F}_s \mathbf{sgn}\,(\dot{\boldsymbol{q}})$
  - ⋆ coppie di bilanciamento di forze di contatto $-\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}$

- Modello dinamico nello spazio dei giunti

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}_v \dot{\boldsymbol{q}} + \boldsymbol{f}_s(\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$$

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_k\dot{q}_j$$

PROPRIETÀ NOTEVOLI DEL MODELLO DINAMICO

Anti-simmetria della matrice $\dot{\boldsymbol{B}} - 2\boldsymbol{C}$

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_k\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial b_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j$$

- Elementi di $\boldsymbol{C}$

$$c_{ij} = \sum_{k=1}^{n} c_{ijk}\dot{q}_k$$

  ⋆ *simboli di Christoffel del primo tipo*

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$$

- Proprietà notevole

$$\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\boldsymbol{N}^T(\boldsymbol{q},\dot{\boldsymbol{q}})$$

- Elementi di $\boldsymbol{C}$

$$
\begin{aligned}
c_{ij} &= \frac{1}{2}\sum_{k=1}^{n}\frac{\partial b_{ij}}{\partial q_k}\dot{q}_k + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k \\
&= \frac{1}{2}\dot{b}_{ij} + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k
\end{aligned}
$$

- Elementi di $\boldsymbol{N}$

$$
n_{ij} = \dot{b}_{ij} - 2c_{ij} = \sum_{k=1}^{n}\left(\frac{\partial b_{jk}}{\partial q_i} - \frac{\partial b_{ik}}{\partial q_j}\right)\dot{q}_k = -n_{ji}
$$

$$\Downarrow$$

$$
\boldsymbol{w}^T\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{w} = 0 \qquad\qquad \forall\boldsymbol{w}
$$

$\star$ se $\boldsymbol{w} = \dot{\boldsymbol{q}}$:

$$
\dot{\boldsymbol{q}}^T\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{0} \qquad\qquad \forall\boldsymbol{C}
$$

- Principio di conservazione dell'energia (*Hamilton*)

$$\frac{1}{2}\frac{d}{dt}\left(\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau} - \boldsymbol{F}_v\dot{\boldsymbol{q}} - \boldsymbol{f}_s(\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e\right)$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) &= \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} \\ &\quad + \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau} - \boldsymbol{F}_v\dot{\boldsymbol{q}} - \boldsymbol{f}_s(\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e\right)\end{aligned}$$

Linearità nei parametri dinamici

- Insieme braccio i e rotore $i+1$

$$\mathcal{T}_i = \mathcal{T}_{\ell_i} + \mathcal{T}_{m_{i+1}}$$

$$\mathcal{T}_{\ell_i} = \frac{1}{2} m_{\ell i} \dot{\boldsymbol{p}}_{\ell_i}^T \dot{\boldsymbol{p}}_{\ell_i} + \frac{1}{2} \boldsymbol{\omega}_i^T \boldsymbol{I}_{\ell_i} \boldsymbol{\omega}_i$$

$$\mathcal{T}_{m_{i+1}} = \frac{1}{2} m_{m_{i+1}} \dot{\boldsymbol{p}}_{m_{i+1}}^T \dot{\boldsymbol{p}}_{m_{i+1}} + \frac{1}{2} \boldsymbol{\omega}_{m_{i+1}}^T \boldsymbol{I}_{m_{i+1}} \boldsymbol{\omega}_{m_{i+1}}$$

⋆ con riferimento al baricentro $\boldsymbol{p}_{C_i}$ dell'insieme braccio–rotore

$$\dot{\boldsymbol{p}}_{\ell_i} = \dot{\boldsymbol{p}}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{C_i,\ell_i}$$

$$\dot{\boldsymbol{p}}_{m_{i+1}} = \dot{\boldsymbol{p}}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{C_i,m_{i+1}}$$

$$\boldsymbol{r}_{C_i,\ell_i} = \boldsymbol{p}_{\ell_i} - \boldsymbol{p}_{C_i}$$

$$\boldsymbol{r}_{C_i,m_{i+1}} = \boldsymbol{p}_{m_{i+1}} - \boldsymbol{p}_{C_i}$$

- Braccio

$$\begin{aligned}\mathcal{T}_{\ell_i} &= \frac{1}{2} m_{\ell i} \dot{\boldsymbol{p}}_{C_i}^T \dot{\boldsymbol{p}}_{C_i} + \dot{\boldsymbol{p}}_{C_i}^T \boldsymbol{S}(\boldsymbol{\omega}_i) m_{\ell_i} \boldsymbol{r}_{C_i,\ell_i} \\ &\quad + \frac{1}{2} m_{\ell_i} \boldsymbol{\omega}_i^T \boldsymbol{S}^T(\boldsymbol{r}_{C_i,\ell_i}) \boldsymbol{S}(\boldsymbol{r}_{C_i,\ell_i}) \boldsymbol{\omega}_i + \frac{1}{2} \boldsymbol{\omega}_i^T \boldsymbol{I}_{\ell_i} \boldsymbol{\omega}_i \\ &= \frac{1}{2} m_{\ell i} \dot{\boldsymbol{p}}_{C_i}^T \dot{\boldsymbol{p}}_{C_i} + \dot{\boldsymbol{p}}_{C_i}^T \boldsymbol{S}(\boldsymbol{\omega}_i) m_{\ell_i} \boldsymbol{r}_{C_i,\ell_i} + \frac{1}{2} \boldsymbol{\omega}_i^T \bar{\boldsymbol{I}}_{\ell_i} \boldsymbol{\omega}_i\end{aligned}$$

  - ⋆ per il teorema di Steiner

$$\bar{\boldsymbol{I}}_{\ell_i} = \boldsymbol{I}_{\ell_i} + m_{\ell_i} \boldsymbol{S}^T(\boldsymbol{r}_{C_i,\ell_i}) \boldsymbol{S}(\boldsymbol{r}_{C_i,\ell_i})$$

- Rotore

$$\begin{aligned}\mathcal{T}_{m_{i+1}} &= \frac{1}{2} m_{m_{i+1}} \dot{\boldsymbol{p}}_{C_i}^T \dot{\boldsymbol{p}}_{C_i} + \dot{\boldsymbol{p}}_{C_i}^T \boldsymbol{S}(\boldsymbol{\omega}_i) m_{m_{i+1}} \boldsymbol{r}_{C_i,m_{i+1}} + \frac{1}{2} \boldsymbol{\omega}_i^T \bar{\boldsymbol{I}}_{m_{i+1}} \boldsymbol{\omega}_i \\ &\quad + k_{r,i+1} \dot{q}_{i+1} \boldsymbol{z}_{m_{i+1}}^T \boldsymbol{I}_{m_{i+1}} \boldsymbol{\omega}_i + \frac{1}{2} k_{r,i+1}^2 \dot{q}_{i+1}^2 \boldsymbol{z}_{m_{i+1}}^T \boldsymbol{I}_{m_{i+1}} \boldsymbol{z}_{m_{i+1}}\end{aligned}$$

  - ⋆ per il teorema di Steiner

$$\bar{\boldsymbol{I}}_{m_{i+1}} = \boldsymbol{I}_{m_{i+1}} + m_{m_{i+1}} \boldsymbol{S}^T(\boldsymbol{r}_{C_i,m_{i+1}}) \boldsymbol{S}(\boldsymbol{r}_{C_i,m_{i+1}})$$

- Energia cinetica

$$\mathcal{T}_i = \frac{1}{2} m_i \dot{\boldsymbol{p}}_{C_i}^T \dot{\boldsymbol{p}}_{C_i} + \frac{1}{2} \boldsymbol{\omega}_i^T \bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i + k_{r,i+1} \dot{q}_{i+1} \boldsymbol{z}_{m_{i+1}}^T \boldsymbol{I}_{m_{i+1}} \boldsymbol{\omega}_i$$
$$+ \frac{1}{2} k_{r,i+1}^2 \dot{q}_{i+1}^2 \boldsymbol{z}_{m_{i+1}}^T \boldsymbol{I}_{m_{i+1}} \boldsymbol{z}_{m_{i+1}}$$

⋆ rotore con distribuzione di massa simmetrica intorno al suo asse di rotazione

$$\boldsymbol{I}_{m_i}^{m_i} = \begin{bmatrix} I_{m_i xx} & 0 & 0 \\ 0 & I_{m_i xx} & 0 \\ 0 & 0 & I_{m_i zz} \end{bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{I}_{m_{i+1}} \boldsymbol{z}_{m_{i+1}} = \boldsymbol{R}_{m_{i+1}} \boldsymbol{I}_{m_{i+1}}^{m_{i+1}} \boldsymbol{R}_{m_{i+1}}^T \boldsymbol{z}_{m_{i+1}} = I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}}$$

- Quantità riferite alle terne solidali ai bracci

$$\mathcal{T}_i = \frac{1}{2} m_i \dot{\boldsymbol{p}}_{C_i}^{iT} \dot{\boldsymbol{p}}_{C_i}^{i} + \frac{1}{2} \boldsymbol{\omega}_i^{iT} \bar{\boldsymbol{I}}_i^i \boldsymbol{\omega}_i^i + k_{r,i+1} \dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}}^{iT} \boldsymbol{\omega}_i^i + \frac{1}{2} k_{r,i+1}^2 \dot{q}_{i+1}^2 I_{m_{i+1}}$$

$$= \frac{1}{2} m_i \dot{\boldsymbol{p}}_i^{iT} \dot{\boldsymbol{p}}_i^i + \dot{\boldsymbol{p}}_i^{iT} \boldsymbol{S}(\boldsymbol{\omega}_i^i) m_i \boldsymbol{r}_{i,C_i}^i + \frac{1}{2} \boldsymbol{\omega}_i^{iT} \hat{\boldsymbol{I}}_i^i \boldsymbol{\omega}_i^i + k_{r,i+1} \dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}}^{iT} \boldsymbol{\omega}_i^i + \frac{1}{2} k_{r,i+1}^2 \dot{q}_{i+1}^2 I_{m_{i+1}}$$

⋆ per il teorema di Steiner

$$\hat{\boldsymbol{I}}_i^i = \bar{\boldsymbol{I}}_i^i + m_i \boldsymbol{S}^T(\boldsymbol{r}_{i,C_i}^i) \boldsymbol{S}(\boldsymbol{r}_{i,C_i}^i)$$

- Energia cinetica *lineare* rispetto a:
  - ⋆ massa m_i
  - ⋆ momento primo di inerzia

$$m_i \boldsymbol{r}^i_{i,C_i} = \begin{bmatrix} m_i \ell_{C_i x} \\ m_i \ell_{C_i y} \\ m_i \ell_{C_i z} \end{bmatrix}$$

  - ⋆ tensore di inerzia

$$\hat{\boldsymbol{I}}^i_i = \begin{bmatrix} \bar{I}_{ixx} + m_i(\ell^2_{C_i y} + \ell^2_{C_i z}) & -\bar{I}_{ixy} - m_i \ell_{C_i x} \ell_{C_i y} & -\bar{I}_{ixz} - m_i \ell_{C_i x} \ell_{C_i z} \\ * & \bar{I}_{iyy} + m_i(\ell^2_{C_i z} + \ell^2_{C_i x}) & -\bar{I}_{iyz} - m_i \ell_{C_i y} \ell_{C_i z} \\ * & * & \bar{I}_{izz} + m_i(\ell^2_{C_i x} + \ell^2_{C_i y}) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{I}_{ixx} & -\hat{I}_{ixy} & -\hat{I}_{ixz} \\ * & \hat{I}_{iyy} & \hat{I}_{iyz} \\ * & * & \hat{I}_{izz} \end{bmatrix}$$

- Energia potenziale

$$\begin{aligned}\mathcal{U}_i &= -m_i \boldsymbol{g}_0^{iT} \boldsymbol{p}_{C_i}^i \\ &= -\boldsymbol{g}_0^{iT} (m_i \boldsymbol{p}_i^i + m_i \boldsymbol{r}_{i,C_i}^i)\end{aligned}$$

lineare rispetto a:

- ⋆ massa m_i
- ⋆ momento primo di inerzia

$$m_i \boldsymbol{r}_{i,C_i}^i = \begin{bmatrix} m_i \ell_{C_i x} \\ m_i \ell_{C_i y} \\ m_i \ell_{C_i z} \end{bmatrix}$$

- Lagrangiana

$$\mathcal{L} = \sum_{i=1}^{n} (\boldsymbol{\beta}_{\mathcal{T}i}^T - \boldsymbol{\beta}_{\mathcal{U}i}^T)\boldsymbol{\pi}_i$$

  - ⋆ vettore (11×1) di parametri dinamici

$$\boldsymbol{\pi}_i = [\, m_i \quad m_i\ell_{C_ix} \quad m_i\ell_{C_iy} \quad m_i\ell_{C_iz} \quad \hat{I}_{ixx} \quad \hat{I}_{ixy} \quad \hat{I}_{ixz} \quad \hat{I}_{iyy} \quad \hat{I}_{iyz} \quad \hat{I}_{izz} \quad I_{m_i} \,]^T$$

- Le operazioni richieste dalle equazioni di Lagrange non alterano la linearità

$$\xi_i = \sum_{j=1}^{n} \boldsymbol{y}_{ij}^T \boldsymbol{\pi}_j$$

ove

$$\boldsymbol{y}_{ij} = \frac{d}{dt}\frac{\partial \boldsymbol{\beta}_{\mathcal{T}j}}{\partial \dot{q}_i} - \frac{\partial \boldsymbol{\beta}_{\mathcal{T}j}}{\partial q_i} + \frac{\partial \boldsymbol{\beta}_{\mathcal{U}j}}{\partial q_i}$$

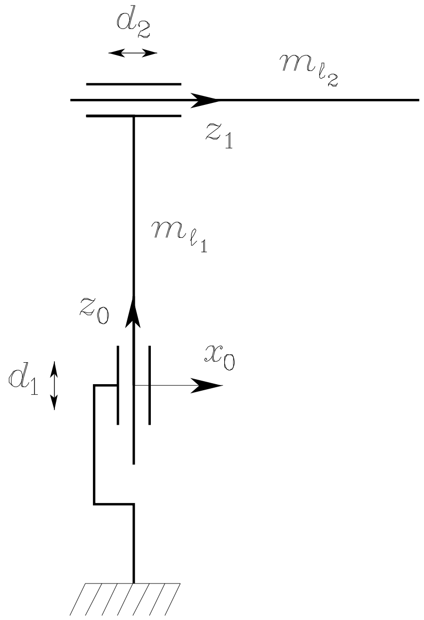
- Proprietà notevole

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{11}^T & \boldsymbol{y}_{12}^T & \dots & \boldsymbol{y}_{1n}^T \\ \boldsymbol{0}^T & \boldsymbol{y}_{22}^T & \dots & \boldsymbol{y}_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0}^T & \boldsymbol{0}^T & \dots & \boldsymbol{y}_{nn}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \\ \vdots \\ \boldsymbol{\pi}_n \end{bmatrix}$$

  - ⋆ in forma compatta:

$$\boldsymbol{\tau} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\pi}$$

Manipolatore cartesiano a due bracci



- Motori
  - ⋆ masse m_{m_i}
  - ⋆ momenti di inerzia I_{m_i}
  - ⋆ $\boldsymbol{p}_{m_i} = \boldsymbol{p}_{i-1} \quad z_{m_i} = z_{i-1}$

⋆ Jacobiani

$$\boldsymbol{J}_P^{(\ell_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{J}_P^{(\ell_2)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_P^{(m_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{J}_P^{(m_2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_O^{(m_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_{r1} & 0 \end{bmatrix} \qquad \boldsymbol{J}_O^{(m_2)} = \begin{bmatrix} 0 & k_{r2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Matrice di inerzia

$$\boldsymbol{B} = \begin{bmatrix} m_{\ell_1} + m_{m_2} + k_{r1}^2 I_{m_1} + m_{\ell_2} & 0 \\ 0 & m_{\ell_2} + k_{r2}^2 I_{m_2} \end{bmatrix}$$

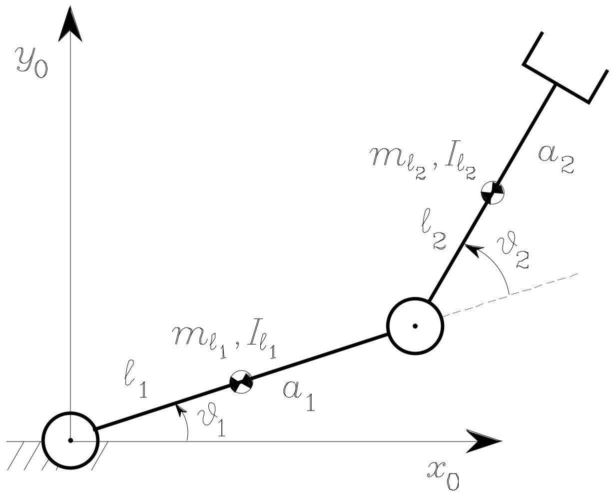
- Forze gravitazionali

$$g_1 = (m_{\ell_1} + m_{m_2} + m_{\ell_2})g \qquad g_2 = 0$$

- Equazioni del moto

$$(m_{\ell_1} + m_{m_2} + k_{r1}^2 I_{m_1} + m_{\ell_2})\ddot{d}_1 + (m_{\ell_1} + m_{m_2} + m_{\ell_2})g = f_1$$
$$(m_{\ell_2} + k_{r2}^2 I_{m_2})\ddot{d}_2 = f_2$$

Manipolatore planare a due bracci



- Motori
  - ⋆ masse m_{m_i}
  - ⋆ momenti di inerzia I_{m_i}
  - ⋆ $\boldsymbol{p}_{m_i} = \boldsymbol{p}_{i-1} \quad z_{m_i} = z_{i-1}$

⋆ Jacobiani

$$\boldsymbol{J}_P^{(\ell_1)} = \begin{bmatrix} -\ell_1 s_1 & 0 \\ \ell_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{J}_P^{(\ell_2)} = \begin{bmatrix} -a_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ a_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_O^{(\ell_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{J}_O^{(\ell_2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{J}_P^{(m_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{J}_P^{(m_2)} = \begin{bmatrix} -a_1 s_1 & 0 \\ a_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_O^{(m_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_{r1} & 0 \end{bmatrix} \qquad \boldsymbol{J}_O^{(m_2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & k_{r2} \end{bmatrix}$$

- Matrice di inerzia

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{bmatrix} b_{11}(\vartheta_2) & b_{12}(\vartheta_2) \\ b_{21}(\vartheta_2) & b_{22} \end{bmatrix}$$

$$\begin{aligned} b_{11} &= I_{\ell_1} + m_{\ell_1}\ell_1^2 + k_{r1}^2 I_{m_1} + I_{\ell_2} + m_{\ell_2}(a_1^2 + \ell_2^2 + 2a_1\ell_2 c_2) \\ &\quad + I_{m_2} + m_{m_2}a_1^2 \\ b_{12} &= b_{21} = I_{\ell_2} + m_{\ell_2}(\ell_2^2 + a_1\ell_2 c_2) + k_{r2} I_{m_2} \\ b_{22} &= I_{\ell_2} + m_{\ell_2}\ell_2^2 + k_{r2}^2 I_{m2} \end{aligned}$$

- Forze centrifughe e di Coriolis

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial b_{11}}{\partial q_1} = 0 \\ c_{112} &= c_{121} = \frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = -m_{\ell_2}a_1\ell_2 s_2 = h \\ c_{122} &= \frac{\partial b_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial b_{22}}{\partial q_1} = h \\ c_{211} &= \frac{\partial b_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = -h \\ c_{212} &= c_{221} = \frac{1}{2}\frac{\partial b_{22}}{\partial q_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial b_{22}}{\partial q_2} = 0 \end{aligned}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} h\dot{\vartheta}_2 & h(\dot{\vartheta}_1+\dot{\vartheta}_2) \\ -h\dot{\vartheta}_1 & 0 \end{bmatrix}$$

⋆ anti-simmetria

$$\begin{aligned} \boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ &= \begin{bmatrix} 2h\dot{\vartheta}_2 & h\dot{\vartheta}_2 \\ h\dot{\vartheta}_2 & 0 \end{bmatrix} - 2\begin{bmatrix} h\dot{\vartheta}_2 & h(\dot{\vartheta}_1+\dot{\vartheta}_2) \\ -h\dot{\vartheta}_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2h\dot{\vartheta}_1 - h\dot{\vartheta}_2 \\ 2h\dot{\vartheta}_1 + h\dot{\vartheta}_2 & 0 \end{bmatrix} \end{aligned}$$

- Forze gravitazionali

$$\begin{aligned} g_1 &= (m_{\ell_1}\ell_1 + m_{m_2}a_1 + m_{\ell_2}a_1)gc_1 + m_{\ell_2}\ell_2 gc_{12} \\ g_2 &= m_{\ell_2}\ell_2 gc_{12} \end{aligned}$$

- Equazioni del moto

$$
\begin{aligned}
&\big(I_{\ell_1} + m_{\ell_1}\ell_1^2 + k_{r1}^2 I_{m_1} + I_{\ell_2} + m_{\ell_2}(a_1^2 + \ell_2^2 + 2a_1\ell_2 c_2) \\
&\qquad\qquad\qquad\qquad + I_{m_2} + m_{m_2} a_1^2\big)\,\ddot{\vartheta}_1 \\
&\quad + \big(I_{\ell_2} + m_{\ell_2}(\ell_2^2 + a_1\ell_2 c_2) + k_{r2} I_{m_2}\big)\,\ddot{\vartheta}_2 \\
&\quad - 2m_{\ell_2} a_1 \ell_2 s_2 \dot{\vartheta}_1 \dot{\vartheta}_2 - m_{\ell_2} a_1 \ell_2 s_2 \dot{\vartheta}_2^2 \\
&\quad + (m_{\ell_1}\ell_1 + m_{m_2} a_1 + m_{\ell_2} a_1) g c_1 + m_{\ell_2}\ell_2 g c_{12} = \tau_1
\end{aligned}
$$

$$
\begin{aligned}
&\big(I_{\ell_2} + m_{\ell_2}\ell_2^2 + k_{r2}^2 I_{m_2}\big)\,\ddot{\vartheta}_2 + \big(I_{\ell_2} + m_{\ell_2}(\ell_2^2 + a_1\ell_2 c_2) + k_{r2} I_{m_2}\big)\,\ddot{\vartheta}_1 \\
&\quad + m_{\ell_2} a_1 \ell_2 s_2 \dot{\vartheta}_1^2 + m_{\ell_2}\ell_2 g c_{12} = \tau_2
\end{aligned}
$$

- Parametrizzazione del modello

$$\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 & \pi_8 \end{bmatrix}^T$$

$$\begin{aligned}
\pi_1 &= m_1 = m_{\ell_1} + m_{m_2} \\
\pi_2 &= m_1 \ell_{C_1} = m_{\ell_1}(\ell_1 - a_1) \\
\pi_3 &= \hat{I}_1 = I_{\ell_1} + m_{\ell_1}(\ell_1 - a_1)^2 + I_{m_2} \\
\pi_4 &= I_{m_1} \\
\pi_5 &= m_2 = m_{\ell_2} \\
\pi_6 &= m_{\ell_2} \ell_{C_2} = m_{\ell_2}(\ell_2 - a_2) \\
\pi_7 &= \hat{I}_2 = I_{\ell_2} + m_{\ell_2}(\ell_2 - a_2)^2 \\
\pi_8 &= I_{m_2}
\end{aligned}$$

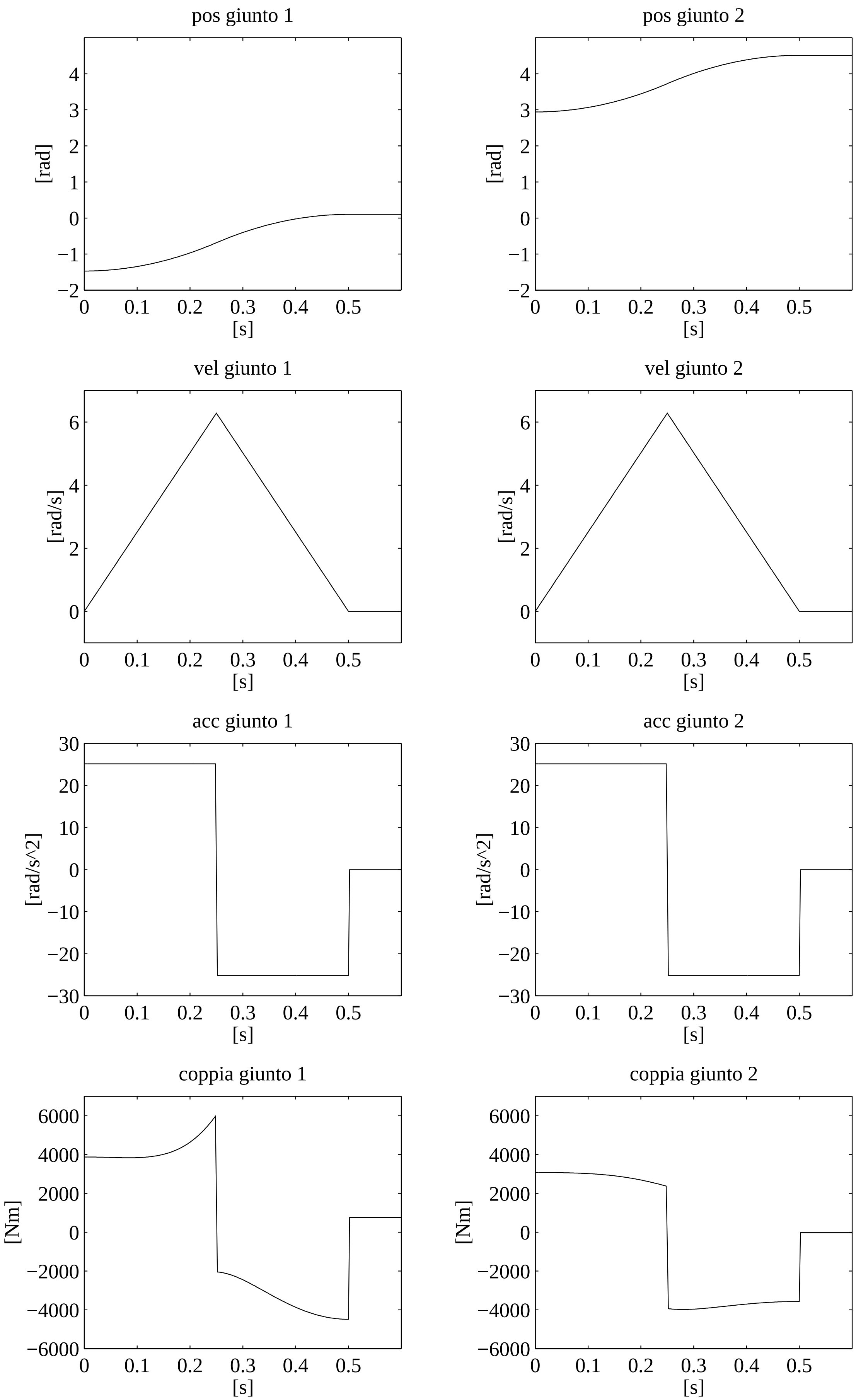
$$\boldsymbol{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} & y_{17} & y_{18} \\ y_{21} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & y_{27} & y_{28} \end{bmatrix}$$

$$\begin{aligned}
y_{11} &= a_1^2\ddot{\vartheta}_1 + a_1 g c_1 \\
y_{12} &= 2a_1\ddot{\vartheta}_1 + g c_1 \\
y_{13} &= \ddot{\vartheta}_1 \\
y_{14} &= k_{r1}^2\ddot{\vartheta}_1 \\
y_{15} &= (a_1^2 + 2a_1a_2c_2 + a_2^2)\ddot{\vartheta}_1 + (a_1a_2c_2 + a_2^2)\ddot{\vartheta}_2 \\
&\quad - 2a_1a_2s_2\dot{\vartheta}_1\dot{\vartheta}_2 - a_1a_2s_2\dot{\vartheta}_2^2 + a_1gc_1 + a_2gc_{12} \\
y_{16} &= (2a_1c_2 + 2a_2)\ddot{\vartheta}_1 + (a_1c_2 + 2a_2)\ddot{\vartheta}_2 - 2a_1s_2\dot{\vartheta}_1\dot{\vartheta}_2 \\
&\quad - a_1s_2\dot{\vartheta}_2^2 + gc_{12} \\
y_{17} &= \ddot{\vartheta}_1 + \ddot{\vartheta}_2 \\
y_{18} &= \ddot{\vartheta}_1 + k_{r2}\ddot{\vartheta}_2
\end{aligned}$$

$$\begin{aligned}
y_{21} &= 0 \\
y_{22} &= 0 \\
y_{23} &= 0 \\
y_{24} &= 0 \\
y_{25} &= (a_1a_2c_2 + a_2^2)\ddot{\vartheta}_1 + a_2^2\ddot{\vartheta}_2 + a_1a_2s_2\dot{\vartheta}_1^2 + a_2gc_{12} \\
y_{26} &= (a_1c_2 + 2a_2)\ddot{\vartheta}_1 + 2a_2\ddot{\vartheta}_2 + a_1s_2\dot{\vartheta}_1^2 + gc_{12} \\
y_{27} &= \ddot{\vartheta}_1 + \ddot{\vartheta}_2 \\
y_{28} &= k_{r2}\ddot{\vartheta}_1 + k_{r2}^2\ddot{\vartheta}_2.
\end{aligned}$$

Esempio

- Manipolatore planare a due bracci
  - $\star$ $a_1 = a_2 = 1\,\mathrm{m} \quad \ell_1 = \ell_2 = 0.5\,\mathrm{m}$
    $m_{\ell_1} = m_{\ell_2} = 50\,\mathrm{kg} \quad I_{\ell_1} = I_{\ell_2} = 10\,\mathrm{kg{\cdot}m^2}$
  - $\star$ $k_{r1} = k_{r2} = 100$
    $m_{m_1} = m_{m_2} = 5\,\mathrm{kg} \quad I_{m_1} = I_{m_2} = 0.01\,\mathrm{kg \cdot m^2}$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[rad]
[s]
pos giunto 2
[rad]
[s]
vel giunto 1
[rad/s]
[s]
vel giunto 2
[rad/s]
[s]
acc giunto 1
[rad/s^2]
[s]
acc giunto 2
[rad/s^2]
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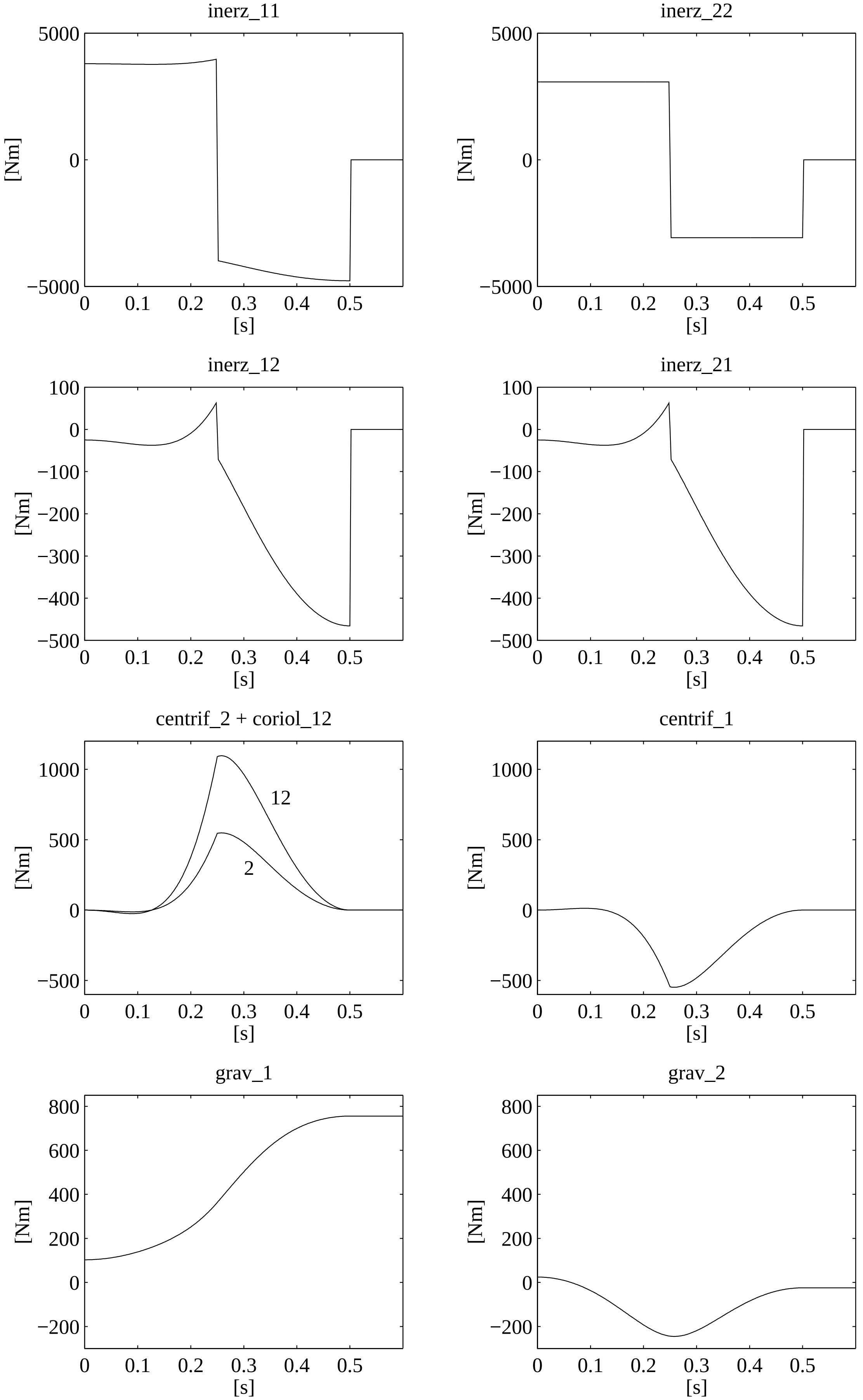
coppia giunto 1
[Nm]
[s]
coppia giunto 2
[Nm]
[s]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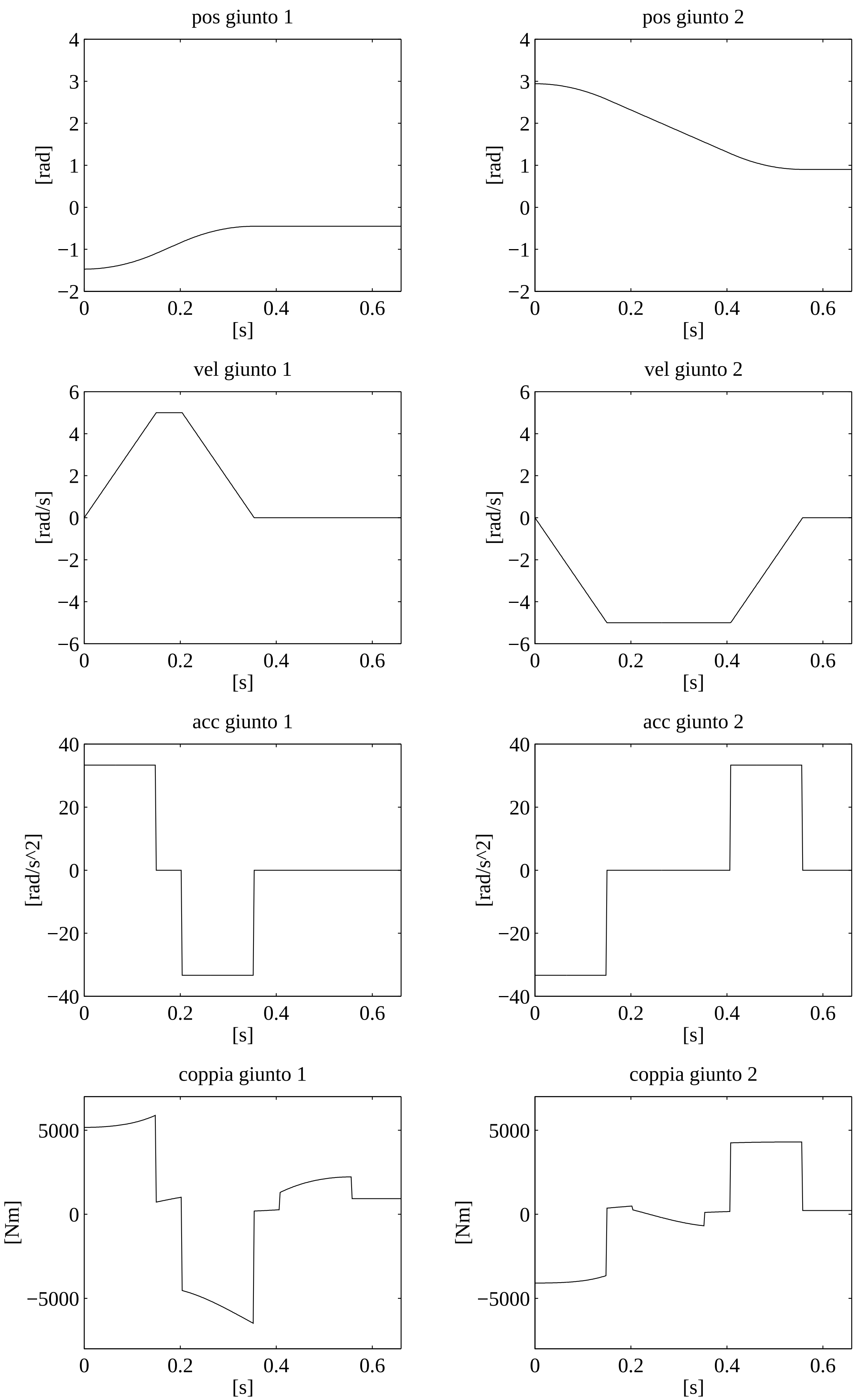
inerz_22
inerz_12
inerz_21
centrif_2 + coriol_12
12
2
centrif_1
grav_1
grav_2
[Nm]
[s]
5000
0
−5000
100
−100
−200
−300
−400
−500
1000
500
800
600
400
200
−200
0.1
0.2
0.3
0.4
0.5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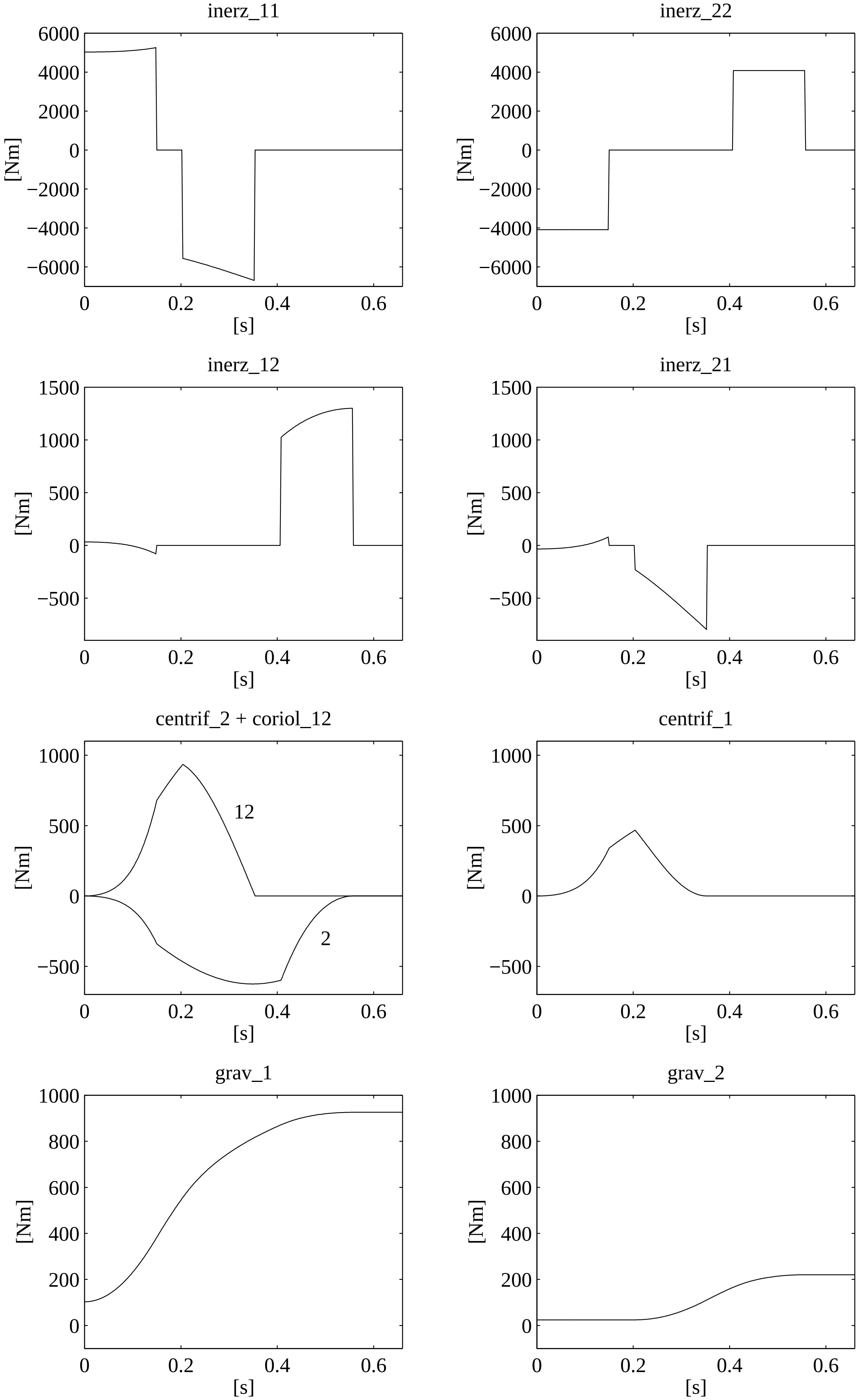
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800
600
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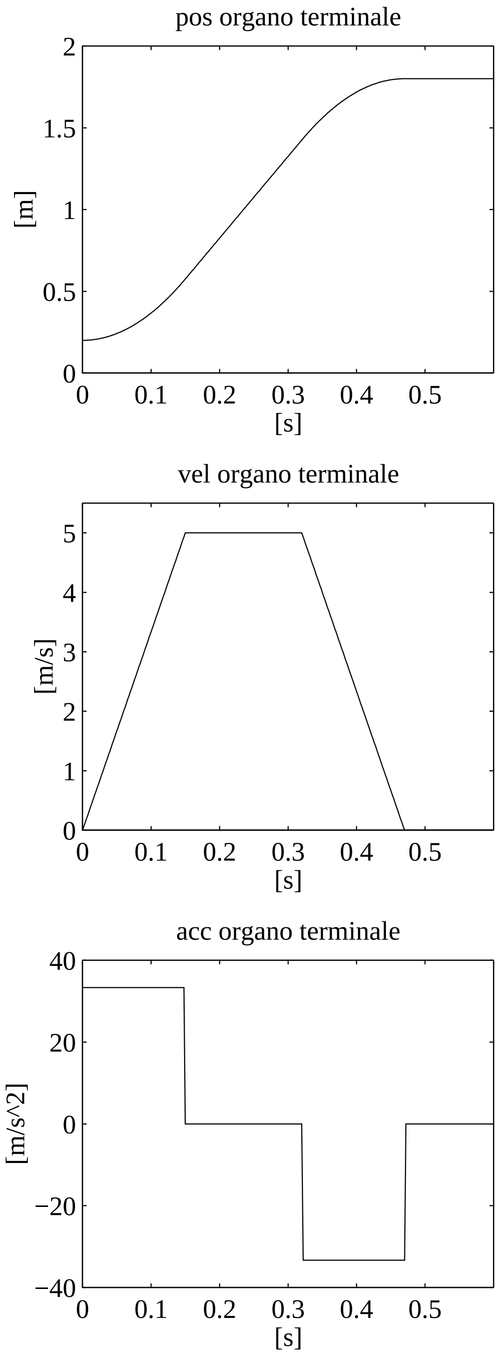
0.2
0.4
0.6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1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
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vel organo terminale
[m/s]
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
[s]
acc organo terminale
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20
0
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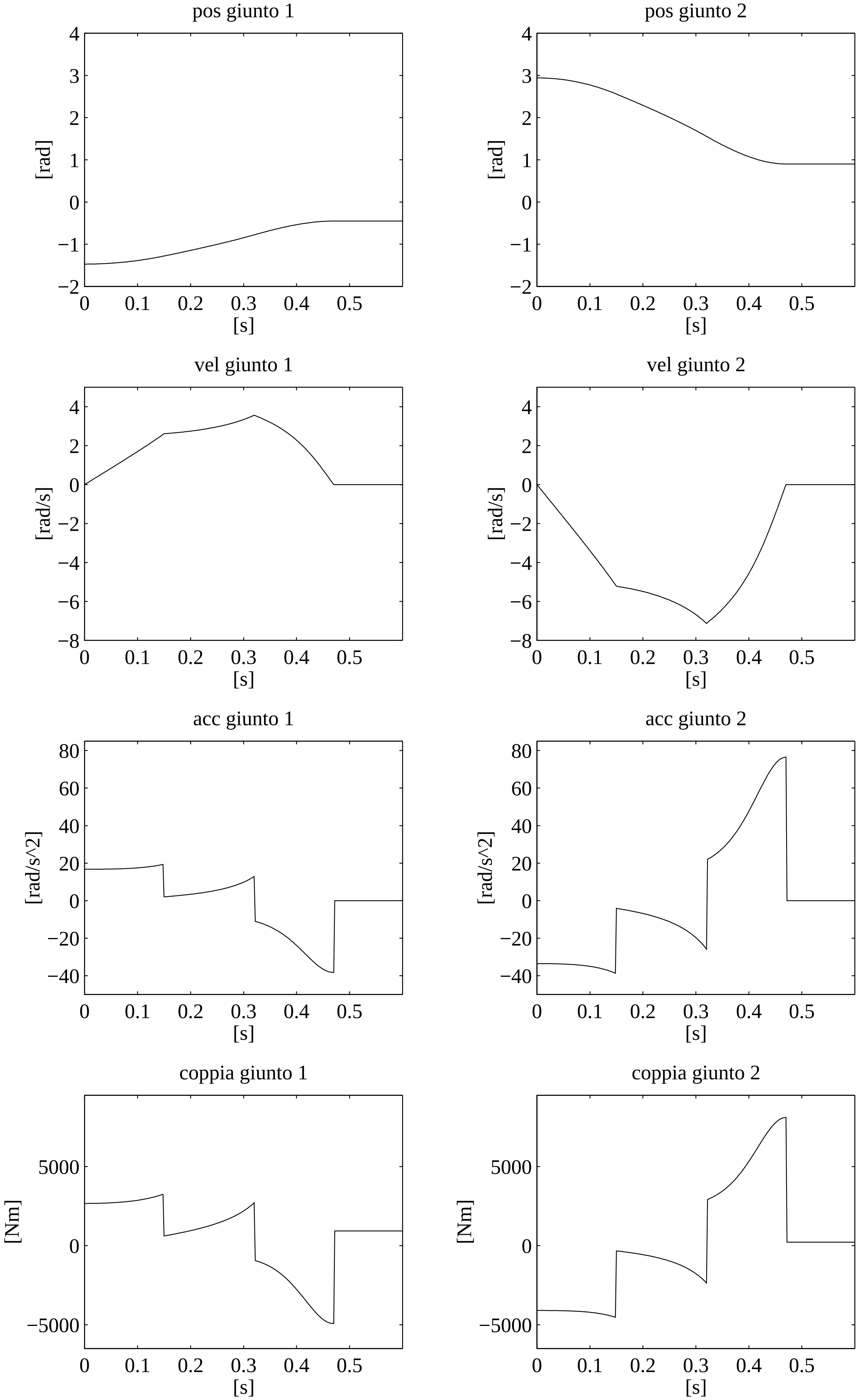
−40
0
0.1
0.2
0.3
0.4
0.5
[s]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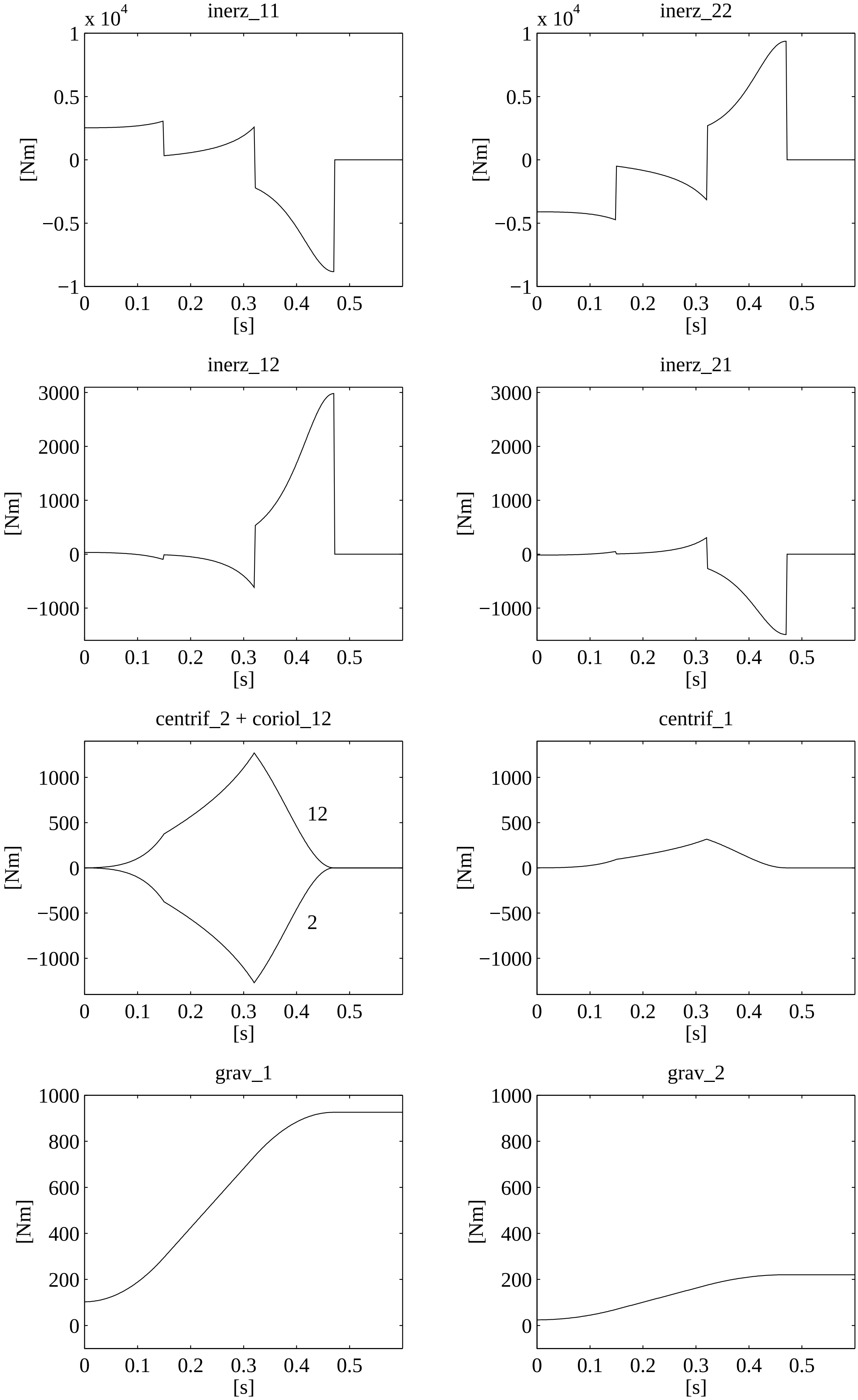
inerz_11
x 10^4
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x 10^4
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inerz_21
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[s]
centrif_2 + coriol_12
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2
[Nm]
[s]
centrif_1
[Nm]
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grav_1
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[s]
grav_2
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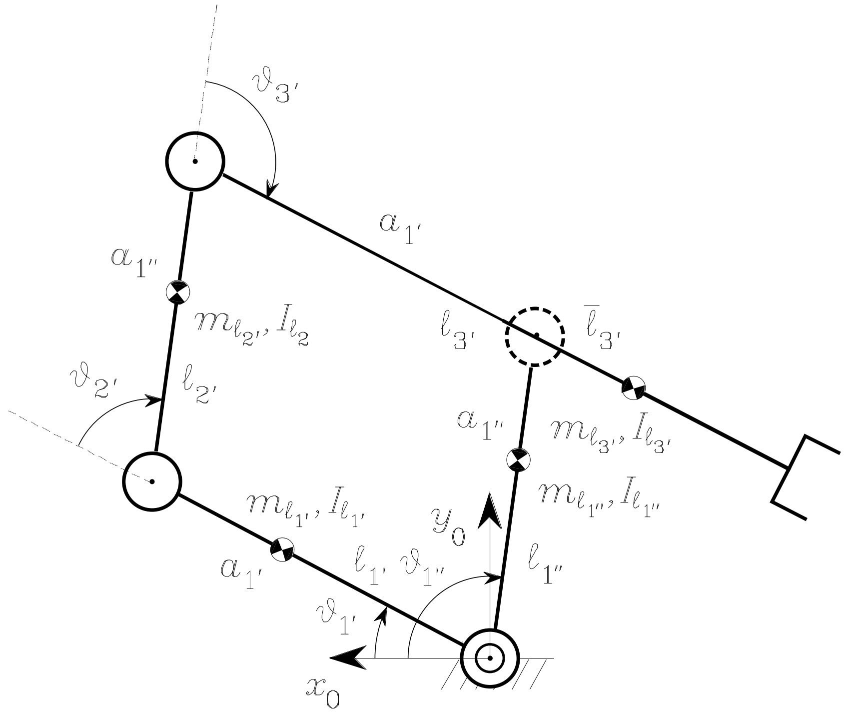
[s]

Manipolatore a parallelogramma



⋆ Jacobiani

$$\boldsymbol{J}_P^{(\ell_{1'})} = \begin{bmatrix} -\ell_{1'} s_{1'} & 0 & 0 \\ \ell_{1'} c_{1'} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{J}_P^{(\ell_{2'})} = \begin{bmatrix} -a_{1'} s_{1'} - \ell_2 s_{1'2'} & -\ell_{2'} s_{1'2'} & 0 \\ a_{1'} c_{1'} + \ell_{2'} c_{1'2'} & \ell_2 c_{1'2'} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_P^{(\ell_{3'})} = \begin{bmatrix} -a_{1'} s_{1'} - a_{2'} s_{1'2'} - \ell_{3'} s_{1'2'3'} & -a_{2'} s_{1'2'} - \ell_{3'} s_{1'2'3'} & -\ell_{3'} s_{1'2'3'} \\ a_{1'} c_{1'} + a_{2'} c_{1'2'} + \ell_{3'} c_{1'2'3'} & a_{2'} c_{1'2'} + \ell_{3'} c_{1'2'3'} & \ell_{3'} c_{1'2'3'} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_P^{(\ell_{1''})} = \begin{bmatrix} -\ell_{1''} s_{1''} \\ \ell_{1''} c_{1''} \\ 0 \end{bmatrix}$$

$$\boldsymbol{J}_O^{(\ell_{1'})} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{J}_O^{(\ell_{2'})} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{J}_O^{(\ell_{3'})} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{J}_O^{(\ell_{1''})} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Matrici di inerzia per i manipolatori virtuali

$$\boldsymbol{B}'(\boldsymbol{q}') = \begin{bmatrix} b_{1'1'}(\vartheta_{2'},\vartheta_{3'}) & b_{1'2'}(\vartheta_{2'},\vartheta_{3'}) & b_{1'3'}(\vartheta_{2'},\vartheta_{3'}) \\ b_{2'1'}(\vartheta_{2'},\vartheta_{3'}) & b_{2'2'}(\vartheta_{3'}) & b_{2'3'}(\vartheta_{3'}) \\ b_{3'1'}(\vartheta_{2'},\vartheta_{3'}) & b_{3'2'}(\vartheta_{3'}) & b_{3'3'} \end{bmatrix}$$

$$\begin{aligned}
b_{1'1'} &= I_{\ell_{1'}} + m_{\ell_{1'}}\ell_{1'}^2 + I_{\ell_{2'}} + m_{\ell_{2'}}(a_{1'}^2 + \ell_{2'}^2 + 2a_{1'}\ell_{2'}c_{2'}) + I_{\ell_{3'}} \\
&\quad + m_{\ell_{3'}}(a_{1'}^2 + a_{2'}^2 + \ell_{3'}^2 + 2a_{1'}a_{2'}c_{2'} + 2a_{1'}\ell_{3'}c_{2'3'} + 2a_{2'}\ell_{3'}c_{3'}) \\
b_{1'2'} &= b_{2'1'} = I_{\ell_{2'}} + m_{\ell_{2'}}(\ell_{2'}^2 + a_{1'}\ell_{2'}c_{2'}) + I_{\ell_{3'}} \\
&\quad + m_{\ell_{3'}}(a_{2'}^2 + \ell_{3'}^2 + a_{1'}a_{2'}c_{2'} + a_{1'}\ell_{3'}c_{2'3'} + 2a_{2'}\ell_{3'}c_{3'}) \\
b_{1'3'} &= b_{31} = I_{\ell_{3'}} + m_{\ell_{3'}}(\ell_{3'}^2 + a_{1'}\ell_{3'}c_{2'3'} + a_{2'}\ell_{3'}c_{3'}) \\
b_{2'2'} &= I_{\ell_{2'}} + m_{\ell_{2'}}\ell_{2'}^2 + I_{\ell_{3'}} + m_{\ell_{3'}}(a_{2'}^2 + \ell_{3'}^2 + 2a_{2'}\ell_{3'}c_{3'}) \\
b_{2'3'} &= I_{\ell_{3'}} + m_{\ell_{3'}}(\ell_{3'}^2 + a_{2'}\ell_{3'}c_{3'}) \\
b_{3'3'} &= I_{\ell_{3'}} + m_{\ell_{3'}}\ell_{3'}^2
\end{aligned}$$

$$b_{1''1''} = I_{\ell_{1''}} + m_{\ell_{1''}}\ell_{1''}^2$$

$$\tau_{i'} = \sum_{j'=1'}^{3'} b_{i'j'}\ddot{\vartheta}_{j'} \qquad \tau_{1''} = b_{1'1'}\ddot{\vartheta}_{1''}$$

- Manipolatore a catena chiusa

$$\boldsymbol{\tau}_a = \boldsymbol{B}_a \ddot{\boldsymbol{q}}_a$$

$$\boldsymbol{B}_a = \begin{bmatrix} b_{a11} & b_{a12} \\ b_{a21} & b_{a22} \end{bmatrix}$$

$$\begin{aligned}
b_{a11} &= I_{\ell_{1'}} + m_{\ell_{1'}} \ell_{1'}^2 + m_{\ell_{2'}} a_{1'}^2 \\
&\quad + I_{\ell_{3'}} + m_{\ell_{3'}} \ell_{3'}^2 + m_{\ell_{3'}} a_{1'}^2 - 2 a_{1'} m_{\ell_{3'}} \ell_{3'} \\
b_{a12} &= b_{a21} = \big(a_{1'} m_{\ell_{2'}} \ell_{2'} + a_{1''} m_{\ell_{3'}} (a_{1'} - \ell_{3'})\big) \cos(\vartheta_{1''} - \vartheta_{1'}) \\
b_{a22} &= I_{\ell_{1'}} + m_{\ell_{1'}} \ell_{1'}^2 + I_{\ell_{2'}} + m_{\ell_{2'}} \ell_{2'}^2 + m_{\ell_{3'}} a_{1''}^2 .
\end{aligned}$$

⋆ matrice costante e diagonale

$$\frac{m_{\ell_{3'}} \bar{\ell}_{3'}}{m_{\ell_{2'}} \ell_{2'}} = \frac{a_{1'}}{a_{1''}}$$

$$\begin{aligned}
b_{a11} &= I_{\ell_{1'}} + m_{\ell_{1'}} \ell_{1'}^2 + m_{\ell_{2'}} a_{1'}^2 \left(1 + \frac{\ell_{2'} \bar{\ell}_{3'}}{a_{1'} a_{1''}}\right) + I_{\ell_{3'}} \\
b_{a22} &= I_{\ell_{1'}} + m_{\ell_{1'}} \ell_{1'}^2 + I_{\ell_{2'}} + m_{\ell_{2'}} \ell_{2'}^2 \left(1 + \frac{a_{1'} a_{1''}}{\ell_{2'} \bar{\ell}_{3'}}\right)
\end{aligned}$$

- Forze gravitazionali

$$
\begin{aligned}
g_{1'} &= (m_{\ell_{1'}}\ell_{1'} + m_{\ell_{2'}}a_{1'} + m_{\ell_{3'}}a_{1'})gc_{1'} + (m_{\ell_{2'}}\ell_{2'} + m_{\ell_{3'}}a_{2'})gc_{1'2'} \\
&\quad + m_{\ell_{3'}}\ell_{3'}gc_{1'2'3} \\
g_{2'} &= (m_{\ell_{2'}}\ell_{2'} + m_{\ell_{3'}}a_{2'})gc_{1'2'} + m_{\ell_{3'}}\ell_{3'}gc_{1'2'3} \\
g_{3'} &= m_{\ell_{3'}}\ell_{3'}gc_{1'2'3}
\end{aligned}
$$

$$
g_{1''} = m_{\ell_{1''}}\ell_{1''}gc_{1''}
$$

$$
\boldsymbol{g}_a = \begin{bmatrix} (m_{\ell_{1'}}\ell_{1'} + m_{\ell_{2'}}a_{1'} - m_{\ell_{3'}}\bar{\ell}_{3'})gc_{1'} \\ (m_{\ell_{1''}}\ell_{1''} + m_{\ell_{2'}}\ell_{2'} + m_{\ell_{3'}}a_{1''})gc_{1''} \end{bmatrix}
$$

IDENTIFICAZIONE DEI PARAMETRI DINAMICI

- Impiego del modello dinamico per simulazione e controllo
- Stime approssimate dei parametri dinamici
  - ⋆ semplificazioni di modellazione geometrica
  - ⋆ effetti dinamici complessi (attriti, etc.)
- Tecniche di identificazione
  - ⋆ proprietà di linearità

$$\boldsymbol{\tau} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\pi}$$

- Misure
  - ⋆ $\boldsymbol{q}$ e $\dot{\boldsymbol{q}}$ dirette (encoder e tachimetri)
  - ⋆ $\ddot{\boldsymbol{q}}$ indirette (filtro ricostruttore)
  - ⋆ $\boldsymbol{\tau}$ dirette (sensore di coppia) o indirette (misure di forza o corrente)

- N insiemi di misure

$$\bar{\boldsymbol{\tau}} = \begin{bmatrix} \boldsymbol{\tau}(t_1) \\ \vdots \\ \boldsymbol{\tau}(t_N) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}(t_1) \\ \vdots \\ \boldsymbol{Y}(t_N) \end{bmatrix} \boldsymbol{\pi} = \bar{\boldsymbol{Y}} \boldsymbol{\pi}$$

- Soluzione (a minimi quadrati)

$$\boldsymbol{\pi} = (\bar{\boldsymbol{Y}}^T \bar{\boldsymbol{Y}})^{-1} \bar{\boldsymbol{Y}}^T \bar{\boldsymbol{\tau}}$$

  - $\star$ $\boldsymbol{Y}$ triangolare $\Longrightarrow$ procedimento iterativo
  - $\star$ se $\varrho(\boldsymbol{Y}) < p$ $\Longrightarrow$ $\boldsymbol{Y}^\star$

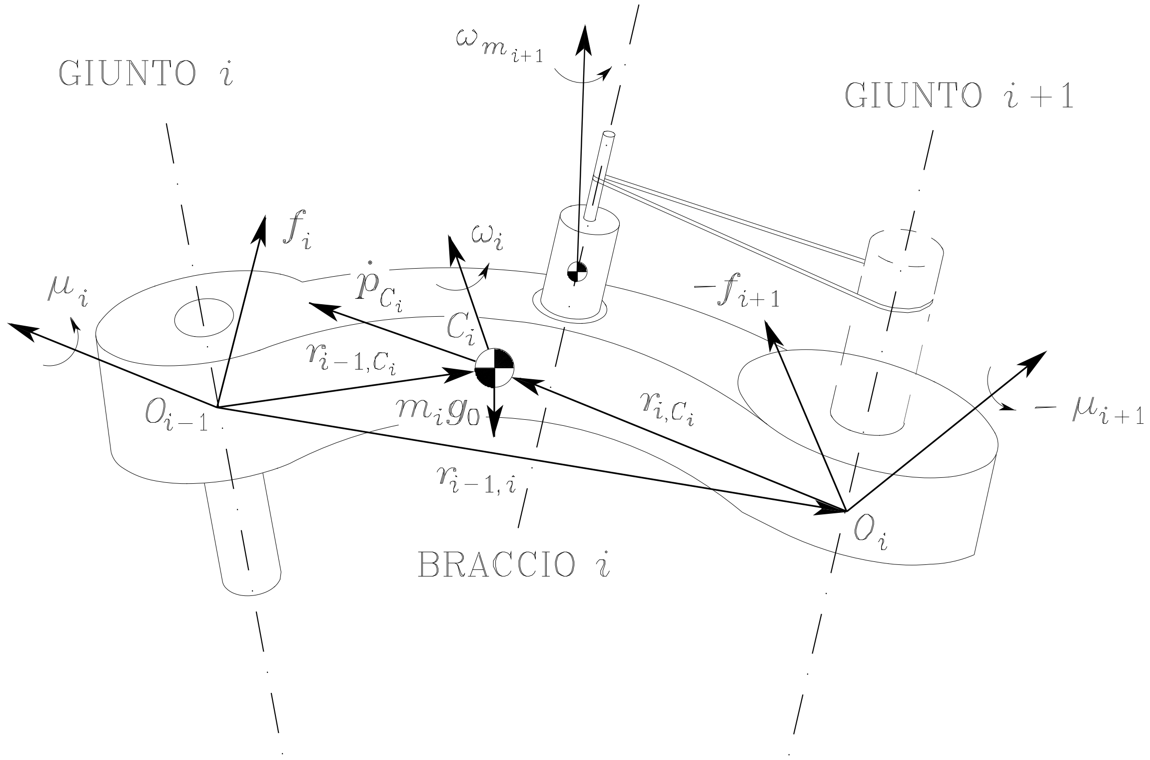
- Parametri
  - $\star$ identificabili
  - $\star$ non identificabili
  - $\star$ identificabili in combinazione lineare

- Carico
  - $\star$ modifica parametri dell'ultimo braccio

- Traiettorie
  - $\star$ di tipo polinomiale (sufficientemente *ricche*)
  - $\star$ attenzione alle dinamiche non modellate

FORMULAZIONE DI NEWTON-EULERO

⋆ bilancio di forze e momenti agenti sul generico braccio



- Equazione di *Newton* (moto *traslazionale* del baricentro)

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} + m_i \boldsymbol{g}_0 = m_i \ddot{\boldsymbol{p}}_{C_i}$$

- Equazione di *Eulero* (moto *rotazionale*)

$$\boldsymbol{\mu}_i + \boldsymbol{f}_i \times \boldsymbol{r}_{i-1,C_i} - \boldsymbol{\mu}_{i+1} - \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,C_i} = \frac{d}{dt}(\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i + k_{r,i+1}\dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}})$$

$$\begin{aligned}\frac{d}{dt}(\bar{\boldsymbol{I}}_i\boldsymbol{\omega}_i) &= \dot{\boldsymbol{R}}_i\bar{\boldsymbol{I}}_i^i\boldsymbol{R}_i^T\boldsymbol{\omega}_i + \boldsymbol{R}_i\bar{\boldsymbol{I}}_i^i\dot{\boldsymbol{R}}_i^T\boldsymbol{\omega}_i + \boldsymbol{R}_i\bar{\boldsymbol{I}}_i^i\boldsymbol{R}_i^T\dot{\boldsymbol{\omega}}_i\\ &= \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{R}_i\bar{\boldsymbol{I}}_i^i\boldsymbol{R}_i^T\boldsymbol{\omega}_i + \boldsymbol{R}_i\bar{\boldsymbol{I}}_i^i\boldsymbol{R}_i^T\boldsymbol{S}^T(\boldsymbol{\omega}_i)\boldsymbol{\omega}_i + \boldsymbol{R}_i\bar{\boldsymbol{I}}_i^i\boldsymbol{R}_i^T\dot{\boldsymbol{\omega}}_i\\ &= \bar{\boldsymbol{I}}_i\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\bar{\boldsymbol{I}}_i\boldsymbol{\omega}_i)\end{aligned}$$

$$\frac{d}{dt}(\dot{q}_{i+1}I_{m_{i+1}}\boldsymbol{z}_{m_{i+1}}) = \ddot{q}_{i+1}I_{m_{i+1}}\boldsymbol{z}_{m_{i+1}} + \dot{q}_{i+1}I_{m_{i+1}}\boldsymbol{\omega}_i \times \boldsymbol{z}_{m_{i+1}}$$

$$\Downarrow$$

- Equazione di Eulero

$$\begin{aligned}\boldsymbol{\mu}_i + \boldsymbol{f}_i \times \boldsymbol{r}_{i-1,C_i} - \boldsymbol{\mu}_{i+1} - \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,C_i} &= \bar{\boldsymbol{I}}_i\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\bar{\boldsymbol{I}}_i\boldsymbol{\omega}_i)\\ &+ k_{r,i+1}\ddot{q}_{i+1}I_{m_{i+1}}\boldsymbol{z}_{m_{i+1}} + k_{r,i+1}\dot{q}_{i+1}I_{m_{i+1}}\boldsymbol{\omega}_i \times \boldsymbol{z}_{m_{i+1}}\end{aligned}$$

- Forza generalizzata al giunto

$$\tau_i = \begin{cases} \boldsymbol{f}_i^T\boldsymbol{z}_{i-1} + k_{ri}I_{mi}\dot{\boldsymbol{\omega}}_{m_i}^T\boldsymbol{z}_{m_i} & \text{per un giunto } \textit{prismatico}\\ \boldsymbol{\mu}_i^T\boldsymbol{z}_{i-1} + k_{ri}I_{mi}\dot{\boldsymbol{\omega}}_{m_i}^T\boldsymbol{z}_{m_i} & \text{per un giunto } \textit{rotoidale}\end{cases}$$

Accelerazioni di un braccio

$$\boldsymbol{\omega}_i = \begin{cases} \boldsymbol{\omega}_{i-1} & \textit{prismatico} \\ \boldsymbol{\omega}_{i-1} + \dot{\vartheta}_i \boldsymbol{z}_{i-1} & \textit{rotoidale} \end{cases}$$

$$\dot{\boldsymbol{p}}_i = \begin{cases} \dot{\boldsymbol{p}}_{i-1} + \dot{d}_i \boldsymbol{z}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i} & \textit{prismatico} \\ \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i} & \textit{rotoidale} \end{cases}$$

$$\dot{\boldsymbol{\omega}}_i = \begin{cases} \dot{\boldsymbol{\omega}}_{i-1} & \textit{prismatico} \\ \dot{\boldsymbol{\omega}}_{i-1} + \ddot{\vartheta}_i \boldsymbol{z}_{i-1} + \dot{\vartheta}_i \boldsymbol{\omega}_{i-1} \times \boldsymbol{z}_{i-1} & \textit{rotoidale} \end{cases}$$

$$\ddot{\boldsymbol{p}}_i = \begin{cases} \ddot{\boldsymbol{p}}_{i-1} + \ddot{d}_i \boldsymbol{z}_{i-1} + 2\dot{d}_i \boldsymbol{\omega}_i \times \boldsymbol{z}_{i-1} & \textit{prismatico} \\ \quad + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i-1,i} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}) & \\ \ddot{\boldsymbol{p}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i-1,i} & \\ \quad + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}) & \textit{rotoidale} \end{cases}$$

$$\ddot{\boldsymbol{p}}_{C_i} = \ddot{\boldsymbol{p}}_i + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i,C_i} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i,C_i})$$

$$\dot{\boldsymbol{\omega}}_{m_i} = \dot{\boldsymbol{\omega}}_{i-1} + k_{ri} \ddot{q}_i \boldsymbol{z}_{mi} + k_{ri} \dot{q}_i \boldsymbol{\omega}_{i-1} \times \boldsymbol{z}_{mi}$$

Algoritmo ricorsivo

- Ricorsione *in avanti* di velocità e accelerazioni
  - ⋆ condizioni iniziali: $\boldsymbol{\omega}_0$, $\ddot{\boldsymbol{p}}_0 - \boldsymbol{g}_0$, $\dot{\boldsymbol{\omega}}_0$
  - ⋆ calcolo di $\boldsymbol{\omega}_i$, $\dot{\boldsymbol{\omega}}_i$, $\ddot{\boldsymbol{p}}_i$, $\ddot{\boldsymbol{p}}_{C_i}$, $\dot{\boldsymbol{\omega}}_{m_i}$
- Ricorsione *all'indietro* di forze e momenti
  - ⋆ condizioni terminali: $\boldsymbol{h} = [\, \boldsymbol{f}_{n+1}^T \quad \boldsymbol{\mu}_{n+1}^T \,]^T$
  - ⋆ calcolo di

$$\boldsymbol{f}_i = \boldsymbol{f}_{i+1} + m_i \ddot{\boldsymbol{p}}_{C_i}$$

$$\begin{aligned}\boldsymbol{\mu}_i = & -\boldsymbol{f}_i \times (\boldsymbol{r}_{i-1,i} + \boldsymbol{r}_{i,C_i}) + \boldsymbol{\mu}_{i+1} + \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,C_i} \\ & + \bar{\boldsymbol{I}}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i) \\ & + k_{r,i+1} \ddot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}} + k_{r,i+1} \dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{\omega}_i \times \boldsymbol{z}_{m_{i+1}}\end{aligned}$$

$$\tau_i = \begin{cases} \boldsymbol{f}_i^T \boldsymbol{z}_{i-1} + k_{ri} I_{mi} \dot{\boldsymbol{\omega}}_{m_i}^T \boldsymbol{z}_{m_i} + F_{vi} \dot{d}_i + F_{si} \operatorname{sgn}(\dot{d}_i) & \textit{prismatico} \\ \boldsymbol{\mu}_i^T \boldsymbol{z}_{i-1} + k_{ri} I_{mi} \dot{\boldsymbol{\omega}}_{m_i}^T \boldsymbol{z}_{m_i} + F_{vi} \dot{\vartheta}_i + F_{si} \operatorname{sgn}(\dot{\vartheta}_i) & \textit{rotoidale} \end{cases}$$

- Conviene riferire tutti i vettori alla terna corrente sul braccio i
  - ⋆ le quantità $\bar{\boldsymbol{I}}_i^i$, $\boldsymbol{r}_{i,C_i}^i$, $\boldsymbol{z}_{m_i}^{i-1}$ sono *costanti*

$$\boldsymbol{\omega}_i^i = \begin{cases} \boldsymbol{R}_i^{i-1T}\boldsymbol{\omega}_{i-1}^{i-1} & \textit{prismatico} \\ \boldsymbol{R}_i^{i-1T}(\boldsymbol{\omega}_{i-1}^{i-1} + \dot{\vartheta}_i \boldsymbol{z}_0) & \textit{rotoidale} \end{cases} \tag{7.107}$$

$$\dot{\boldsymbol{\omega}}_i^i = \begin{cases} \boldsymbol{R}_i^{i-1T}\dot{\boldsymbol{\omega}}_{i-1}^{i-1} & \textit{prismatico} \\ \boldsymbol{R}_i^{i-1T}(\dot{\boldsymbol{\omega}}_{i-1}^{i-1} + \ddot{\vartheta}_i \boldsymbol{z}_0 + \dot{\vartheta}_i \boldsymbol{\omega}_{i-1}^{i-1} \times \boldsymbol{z}_0) & \textit{rotoidale} \end{cases} \tag{7.108}$$

$$\ddot{\boldsymbol{p}}_i^i = \begin{cases} \boldsymbol{R}_i^{i-1T}(\ddot{\boldsymbol{p}}_{i-1}^{i-1} + \ddot{d}_i \boldsymbol{z}_0) + 2\dot{d}_i \boldsymbol{\omega}_i^i \times \boldsymbol{R}_i^{i-1T}\boldsymbol{z}_0 & \\ \quad + \dot{\boldsymbol{\omega}}_i^i \times \boldsymbol{r}_{i-1,i}^i + \boldsymbol{\omega}_i^i \times (\boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i-1,i}^i) & \textit{prismatico} \\ \boldsymbol{R}_i^{i-1T}\ddot{\boldsymbol{p}}_{i-1}^{i-1} + \dot{\boldsymbol{\omega}}_i^i \times \boldsymbol{r}_{i-1,i}^i & \\ \quad + \boldsymbol{\omega}_i^i \times (\boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i-1,i}^i) & \textit{rotoidale} \end{cases} \tag{7.109}$$

$$\ddot{\boldsymbol{p}}_{C_i}^i = \ddot{\boldsymbol{p}}_i^i + \dot{\boldsymbol{\omega}}_i^i \times \boldsymbol{r}_{i,C_i}^i + \boldsymbol{\omega}_i^i \times (\boldsymbol{\omega}_i^i \times \boldsymbol{r}_{i,C_i}^i) \tag{7.110}$$

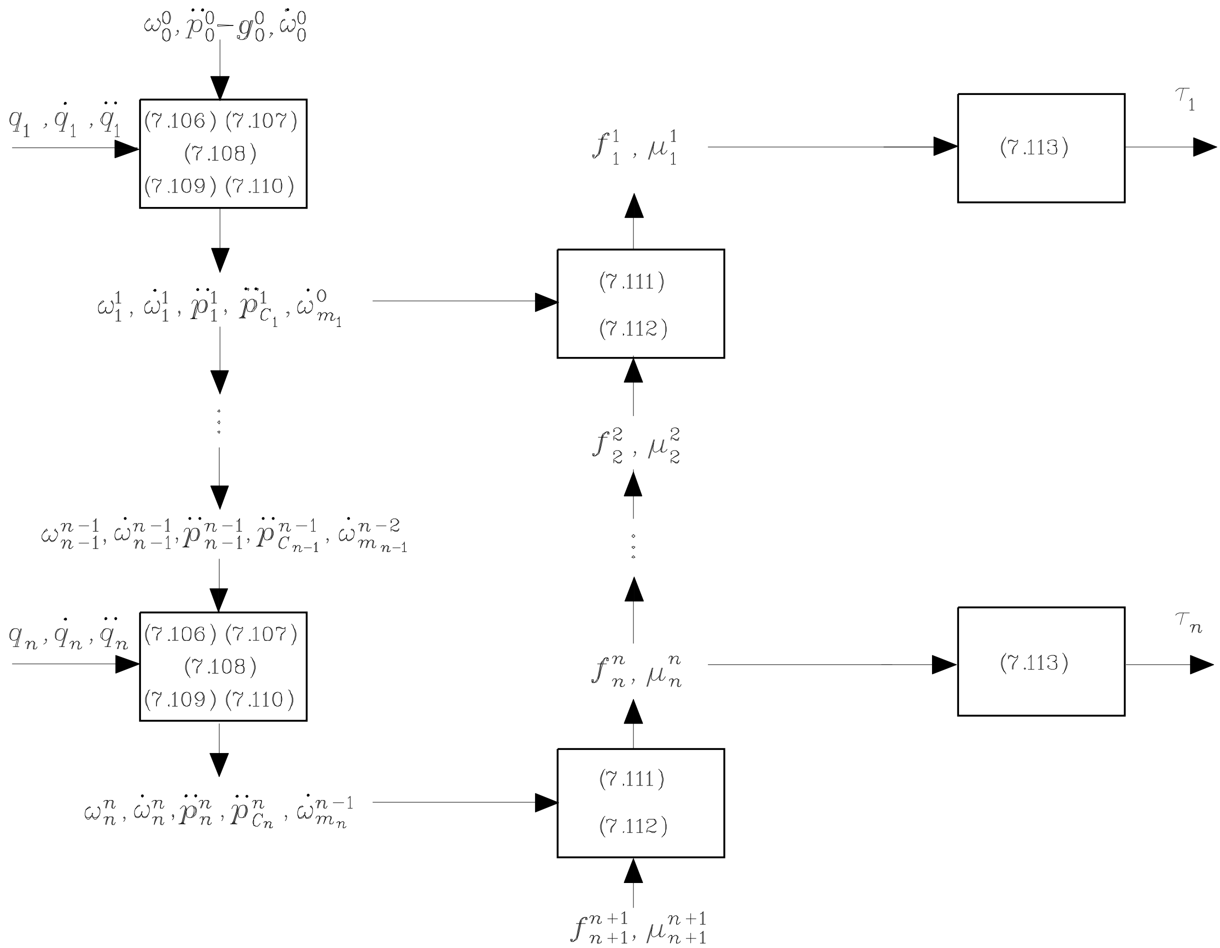
$$\dot{\boldsymbol{\omega}}_{m_i}^{i-1} = \dot{\boldsymbol{\omega}}_{i-1}^{i-1} + k_{ri}\ddot{q}_i \boldsymbol{z}_{m_i}^{i-1} + k_{ri}\dot{q}_i \boldsymbol{\omega}_{i-1}^{i-1} \times \boldsymbol{z}_{m_i}^{i-1} \tag{7.111}$$

$$\boldsymbol{f}_i^i = \boldsymbol{R}_{i+1}^i \boldsymbol{f}_{i+1}^{i+1} + m_i \ddot{\boldsymbol{p}}_{C_i}^i \tag{7.112}$$

$$\begin{aligned} \boldsymbol{\mu}_i^i = & -\boldsymbol{f}_i^i \times (\boldsymbol{r}_{i-1,i}^i + \boldsymbol{r}_{i,C_i}^i) + \boldsymbol{R}_{i+1}^i \boldsymbol{\mu}_{i+1}^{i+1} \\ & + \boldsymbol{R}_{i+1}^i \boldsymbol{f}_{i+1}^{i+1} \times \boldsymbol{r}_{i,C_i}^i + \bar{\boldsymbol{I}}_i^i \dot{\boldsymbol{\omega}}_i^i + \boldsymbol{\omega}_i^i \times (\bar{\boldsymbol{I}}_i^i \boldsymbol{\omega}_i^i) \\ & + k_{r,i+1}\ddot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}}^i + k_{r,i+1}\dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{\omega}_i^i \times \boldsymbol{z}_{m_{i+1}}^i \end{aligned} \tag{7.113}$$

$$\tau_i = \begin{cases} \boldsymbol{f}_i^{iT}\boldsymbol{R}_i^{i-1T}\boldsymbol{z}_0 + k_{ri} I_{m_i} \dot{\boldsymbol{\omega}}_{m_i}^{i-1T} \boldsymbol{z}_{m_i}^{i-1} & \\ \quad + F_{vi}\dot{d}_i + F_{si}\,\mathrm{sgn}\,(\dot{d}_i) & \textit{prismatico} \\ \boldsymbol{\mu}_i^{iT}\boldsymbol{R}_i^{i-1T}\boldsymbol{z}_0 + k_{ri} I_{m_i} \dot{\boldsymbol{\omega}}_{m_i}^{i-1T} \boldsymbol{z}_{m_i}^{i-1} & \\ \quad + F_{vi}\dot{\vartheta}_i + F_{si}\,\mathrm{sgn}\,(\dot{\vartheta}_i) & \textit{rotoidale} \end{cases} \tag{7.114}$$

- Struttura computazionale



Manipolatore planare a due bracci

- Condizioni iniziali

$$\ddot{\boldsymbol{p}}_0^0 - \boldsymbol{g}_0^0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \qquad \boldsymbol{\omega}_0^0 = \dot{\boldsymbol{\omega}}_0^0 = \mathbf{0}$$

- Condizioni terminali

$$\boldsymbol{f}_3^3 = \mathbf{0} \qquad \boldsymbol{\mu}_3^3 = \mathbf{0}$$

- Quantità riferite alla terna corrente

$$\boldsymbol{r}_{1,C_1}^1 = \begin{bmatrix} \ell_{C_1} \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{r}_{0,1}^1 = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{r}_{2,C_2}^2 = \begin{bmatrix} \ell_{C_2} \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{r}_{1,2}^2 = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{z}_{m_1}^1 = \boldsymbol{z}_0^1 = \boldsymbol{z}_{m_2}^1 = \boldsymbol{z}_{m_2}^2 = \boldsymbol{z}_1^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - ⋆ matrici di rotazione

$$\boldsymbol{R}_i^{i-1} = \begin{bmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad i = 1,2 \qquad \boldsymbol{R}_3^2 = \boldsymbol{I}$$

- Ricorsione in avanti: braccio 1

$$\boldsymbol{\omega}_1^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\vartheta}_1 \end{bmatrix}$$

$$\dot{\boldsymbol{\omega}}_1^1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\vartheta}_1 \end{bmatrix}$$

$$\ddot{\boldsymbol{p}}_1^1 = \begin{bmatrix} -a_1\dot{\vartheta}_1^2 + gs_1 \\ a_1\ddot{\vartheta}_1 + gc_1 \\ 0 \end{bmatrix}$$

$$\ddot{\boldsymbol{p}}_{C_1}^1 = \begin{bmatrix} -(\ell_{C_1} + a_1)\dot{\vartheta}_1^2 + gs_1 \\ (\ell_{C_1} + a_1)\ddot{\vartheta}_1 + gc_1 \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{\omega}}_{m_1}^0 = \begin{bmatrix} 0 \\ 0 \\ k_{r1}\ddot{\vartheta}_1 \end{bmatrix}$$

- Ricorsione in avanti: braccio 2

$$\boldsymbol{\omega}_2^2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\vartheta}_1 + \dot{\vartheta}_2 \end{bmatrix}$$

$$\dot{\boldsymbol{\omega}}_2^2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\vartheta}_1 + \ddot{\vartheta}_2 \end{bmatrix}$$

$$\ddot{\boldsymbol{p}}_2^2 = \begin{bmatrix} a_1 s_2 \ddot{\vartheta}_1 - a_1 c_2 \dot{\vartheta}_1^2 - a_2(\dot{\vartheta}_1 + \dot{\vartheta}_2)^2 + g s_{12} \\ a_1 c_2 \ddot{\vartheta}_1 + a_2(\ddot{\vartheta}_1 + \ddot{\vartheta}_2) + a_1 s_2 \dot{\vartheta}_1^2 + g c_{12} \\ 0 \end{bmatrix}$$

$$\ddot{\boldsymbol{p}}_{C_2}^2 = \begin{bmatrix} a_1 s_2 \ddot{\vartheta}_1 - a_1 c_2 \dot{\vartheta}_1^2 - (\ell_{C_2} + a_2)(\dot{\vartheta}_1 + \dot{\vartheta}_2)^2 + g s_{12} \\ a_1 c_2 \ddot{\vartheta}_1 + (\ell_{C_2} + a_2)(\ddot{\vartheta}_1 + \ddot{\vartheta}_2) + a_1 s_2 \dot{\vartheta}_1^2 + g c_{12} \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{\omega}}_{m_2}^1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\vartheta}_1 + k_{r2} \ddot{\vartheta}_2 \end{bmatrix}$$

- Ricorsione all'indietro: braccio 2

$$\boldsymbol{f}_2^2 = m_2 \ddot{\boldsymbol{p}}_{C_2}^2$$

$$\boldsymbol{\mu}_2^2 = \begin{bmatrix} * \\ * \\ \bar{I}_{2zz}(\ddot{\vartheta}_1 + \ddot{\vartheta}_2) + m_2(\ell_{C_2} + a_2)^2(\ddot{\vartheta}_1 + \ddot{\vartheta}_2) \\ +m_2 a_1(\ell_{C_2} + a_2)c_2\ddot{\vartheta}_1 \\ +m_2 a_1(\ell_{C_2} + a_2)s_2\dot{\vartheta}_1^2 + m_2(\ell_{C_2} + a_2)gc_{12} \end{bmatrix}$$

$$\begin{aligned} \tau_2 = &\big(\bar{I}_{2zz} + m_2\big((\ell_{C_2} + a_2)^2 + a_1(\ell_{C_2} + a_2)c_2\big) + k_{r2}I_{m_2}\big)\ddot{\vartheta}_1 \\ &+ \big(\bar{I}_{2zz} + m_2(\ell_{C_2} + a_2)^2 + k_{r2}^2 I_{m_2}\big)\,\ddot{\vartheta}_2 \\ &+ m_2 a_1(\ell_{C_2} + a_2)s_2\dot{\vartheta}_1^2 + m_2(\ell_{C_2} + a_2)gc_{12} \end{aligned}$$

- Ricorsione all'indietro: braccio 1

$$\boldsymbol{f}_1^1 = \begin{bmatrix} -m_2(\ell_{C_2}+a_2)s_2(\ddot{\vartheta}_1+\ddot{\vartheta}_2) - m_1(\ell_{C_1}+a_1)\dot{\vartheta}_1^2 - m_2 a_1 \dot{\vartheta}_1^2 \\ -m_2(\ell_{C_2}+a_2)c_2(\dot{\vartheta}_1+\dot{\vartheta}_2)^2 + (m_1+m_2)g s_1 \\ m_1(\ell_{C_1}+a_1)\ddot{\vartheta}_1 + m_2 a_1\ddot{\vartheta}_1 + m_2(\ell_{C_2}+a_2)c_2(\ddot{\vartheta}_1+\ddot{\vartheta}_2) \\ -m_2(\ell_{C_2}+a_2)s_2(\dot{\vartheta}_1+\dot{\vartheta}_2)^2 + (m_1+m_2)g c_1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\mu}_1^1 = \begin{bmatrix} * \\ * \\ \bar{I}_{1zz}\ddot{\vartheta}_1 + m_2 a_1^2\ddot{\vartheta}_1 + m_1(\ell_{C_1}+a_1)^2\ddot{\vartheta}_1 + m_2 a_1(\ell_{C_2}+a_2)c_2\ddot{\vartheta}_1 \\ +\bar{I}_{2zz}(\ddot{\vartheta}_1+\ddot{\vartheta}_2) + m_2 a_1(\ell_{C_2}+a_2)c_2(\ddot{\vartheta}_1+\ddot{\vartheta}_2) \\ +m_2(\ell_{C_2}+a_2)^2(\ddot{\vartheta}_1+\ddot{\vartheta}_2) + k_{r2}I_{m_2}\ddot{\vartheta}_2 \\ +m_2 a_1(\ell_{C_2}+a_2)s_2\dot{\vartheta}_1^2 - m_2 a_1(\ell_{C_2}+a_2)s_2(\dot{\vartheta}_1+\dot{\vartheta}_2)^2 \\ +m_1(\ell_{C_1}+a_1)g c_1 + m_2 a_1 g c_1 + m_2(\ell_{C_2}+a_2)g c_{12} \end{bmatrix}$$

$$\begin{aligned} \tau_1 = \Big(&\bar{I}_{1zz} + m_1(\ell_{C_1}+a_1)^2 + k_{r1}^2 I_{m_1} + \bar{I}_{2zz} \\ &+m_2\left(a_1^2 + (\ell_{C_2}+a_2)^2 + 2a_1(\ell_{C_2}+a_2)c_2\right)\Big)\ddot{\vartheta}_1 \\ &+\Big(\bar{I}_{2zz} + m_2\left((\ell_{C_2}+a_2)^2 + a_1(\ell_{C_2}+a_2)c_2\right) + k_{r2}I_{m_2}\Big)\ddot{\vartheta}_2 \\ &-2m_2 a_1(\ell_{C_2}+a_2)s_2\dot{\vartheta}_1\dot{\vartheta}_2 - m_2 a_1(\ell_{C_2}+a_2)s_2\dot{\vartheta}_2^2 \\ &+\big(m_1(\ell_{C_1}+a_1) + m_2 a_1\big)g c_1 + m_2(\ell_{C_2}+a_2)g c_{12} \end{aligned}$$

- Equivalenza col modello di Lagrange
  - ⋆ parametri di bracci e rotori

$$
\begin{aligned}
m_1 &= m_{\ell_1} + m_{m_2} \\
m_1 \ell_{C_1} &= m_{\ell_1}(\ell_1 - a_1) \\
\bar{I}_{1zz} + m_1 \ell_{C_1}^2 = \hat{I}_1 &= I_{\ell_1} + m_{\ell_1}(\ell_1 - a_1)^2 + I_{m_2} \\
m_2 &= m_{\ell_2} \\
m_2 \ell_{C_2} &= m_{\ell_2}(\ell_2 - a_2) \\
\bar{I}_{2zz} + m_2 \ell_{C_2}^2 = \hat{I}_2 &= I_{\ell_2} + m_{\ell_2}(\ell_2 - a_2)^2
\end{aligned}
$$

DINAMICA DIRETTA E DINAMICA INVERSA

- Formulazione di *Lagrange*
  - ⋆ è *sistematica* e di facile comprensione
  - ⋆ fornisce le equazioni del moto in una *forma analitica compatta* che evidenzia la matrice di inerzia, la matrice a fattore delle forze centrifughe e di Coriolis e il vettore delle forze gravitazionali (utile ai fini della *sintesi del controllo*)
  - ⋆ è efficace se si vogliono portare in conto effetti meccanici più complessi quali ad esempio le deformazioni elastiche dei bracci

- Formulazione di *Newton-Eulero*
  - ⋆ è intrinsecamente un metodo *ricorsivo* che risulta efficiente da un punto di vista computazionale

- Dinamica *diretta*
  - note $\boldsymbol{q}(t_0)$, $\dot{\boldsymbol{q}}(t_0)$, $\boldsymbol{\tau}(t)$ (e $\boldsymbol{h}_e(t)$), determinare $\ddot{\boldsymbol{q}}(t)$, $\dot{\boldsymbol{q}}(t)$, $\boldsymbol{q}(t)$ per $t > t_0$
  - utile in simulazione
- Dinamica *inversa*
  - note $\ddot{\boldsymbol{q}}(t)$, $\dot{\boldsymbol{q}}(t)$, $\boldsymbol{q}(t)$ (e $\boldsymbol{h}_e(t)$), determinare $\boldsymbol{\tau}(t)$
  - utile per la pianificazione e il controllo
- Per un manipolatore ad n giunti, il numero di operazioni per il calcolo della dinamica è:
  - $O(n^2)$ per la *dinamica diretta*
  - $O(n)$ per la *dinamica inversa*

- Dinamica diretta con Lagrange

$$\ddot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})(\boldsymbol{\tau} - \boldsymbol{\tau}')$$

$$\boldsymbol{\tau}'(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}_v\dot{\boldsymbol{q}} + \boldsymbol{f}_s(\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$$

  - ⋆ note $\boldsymbol{q}(t_k)$, $\dot{\boldsymbol{q}}(t_k)$, $\boldsymbol{\tau}(t_k)$, si calcola $\ddot{\boldsymbol{q}}(t_k)$
  - ⋆ si integra numericamente con passo Δt: $\dot{\boldsymbol{q}}(t_{k+1})$, $\boldsymbol{q}(t_{k+1})$

- Dinamica diretta con Newton-Eulero
  - ⋆ note $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, si calcola:

    $\boldsymbol{\tau}'$ come $\boldsymbol{\tau}$ per $\ddot{\boldsymbol{q}} = \boldsymbol{0}$

    colonna $\boldsymbol{b}_i$ di $\boldsymbol{B}$ come $\boldsymbol{\tau}$ per $\boldsymbol{g}_0 = \boldsymbol{0}$, $\dot{\boldsymbol{q}} = \boldsymbol{0}$, $\ddot{q}_i = 1$ e $\ddot{q}_j = 0$ per $j \neq i$
  - ⋆ nota $\boldsymbol{\tau}$, si calcola $\ddot{\boldsymbol{q}}$ e si integra numericamente

MODELLO DINAMICO NELLO SPAZIO OPERATIVO

- Ridondanza
  - ⋆ coordinate generalizzate (moti interni?)
- Singolarità
- Modello nello spazio dei giunti

$$\ddot{\boldsymbol{q}} = -\boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})(\boldsymbol{\gamma} - \boldsymbol{h}_e)$$

- Cinematica differenziale del secondo ordine

$$\begin{aligned}\ddot{\boldsymbol{x}}_e &= \boldsymbol{J}_A(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_A(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \\ &= -\boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{g} + \dot{\boldsymbol{J}}_A\dot{\boldsymbol{q}} + \boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{J}_A^T(\boldsymbol{\gamma}_A - \boldsymbol{h}_A)\end{aligned}$$

- Trasformazioni

$$
\begin{aligned}
\boldsymbol{B}_A &= (\boldsymbol{J}_A \boldsymbol{B}^{-1} \boldsymbol{J}_A^T)^{-1} \\
\boldsymbol{C}_A \dot{\boldsymbol{x}}_e &= \boldsymbol{B}_A \boldsymbol{J}_A \boldsymbol{B}^{-1} \boldsymbol{C} \dot{\boldsymbol{q}} - \boldsymbol{B}_A \dot{\boldsymbol{J}}_A \dot{\boldsymbol{q}} \\
\boldsymbol{g}_A &= \boldsymbol{B}_A \boldsymbol{J}_A \boldsymbol{B}^{-1} \boldsymbol{g}
\end{aligned}
$$

  ⋆ manipolatore non ridondante

$$
\begin{aligned}
\boldsymbol{B}_A &= \boldsymbol{J}_A^{-T} \boldsymbol{B} \boldsymbol{J}_A^{-1} \\
\boldsymbol{C}_A \dot{\boldsymbol{x}}_e &= \boldsymbol{J}_A^{-T} \boldsymbol{C} \dot{\boldsymbol{q}} - \boldsymbol{B}_A \dot{\boldsymbol{J}}_A \dot{\boldsymbol{q}} \\
\boldsymbol{g}_A &= \boldsymbol{J}_A^{-T} \boldsymbol{g}
\end{aligned}
$$

- Equazioni del moto

$$
\boldsymbol{B}_A(\boldsymbol{x}_e)\ddot{\boldsymbol{x}}_e + \boldsymbol{C}_A(\boldsymbol{x}_e, \dot{\boldsymbol{x}}_e)\dot{\boldsymbol{x}}_e + \boldsymbol{g}_A(\boldsymbol{x}_e) = \boldsymbol{\gamma}_A - \boldsymbol{h}_A
$$

- Dinamica diretta
  - ⋆ note $\boldsymbol{\tau}(t)$ (e $\boldsymbol{h}_e(t)$), determinare $\ddot{\boldsymbol{x}}_e(t)$, $\dot{\boldsymbol{x}}_e(t)$, $\boldsymbol{x}_e(t)$
- Soluzione
  - ⋆ dinamica diretta ai giunti
  - ⋆ cinematica diretta

- Dinamica inversa
  - ⋆ note $\boldsymbol{x}_e(t)$, $\dot{\boldsymbol{x}}_e(t)$, $\ddot{\boldsymbol{x}}_e(t)$ (e $\boldsymbol{h}_e(t)$), determinare $\boldsymbol{\tau}(t)$
- Soluzione (ridondanza a livello cinematico)
  - ⋆ cinematica inversa
  - ⋆ dinamica inversa ai giunti
- Soluzione (ridondanza a livello dinamico)
  - ⋆ modello dinamico nello spazio operativo
  - ⋆ coppie con gestione di moti interni

- Dinamica inversa

$$
\begin{aligned}
\boldsymbol{\gamma}_A - \boldsymbol{h}_A &= \boldsymbol{B}_A(\boldsymbol{x}_e)\ddot{\boldsymbol{x}}_e + \boldsymbol{C}_A(\boldsymbol{x}_e, \dot{\boldsymbol{x}}_e)\dot{\boldsymbol{x}}_e + \boldsymbol{g}_A(\boldsymbol{x}_e) \\
&= \boldsymbol{B}_A(\ddot{\boldsymbol{x}}_e - \dot{\boldsymbol{J}}_A\dot{\boldsymbol{q}}) + \boldsymbol{B}_A\boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{B}_A\boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{g} \\
&= \boldsymbol{B}_A\boldsymbol{J}_A\ddot{\boldsymbol{q}} + \boldsymbol{B}_A\boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{B}_A\boldsymbol{J}_A\boldsymbol{B}^{-1}\boldsymbol{g}
\end{aligned}
$$

⋆ $\boldsymbol{J}_A^T$: pseudo-inversa destra pesata secondo $\boldsymbol{B}^{-1}$ di

$$
\bar{\boldsymbol{J}}_A^T(\boldsymbol{q}) = \boldsymbol{B}_A(\boldsymbol{q})\boldsymbol{J}_A(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})
$$

$$\Downarrow$$

$$
\begin{aligned}
\boldsymbol{\gamma}_A - \boldsymbol{h}_A &= \bar{\boldsymbol{J}}_A^T(\boldsymbol{B}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{g}) \\
&= \bar{\boldsymbol{J}}_A^T(\boldsymbol{\tau} - \boldsymbol{J}_A^T\boldsymbol{h}_A)
\end{aligned}
$$

$$
\boldsymbol{\tau} = \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{\gamma}_A + \big(\boldsymbol{I} - \boldsymbol{J}_A^T(\boldsymbol{q})\bar{\boldsymbol{J}}_A^T(\boldsymbol{q})\big)\boldsymbol{\tau}_0
$$

ELLISSOIDE DI MANIPOLABILITÀ DINAMICA

- Sfera nello spazio dei giunti

$$\boldsymbol{\tau}^T\boldsymbol{\tau} = 1$$

  - ⋆ modello dinamico con $\dot{\boldsymbol{q}} = \boldsymbol{0}$, $\boldsymbol{h} = \boldsymbol{0}$

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

  - ⋆ accelerazioni ai giunti ($\dot{\boldsymbol{q}} = \boldsymbol{0}$)

$$\dot{\boldsymbol{v}}_e = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}}$$

$$\Downarrow$$

$$\boldsymbol{\tau} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{J}^{\dagger}(\boldsymbol{q})\dot{\boldsymbol{v}}_e + \boldsymbol{g}(\boldsymbol{q})$$

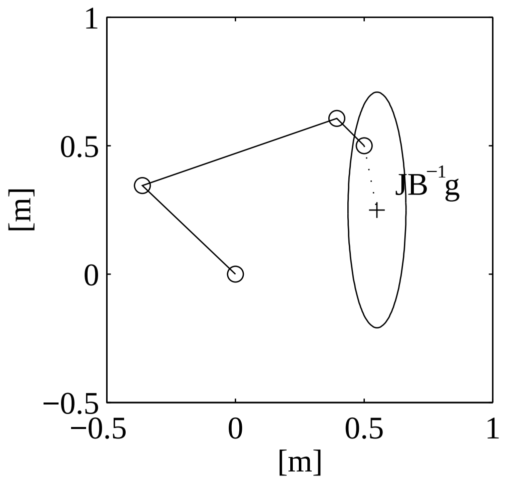
- Ellissoide nello spazio operativo

$$\big(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{J}^{\dagger}(\boldsymbol{q})\dot{\boldsymbol{v}}_e + \boldsymbol{g}(\boldsymbol{q})\big)^T\big(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{J}^{\dagger}(\boldsymbol{q})\dot{\boldsymbol{v}}_e + \boldsymbol{g}(\boldsymbol{q})\big) = 1$$

$$\begin{aligned}\boldsymbol{B}\boldsymbol{J}^{\dagger}\dot{\boldsymbol{v}}_e + \boldsymbol{g} &= \boldsymbol{B}(\boldsymbol{J}^{\dagger}\dot{\boldsymbol{v}}_e + \boldsymbol{B}^{-1}\boldsymbol{g}) \\ &= \boldsymbol{B}(\boldsymbol{J}^{\dagger}\dot{\boldsymbol{v}}_e + \boldsymbol{B}^{-1}\boldsymbol{g} + \boldsymbol{J}^{\dagger}\boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g} - \boldsymbol{J}^{\dagger}\boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g}) \\ &= \boldsymbol{B}\big(\boldsymbol{J}^{\dagger}\dot{\boldsymbol{v}}_e + \boldsymbol{J}^{\dagger}\boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g} + (\boldsymbol{I} - \boldsymbol{J}^{\dagger}\boldsymbol{J})\boldsymbol{B}^{-1}\boldsymbol{g}\big) \\ &= \boldsymbol{B}\boldsymbol{J}^{\dagger}(\dot{\boldsymbol{v}}_e + \boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g})\end{aligned}$$

$$\Downarrow$$

$$(\dot{\boldsymbol{v}}_e + \boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g})^T \boldsymbol{J}^{\dagger T}\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{J}^{\dagger}(\dot{\boldsymbol{v}}_e + \boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g}) = 1$$



⋆ manipolatore non ridondante

$$(\dot{\boldsymbol{v}}_e + \boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g})^T \boldsymbol{J}^{-T}\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{J}^{-1}(\dot{\boldsymbol{v}}_e + \boldsymbol{J}\boldsymbol{B}^{-1}\boldsymbol{g}) = 1$$

SCALATURA DINAMICA DI TRAIETTORIE

⋆ vincoli dinamici (coppie troppo elevate)

$$
\begin{aligned}
\boldsymbol{\tau}(t) &= \boldsymbol{B}(\boldsymbol{q}(t))\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))\dot{\boldsymbol{q}}(t) + \boldsymbol{g}(\boldsymbol{q}(t)) \\
&= \boldsymbol{B}(\boldsymbol{q}(t))\ddot{\boldsymbol{q}}(t) + \boldsymbol{\Gamma}(\boldsymbol{q}(t))[\dot{\boldsymbol{q}}(t)\dot{\boldsymbol{q}}(t)] + \boldsymbol{g}(\boldsymbol{q}(t)) \\
&= \boldsymbol{\tau}_s(t) + \boldsymbol{g}(\boldsymbol{q}(t))
\end{aligned}
$$

$$
\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{\Gamma}(\boldsymbol{q})[\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}]
$$

$$
[\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}] = [\, \dot{q}_1^2 \quad \dot{q}_1\dot{q}_2 \quad \dots \quad \dot{q}_{n-1}\dot{q}_n \quad \dot{q}_n^2 \,]^T
$$

$\star$ scalatura temporale $r(t)$: $\quad r(0) = 0 \quad r(t_f) = \bar{t}_f$

$$\boldsymbol{q}(t) = \bar{\boldsymbol{q}}(r(t))$$
$$\dot{\boldsymbol{q}} = \dot{r}\bar{\boldsymbol{q}}'(r)$$
$$\ddot{\boldsymbol{q}} = \dot{r}^2\bar{\boldsymbol{q}}''(r) + \ddot{r}\bar{\boldsymbol{q}}'(r)$$

$$\Downarrow$$

$$\boldsymbol{\tau} = \dot{r}^2\Big(\boldsymbol{B}(\bar{\boldsymbol{q}}(r))\bar{\boldsymbol{q}}''(r) + \boldsymbol{\Gamma}(\bar{\boldsymbol{q}}(r))[\bar{\boldsymbol{q}}'(r)\bar{\boldsymbol{q}}'(r)]\Big) + \ddot{r}\boldsymbol{B}(\bar{\boldsymbol{q}}(r))\bar{\boldsymbol{q}}'(r) + \boldsymbol{g}(\bar{\boldsymbol{q}}(r))$$

$$= \boldsymbol{\tau}_s(t) + \boldsymbol{g}(\bar{\boldsymbol{q}}(r))$$

$$\bar{\boldsymbol{\tau}}_s(r) = \boldsymbol{B}(\bar{\boldsymbol{q}}(r))\bar{\boldsymbol{q}}''(r) + \boldsymbol{\Gamma}(\bar{\boldsymbol{q}}(r))[\bar{\boldsymbol{q}}'(r)\bar{\boldsymbol{q}}'(r)]$$

$$\Downarrow$$

$$\boldsymbol{\tau}_s(t) = \dot{r}^2\bar{\boldsymbol{\tau}}_s(r) + \ddot{r}\boldsymbol{B}(\bar{\boldsymbol{q}}(r))\bar{\boldsymbol{q}}'(r)$$

$\star$ scelta semplice $r(t) = ct$

$$\boldsymbol{\tau}_s(t) = c^2\bar{\boldsymbol{\tau}}_s(ct)$$

giunto q_i in corrispondenza della violazione maggiore

$$\frac{|\tau_s|}{|\bar{\tau}_i - g(q_i)|} = c^2$$