FOUNDATIONS OF ROBOTICS

Dynamics

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Outline

- Direct Dynamics and Inverse Dynamics
- Euler–Lagrange Formulation
- Properties of Dynamic Model
- Dynamic Model of Simple Manipulator Structures
- Identification of Dynamic Parameters
- Newton–Euler Formulation
- Summary of Dynamic Formulations
- Dynamic Scaling of Trajectories
- Task-Space Dynamic Model
- Constrained Dynamic Model

Direct Dynamics and Inverse Dynamics

- The dynamic model of a robot provides a description of the relationship between the generalized forces $\boldsymbol{\xi}$ acting on the system and the motion of the structure, as described by the time evolution of the generalized coordinates $\boldsymbol{q}$
- In this sense, dynamics connects the causes (input) to their effects (output) on the mechanical system, while kinematics is intended to describe robot motion without accounting for its cause
- Consider for example a 2R planar arm with revolute joints, initially at rest on a horizontal plane (without gravity)
- A desired motion is specified for the second joint as $q_2(t)$, with $t \in [0, T]$, while the first joint should remain at rest
- At a kinematic level, one can generate $\dot{q}_2(t)$ or $\ddot{q}_2(t)$, with $\dot{q}_1(t) = \ddot{q}_1(t) = 0$
- At a dynamic level, one needs a suitable torque $\tau_2(t)$, but also a torque $\tau_1(t)$ to avoid the first link moving due to the inertial couplings

- Availability of a dynamic model is important for the following reasons
  - used at the design stage to optimize the distribution of masses and assess the inertial and gravitational loads acting on the joints and links
  - allows proper choice of the size of the motors to deliver the torque needed for the desired level of velocity/acceleration of the joints or of end-effector tools
  - used as a digital twin for simulating the behavior of a physical robot under different inputs, both in open space and during interaction with the environment
  - used to plan feasible optimal trajectories and interaction forces, and to design control laws that are provably good for achieving the target robot behaviors
- Assuming for simplicity that neither interaction nor dissipative forces are present ($\boldsymbol{\xi} = \boldsymbol{\tau}$), the dynamic model of a robot with n generalized coordinates $\boldsymbol{q}$ can be expressed in the generic form

  $$\boldsymbol{\phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

  a set of n second-order differential equations, with actuating torques $\boldsymbol{\tau} \in I\!R^n$ on the right-hand side
- This form allows formulating the direct dynamics and inverse dynamics problems

- The direct dynamics problem consists in determining, at any time t and with the robot in a state $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$, the joint accelerations $\ddot{\boldsymbol{q}}(t)$ resulting from the application of known joint torques $\boldsymbol{\tau}(t)$
- A simulation of robot motion over a time interval $[t_0, t_f]$ is the repeated solution of the direct dynamics problem with time step T_s, followed by numerical integration of the computed $\ddot{\boldsymbol{q}}(t)$ to obtain $\dot{\boldsymbol{q}}(t + T_s)$ and $\boldsymbol{q}(t + T_s)$
- The experimental solution is obtained by commanding the chosen torque profiles $\boldsymbol{\tau}(t)$ and recording the measured $\boldsymbol{q}(t)$
- If the robot is not available, the solution $\boldsymbol{q}(t)$ can be found by resorting to the numerical simulation of the dynamic model

- The inverse dynamics problem consists in determining, at any time t, the joint torque $\boldsymbol{\tau}(t)$ that is needed to generate a joint acceleration $\ddot{\boldsymbol{q}}(t)$ when the robot is at a configuration $\boldsymbol{q}(t)$ with velocity $\dot{\boldsymbol{q}}(t)$
- The straightforward solution requires the algebraic evaluation of the dynamic model by plugging in $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), \ddot{\boldsymbol{q}}(t))$
- Two main formulations are considered
  - Euler–Lagrange (EL) method, an energy-based approach
  - Newton–Euler (NE) method, based on the inertial balance of forces and moments along the structure

Euler–Lagrange Formulation

- In the EL formulation, the equations of motion are derived in a systematic way from energy arguments
- Once a set of generalized coordinates $\boldsymbol{q} = (q_1, \ldots, q_n)$ has been chosen, the Lagrangian function of the mechanical system is defined as

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q})$$

  where $\mathcal{T}$ is the kinetic energy and $\mathcal{U}$ is the potential energy of the system
- EL equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \xi_i \qquad i = 1, \ldots, n$$

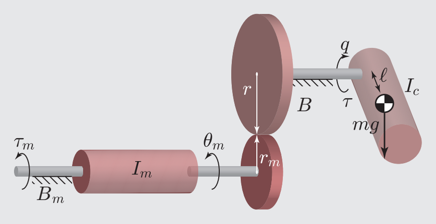
  where ξ_i is the nonconservative generalized force that affects coordinate q_i
- Vector form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{\xi}$$

Example

- Single link with gear
  - reduction ratio

$$n_r = \frac{r}{r_m} = \frac{\dot\theta_m}{\dot q} > 1$$



  - energies

$$\mathcal{T} = \frac{1}{2} I \dot q^2 + \frac{1}{2} I_m \dot\theta_m^2 = \frac{1}{2}\left(I + I_m n_r^2\right) \dot q^2$$

$$\mathcal{U} = mgh = mg\ell(1 - \cos q) \qquad \Rightarrow \qquad \mathcal{L} = \frac{1}{2}\left(I + I_m n_r^2\right) \dot q^2 - mg\ell(1 - \cos q)$$

$$\left(I + I_m n_r^2\right) \ddot q + mg\ell \sin q = \xi \qquad \text{with} \qquad \xi = \tau - \left(B + B_m n_r^2\right) \dot q$$

$$\left(I + I_m n_r^2\right) \ddot q + mg\ell \sin q + \left(B + B_m n_r^2\right) \dot q = \tau$$

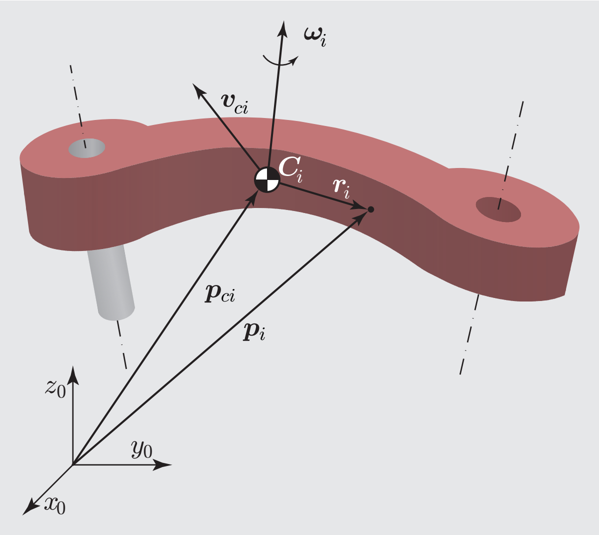
Kinetic Energy

- Assume that all elements of the manipulator (links, motors, transmissions) are rigid
- The total kinetic energy is given by the sum of the contributions due to the motion of each link and of each actuator

$$\mathcal{T} = \sum_{i=1}^{n} (\mathcal{T}_i + \mathcal{T}_{mi})$$

  - $\mathcal{T}_i$ is the kinetic energy of link i
  - $\mathcal{T}_{mi}$ is the kinetic energy of the corresponding motor i

- Kinematic description of link for kinetic energy computation



- Link contributions
- The generic i-th link is a body $\mathcal{B}_i$ with mass

$$m_i = \int_{\mathcal{B}_i} dm$$

- The position $\boldsymbol{p}_{ci} \in \mathbb{R}^3$ of its center of mass (CoM) is

$$\boldsymbol{p}_{ci} = \frac{1}{m_i} \int_{\mathcal{B}_i} \boldsymbol{p}_i \, dm$$

  - $\boldsymbol{p}_i \in \mathbb{R}^3$ is the position of an elementary particle of mass dm in the body $\mathcal{B}_i$
  - all vectors are expressed in the base frame of the robot

- The kinetic energy of link i is

$$\mathcal{T}_i = \frac{1}{2}\int_{\mathcal{B}_i} \boldsymbol{v}_i^T \boldsymbol{v}_i \, dm$$

  where $\boldsymbol{v}_i = \dot{\boldsymbol{p}}_i$ is the velocity of the elementary particle of mass dm
- This velocity can be expressed as

$$\boldsymbol{v}_i = \boldsymbol{v}_{ci} + \boldsymbol{\omega}_i \times \boldsymbol{r}_i = \boldsymbol{v}_{ci} + \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i$$

  - $\boldsymbol{v}_{ci} = \dot{\boldsymbol{p}}_{ci}$
  - $\boldsymbol{\omega}_i$ is the angular velocity of link i
  - $\boldsymbol{r}_i = \boldsymbol{p}_i - \boldsymbol{p}_{ci}$

- The translational contribution is

$$\frac{1}{2}\int_{\mathcal{B}_i} \boldsymbol{v}_{ci}^T \boldsymbol{v}_{ci}\, dm = \frac{1}{2} m_i \boldsymbol{v}_{ci}^T \boldsymbol{v}_{ci}$$

- The mutual contribution is

$$\frac{1}{2}\left(2\int_{\mathcal{B}_i} \boldsymbol{v}_{ci}^T \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i\, dm\right) = \boldsymbol{v}_{ci}^T \boldsymbol{S}(\boldsymbol{\omega}_i)\int_{\mathcal{B}_i} (\boldsymbol{p}_i - \boldsymbol{p}_{ci})\, dm = 0$$

  being

$$\int_{\mathcal{B}_i} \boldsymbol{p}_i\, dm = \int_{\mathcal{B}_i} dm$$

- The rotational contribution is

$$\frac{1}{2}\int_{\mathcal{B}_i} \boldsymbol{r}_i^T \boldsymbol{S}^T(\boldsymbol{\omega}_i)\boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i\, dm = \frac{1}{2}\boldsymbol{\omega}_i^T \left(\int_{\mathcal{B}_i} \boldsymbol{S}^T(\boldsymbol{r}_i)\boldsymbol{S}(\boldsymbol{r}_i)\, dm\right)\boldsymbol{\omega}_i$$

- The angular velocity of link i is expressed in a frame attached to the link

$$ {}^{i}\boldsymbol{\omega}_i = \boldsymbol{R}_i^T(\boldsymbol{q})\,\boldsymbol{\omega}_i $$

$$ \boldsymbol{\omega}_i^T \boldsymbol{S}^T(\boldsymbol{r}_i)\boldsymbol{S}(\boldsymbol{r}_i)\boldsymbol{\omega}_i = {}^{i}\boldsymbol{\omega}_i^T \boldsymbol{S}^T({}^{i}\boldsymbol{r}_i)\boldsymbol{S}({}^{i}\boldsymbol{r}_i){}^{i}\boldsymbol{\omega}_i $$

  where property $\boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{r}_i = -\boldsymbol{S}(\boldsymbol{r}_i)\boldsymbol{\omega}_i$ has been used

- Vector ${}^{i}\boldsymbol{r}_i = (r_{ix}, r_{iy}, r_{iz})$ does not depend on $\boldsymbol{q}$

$$ \boldsymbol{S}({}^{i}\boldsymbol{r}_i) = \begin{pmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{pmatrix} $$

- The rotational contribution can be written as

$$\frac{1}{2}\,{}^i\boldsymbol{\omega}_i^T\left(\int_{\mathcal{B}_i}\boldsymbol{S}^T({}^i\boldsymbol{r}_i)\boldsymbol{S}({}^i\boldsymbol{r}_i)\,dm\right){}^i\boldsymbol{\omega}_i=\frac{1}{2}\,{}^i\boldsymbol{\omega}_i^T\,{}^i\boldsymbol{I}_i\,{}^i\boldsymbol{\omega}_i$$

- The constant symmetric matrix represents the inertia tensor relative to the CoM

$$\begin{aligned}{}^i\boldsymbol{I}_i &= \begin{pmatrix}\int\left(r_{iy}^2+r_{iz}^2\right)dm & -\int r_{ix}r_{iy}\,dm & -\int r_{ix}r_{iz}\,dm\\ * & \int\left(r_{ix}^2+r_{iz}^2\right)dm & -\int r_{iy}r_{iz}\,dm\\ * & * & \int\left(r_{ix}^2+r_{iy}^2\right)dm\end{pmatrix}\\ &= \begin{pmatrix}I_{ixx} & I_{ixy} & I_{ixz}\\ * & I_{iyy} & I_{iyz}\\ * & * & I_{izz}\end{pmatrix}\end{aligned}$$

- Properties of the link inertia tensor
  - it is the inertia tensor relative to the CoM of link i, expressed in a frame attached to the link
  - if the frame axes coincide with the central axes of inertia, the off-diagonal terms are zero and ${}^{i}\boldsymbol{I}_i$ is diagonal
  - the rotational contribution can also be written in the base frame as

$$\frac{1}{2}\,{}^{i}\boldsymbol{\omega}_i^T\,{}^{i}\boldsymbol{I}_i\,{}^{i}\boldsymbol{\omega}_i = \frac{1}{2}\,\boldsymbol{\omega}_i^T\,\boldsymbol{I}_i\,\boldsymbol{\omega}_i$$

with the link inertia tensor in the base frame given by

$$\boldsymbol{I}_i = \boldsymbol{R}_i(\boldsymbol{q})\,{}^{i}\boldsymbol{I}_i\,\boldsymbol{R}_i^T(\boldsymbol{q})$$

- The kinetic energy of link i is

$$\mathcal{T}_i = \frac{1}{2}\,m_i\,\boldsymbol{v}_{ci}^T\boldsymbol{v}_{ci} + \frac{1}{2}\,\boldsymbol{\omega}_i^T\,\boldsymbol{I}_i\,\boldsymbol{\omega}_i$$

- The CoM linear velocity and the link angular velocity can be written as

$$\boldsymbol{v}_{ci} = \boldsymbol{\jmath}_{p1}^{i}\dot{q}_1 + \cdots + \boldsymbol{\jmath}_{pi}^{i}\dot{q}_i = \boldsymbol{J}_p^i\,\dot{\boldsymbol{q}}$$
$$\boldsymbol{\omega}_i = \boldsymbol{\jmath}_{o1}^{i}\dot{q}_1 + \cdots + \boldsymbol{\jmath}_{oi}^{i}\dot{q}_i = \boldsymbol{J}_o^i\,\dot{\boldsymbol{q}}$$

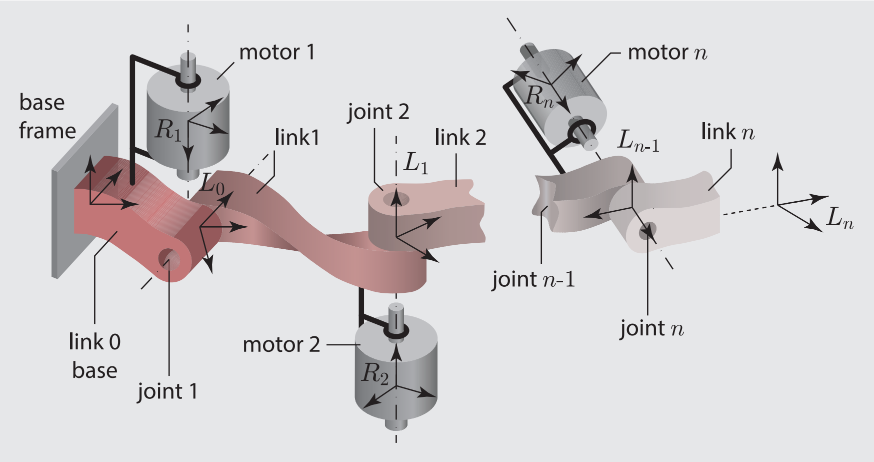
- By construction, the last $n-i$ columns of $\boldsymbol{J}_p^i$ and $\boldsymbol{J}_o^i$ are zero

$$\begin{pmatrix} \boldsymbol{\jmath}_{pj}^{i} \\ \boldsymbol{\jmath}_{oj}^{i} \end{pmatrix} = \begin{cases} \begin{pmatrix} \boldsymbol{z}_{j-1} \\ \boldsymbol{0} \end{pmatrix} & \text{for a prismatic joint} \\ \begin{pmatrix} \boldsymbol{z}_{j-1} \times (\boldsymbol{p}_{ci} - \boldsymbol{p}_{j-1}) \\ \boldsymbol{z}_{j-1} \end{pmatrix} & \text{for a revolute joint} \end{cases}$$

- Making explicit the dependence on $\boldsymbol{q}$, the kinetic energy is written as

$$\mathcal{T}_i = \frac{1}{2}\,m_i\,\dot{\boldsymbol{q}}^T \boldsymbol{J}_p^{i\,T}(\boldsymbol{q})\boldsymbol{J}_p^i(\boldsymbol{q})\,\dot{\boldsymbol{q}} + \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{J}_o^{i\,T}(\boldsymbol{q})\boldsymbol{R}_i(\boldsymbol{q})\,{}^{i}\boldsymbol{I}_i\,\boldsymbol{R}_i^T(\boldsymbol{q})\,\boldsymbol{J}_o^i(\boldsymbol{q})\,\dot{\boldsymbol{q}}$$

- Motor contributions



- Consider n electric motors, each driving a single joint
- Standard assumptions
  A1. the rotors are uniform bodies having their CoM on the rotation axis
  A2. motor i moves link i and is mounted before the link itself, usually on link $i-1$
  A3. the angular velocity of each rotor is only due to its own spinning
- The static mass of each motor can be included in the mass of the carrying link, modifying the link CoM and inertia
- The transmission is rigid, so the rotor speed is proportional to the joint speed

$$\dot{\theta}_{mi} = n_{ri}\dot{q}_i$$

- The angular kinetic energy of rotor i is

$$\mathcal{T}_{mi} = \frac{1}{2} I_{mi} n_{ri}^2 \, \dot{q}_i^2$$

where I_{mi} is the rotor inertia about its spinning axis

- The total kinetic energy of a manipulator is written as the quadratic form

$$\mathcal{T} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- $\boldsymbol{M}(\boldsymbol{q})$ is the $n \times n$ robot inertia matrix, which can be decomposed as

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}_l(\boldsymbol{q}) + \boldsymbol{M}_m$$

- Configuration-dependent inertia matrix of the links

$$\boldsymbol{M}_l(\boldsymbol{q}) = \sum_{i=1}^{n}\left(m_i\boldsymbol{J}_p^{i\,T}(\boldsymbol{q})\boldsymbol{J}_p^i(\boldsymbol{q}) + \boldsymbol{J}_o^{i\,T}(\boldsymbol{q})\boldsymbol{R}_i(\boldsymbol{q})^i\boldsymbol{I}_i\,\boldsymbol{R}_i^T(\boldsymbol{q})\,\boldsymbol{J}_o^i(\boldsymbol{q})\right)$$

- Constant inertia mastrix of the motors

$$\boldsymbol{M}_m = \boldsymbol{N}_r\boldsymbol{I}_m\boldsymbol{N}_r \qquad \boldsymbol{N}_r = \mathrm{diag}\{n_{r1},\dots,n_{rn}\} \qquad \boldsymbol{I}_m = \mathrm{diag}\{I_{m1},\dots,I_{mn}\}$$

- Properties of robot inertia matrix
  - $\boldsymbol{M}(\boldsymbol{q})$ is symmetric
  - $\boldsymbol{M}(\boldsymbol{q})$ is positive definite
  - for serial manipulators, $\boldsymbol{M}(\boldsymbol{q})$ is independent of the first joint variable q_1
  - for serial manipulators, the element m_{nn} is constant
  - efficient computation can be obtained using a recursion for the link velocities

Potential Energy

- Under the rigid-structure assumption, the potential energy $\mathcal{U}$ is only due to gravity
- The total potential energy is the sum of the link contributions

$$\mathcal{U} = \sum_{i=1}^{n} \mathcal{U}_i$$

- The potential energy of the motors does not appear explicitly because each motor mass is included in the mass of the link on which it is mounted
  - due to assumption A1, when the motors are spinning the CoMs of the rotors do not change position (height in the gravity potential field)

- The potential energy of link i is

$$\mathcal{U}_i = -\int_{\mathcal{B}_i} \boldsymbol{g}_0^T \boldsymbol{p}_i \, dm = -m_i \, \boldsymbol{g}_0^T \boldsymbol{p}_{ci}$$

  - $\boldsymbol{g}_0$ is the gravity acceleration vector expressed in the base frame
  - $\boldsymbol{g}_0 = (0, 0, -g)$ when the upward vertical axis is z_0

- The total potential energy can be written as

$$\mathcal{U} = -\sum_{i=1}^{n} m_i \, \boldsymbol{g}_0^T \boldsymbol{p}_{ci} + \mathcal{U}_0$$

  - $\mathcal{U}$ depends on the joint variables $\boldsymbol{q}$ through the positions $\boldsymbol{p}_{ci}$
  - $\mathcal{U}_0$ is a constant used to set the zero level of potential energy at a convenient configuration

Equations of Motion

- Lagrangian

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q})$$

- The Euler–Lagrange derivatives are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T &= \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} \\ \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T &= \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T - \boldsymbol{g}(\boldsymbol{q}) \end{aligned}$$

with the gravitational vector

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{q}}\right)^T = \nabla_{\boldsymbol{q}} \mathcal{U}(\boldsymbol{q})$$

- The dynamic model is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\xi}$$

with the velocity vector

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T$$

- Componentwise, we have for $i = 1, \ldots, n$

$$\begin{aligned} c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \sum_{j=1}^{n} \frac{dm_{ij}}{dt}\,\dot{q}_j - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial m_{jk}}{\partial q_i}\,\dot{q}_j\dot{q}_k \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k \end{aligned}$$

$$g_i(\boldsymbol{q}) = -\sum_{j=1}^{n} m_j\,\boldsymbol{g}_0^T\,\frac{\partial \boldsymbol{p}_{cj}}{\partial q_i} = -\sum_{j=1}^{n} m_j\,\boldsymbol{g}_0^T\,\boldsymbol{j}_{pi}^{j}(\boldsymbol{q})$$

- Putting all terms together

$$\sum_{j=1}^{n} m_{ij}(\boldsymbol{q})\,\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}(\boldsymbol{q})\,\dot{q}_j\dot{q}_k + g_i(\boldsymbol{q}) = \xi_i \qquad i = 1, \ldots, n$$

where the coefficients $c_{ijk}(\boldsymbol{q})$ are the Christoffel symbols of the first kind

$$c_{ijk}(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i}\right)$$

- The model is linear in $\ddot{\boldsymbol{q}}$, quadratic in $\dot{\boldsymbol{q}}$, and nonlinear in $\boldsymbol{q}$

- Inertial terms
  - $m_{ii}(\boldsymbol{q})$ is the inertia seen at joint i with all other joints blocked
  - $m_{ij}(\boldsymbol{q})$ is inertial coupling between joints i and j, with $m_{ji} = m_{ij}$
- Velocity terms
  - $c_{ijj}(\boldsymbol{q})\dot{q}_j^2$ accounts for centrifugal effect on joint i due to joint j
  - $c_{iii} = 0$ since $\partial m_{ii}/\partial q_i = 0$
  - $c_{ijk}(\boldsymbol{q})\dot{q}_j\dot{q}_k$ accounts for Coriolis effect on joint i due to joints j and k
  - $c_{ijk} = c_{ikj}$
- Gravitational terms
  - $g_i(\boldsymbol{q})$ accounts for gravity effect on joint i
- Nonconservative terms
  - ξ_i in the right-hand side includes active and dissipative torques performing work on q_i

- Nonconservative torques

$$\boldsymbol{\xi} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{int}} + \boldsymbol{\tau}_B$$

  - $\boldsymbol{\tau}$: actuator torques produced by the motors
  - $\boldsymbol{\tau}_{\text{int}}$: interaction torques resulting from interaction forces and moments with the environment
  - $\boldsymbol{\tau}_B$: dissipative torques due to friction
- Viscous friction $\boldsymbol{\tau}_{B,v} = -\boldsymbol{B}_v \dot{\boldsymbol{q}}$
  - when considering viscous friction on both the motor and the link side of the transmission

$$\boldsymbol{B}_v = \boldsymbol{B}_j + \boldsymbol{N}_r \boldsymbol{B}_m \boldsymbol{N}_r$$

- Coulomb friction $\boldsymbol{\tau}_{B,c} = -\boldsymbol{B}_c \,\textbf{sign}\left(\dot{\boldsymbol{q}}\right)$
- Static friction $\boldsymbol{\tau}_{B,s}$
- $\boldsymbol{B}_v$ and $\boldsymbol{B}_c$ are diagonal with nonnegative elements

$$\Downarrow$$

$$\boldsymbol{\xi} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{int}} + \boldsymbol{\tau}_B = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{int}} - \boldsymbol{B}_v \dot{\boldsymbol{q}} - \boldsymbol{B}_c \,\textbf{sign}\left(\dot{\boldsymbol{q}}\right) - \boldsymbol{\tau}_{B,s}$$

- Resulting equations of motion

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{int}} + \boldsymbol{\tau}_B$$

- In the absence of interaction forces and friction

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

- Furthermore, for $i = 1, \ldots, n$

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad i = 1, \ldots, n$$

  with the symmetric matrix $\boldsymbol{C}_i(\boldsymbol{q})$ given by

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_i}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_i}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_i}\right)\right)$$

  where $\boldsymbol{m}_i$ is the i-th column of $\boldsymbol{M}$

- Constant elements in $\boldsymbol{M}(\boldsymbol{q})$ do not contribute to $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Closed Kinematic Chains

- A closed-chain manipulator can be viewed through an equivalent tree-structured open-chain robot with n joints
- The closed-chain robot has m DoFs and $k = n - m$ independent constraints
- Assume the constraints can be explicitly solved as

$$\boldsymbol{\varrho} = \boldsymbol{\varrho}(\boldsymbol{q})$$

  - $\boldsymbol{q} \in I\!R^m$ independent joint variables
  - $\boldsymbol{\varrho} \in I\!R^k$ dependent joint variables

- Equivalent open-chain joint vector

$$\boldsymbol{q}_o = (\boldsymbol{q}, \boldsymbol{\varrho}) \in I\!R^n$$

- Assume also that only the independent joints $\boldsymbol{q}$ are actuated

- Consider a pair of twin robots
  - the closed-chain robot actuated only in $\boldsymbol{q}$
  - an equivalent open-chain robot fully actuated in $\boldsymbol{q}_o$
- Neglect dissipative and interaction forces
- Constraint forces perform no virtual work on admissible virtual displacements

$$\boldsymbol{\tau}^T \delta \boldsymbol{q} = \boldsymbol{\tau}_o^T \delta \boldsymbol{q}_o$$

  - $\boldsymbol{\tau} \in \mathbb{R}^m$ torques for the closed-chain robot
  - $\boldsymbol{\tau}_o \in \mathbb{R}^n$ torques for the open-chain twin

- Generalized velocities are related by

$$\dot{\boldsymbol{q}}_o = \boldsymbol{\Upsilon}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

with

$$\boldsymbol{\Upsilon}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{I} \\ \dfrac{\partial \boldsymbol{\varrho}(\boldsymbol{q})}{\partial \boldsymbol{q}} \end{pmatrix}$$

and hence $\delta \boldsymbol{q}_o = \boldsymbol{\Upsilon}(\boldsymbol{q})\delta \boldsymbol{q}$

- Substituting $\delta \boldsymbol{q}_o = \boldsymbol{\Upsilon}(\boldsymbol{q})\delta \boldsymbol{q}$ into the virtual-work equality

$$\boldsymbol{\tau}^T \delta \boldsymbol{q} = \boldsymbol{\tau}_o^T \boldsymbol{\Upsilon}(\boldsymbol{q})\delta \boldsymbol{q}$$

yields the torque relationship

$$\boldsymbol{\tau} = \boldsymbol{\Upsilon}^T(\boldsymbol{q})\boldsymbol{\tau}_o$$

- The torque of the equivalent fully actuated open-chain robot is

$$\boldsymbol{\tau}_o = \boldsymbol{M}_o(\boldsymbol{q}_o)\ddot{\boldsymbol{q}}_o + \boldsymbol{c}_o(\boldsymbol{q}_o, \dot{\boldsymbol{q}}_o) + \boldsymbol{g}_o(\boldsymbol{q}_o)$$

- Use the time derivative of $\dot{\boldsymbol{q}}_o = \boldsymbol{\Upsilon}(\boldsymbol{q})\dot{\boldsymbol{q}}$

$$\ddot{\boldsymbol{q}}_o = \boldsymbol{\Upsilon}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Upsilon}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Combine torque mapping and open-chain dynamics

$$\boldsymbol{\tau} = \boldsymbol{\Upsilon}^T(\boldsymbol{q})\boldsymbol{M}_o(\boldsymbol{q}_o)\boldsymbol{\Upsilon}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{\Upsilon}^T(\boldsymbol{q})\left(\boldsymbol{M}_o(\boldsymbol{q}_o)\dot{\boldsymbol{\Upsilon}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{c}_o(\boldsymbol{q}_o, \dot{\boldsymbol{q}}_o) + \boldsymbol{g}_o(\boldsymbol{q}_o)\right)$$

- Using $\boldsymbol{q}_o = (\boldsymbol{q}, \boldsymbol{\varrho}(\boldsymbol{q}))$ and $\dot{\boldsymbol{q}}_o = \boldsymbol{\Upsilon}(\boldsymbol{q})\dot{\boldsymbol{q}}$, this expression can be rewritten using only independent state variables $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

- Closed-chain dynamic model
  - add motor inertia for the m actuated joints as an additive term
  - equations of motion in m independent coordinates

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

where

$$\begin{aligned}
\boldsymbol{M}(\boldsymbol{q}) &= \boldsymbol{\Upsilon}^T(\boldsymbol{q})\boldsymbol{M}_o(\boldsymbol{q})\boldsymbol{\Upsilon}(\boldsymbol{q}) + \boldsymbol{M}_m \\
\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{\Upsilon}^T(\boldsymbol{q})\left(\boldsymbol{c}_o(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{M}_o(\boldsymbol{q})\dot{\boldsymbol{\Upsilon}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\
\boldsymbol{g}(\boldsymbol{q}) &= \boldsymbol{\Upsilon}^T(\boldsymbol{q})\,\boldsymbol{g}_o(\boldsymbol{q})
\end{aligned}$$

with the (reflected) motor inertias

$$\boldsymbol{M}_m = \boldsymbol{N}_r\boldsymbol{I}_m\boldsymbol{N}_r$$

Properties of Dynamic Model

- Two notable properties of the dynamic model are presented
  - skew-symmetry of a special matrix, for a suitable factorization of the velocity terms
  - linearity of the dynamic model in terms of a set of parameters
- These properties are useful for dynamic parameter identification and for designing control algorithms

Skew-Symmetry and Passivity

- The quadratic dependence on $\dot{\boldsymbol{q}}$ of $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$ implies

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

  with infinitely many valid factorizations, each with a corresponding matrix $\boldsymbol{N}$

- The particular choice

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \qquad \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T\boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T\boldsymbol{C}_2(\boldsymbol{q}) \\ \vdots \\ \dot{\boldsymbol{q}}^T\boldsymbol{C}_n(\boldsymbol{q}) \end{pmatrix} \qquad c_{ik}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \sum_{j=1}^{n} c_{ijk}(\boldsymbol{q})\,\dot{q}_j$$

  with Christoffel symbols c_{ijk}, has the property that following matrix is skew-symmetric

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\,\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$$

$$\Rightarrow \quad \boldsymbol{x}^T\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\,\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\boldsymbol{x} = 0 \quad \text{for all } \ \boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{x} \in I\!R^n$$

- Proof

$$\dot{m}_{ij}(\boldsymbol{q}) = \sum_{k=1}^{n} \frac{\partial m_{ij}}{\partial q_k}\, \dot{q}_k$$

$$2\, c_{ij}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{k=1}^{n} 2\, c_{ikj}(\boldsymbol{q})\, \dot{q}_k = \sum_{k=1}^{n} \left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i} \right) \dot{q}_k$$

$$s_{ij}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{m}_{ij}(\boldsymbol{q}) - 2\, c_{ij}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{k=1}^{n} \left(\frac{\partial m_{kj}}{\partial q_i} - \frac{\partial m_{ik}}{\partial q_j} \right) \dot{q}_k$$

$$s_{ji}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{k=1}^{n} \left(\frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i} \right) \dot{q}_k = -s_{ij}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- An exchange property is also valid in this case

$$\boldsymbol{C}(\boldsymbol{q}, \boldsymbol{v})\boldsymbol{w} = \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{v} \quad \text{for all } \ \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$$

- Principle of energy conservation
  - total energy (Hamiltonian function)

$$\mathcal{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathcal{U}(\boldsymbol{q}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \mathcal{U}(\boldsymbol{q})$$

  - its time derivative in the absence of nonconservative torques is

$$\begin{aligned}\dot{\mathcal{H}} &= \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\,\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} \\ &= -\,\dot{\boldsymbol{q}}^T \left(\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})\right) + \frac{1}{2}\,\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}) \\ &= \frac{1}{2}\,\dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}}\end{aligned}$$

- $\dot{\mathcal{H}} = 0$ is equivalent to

$$\dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \right) \dot{\boldsymbol{q}} = 0 \quad \text{for all} \;\; \boldsymbol{q}, \dot{\boldsymbol{q}} \in I\!R^n$$

- This identity is related to the passivity property of the robot dynamics
- It is weaker than the skew-symmetry property obtained for $\boldsymbol{N} = \boldsymbol{C}$
- In the presence of nonconservative torques $\boldsymbol{\xi}$

$$\dot{\mathcal{H}} = \dot{\boldsymbol{q}}^T \boldsymbol{\xi}$$

- In particular
    - when $\boldsymbol{\tau} = \boldsymbol{0}$ and $\boldsymbol{\xi} = -\boldsymbol{B}_v \dot{\boldsymbol{q}}$, then $\dot{\mathcal{H}} = -\dot{\boldsymbol{q}}^T \boldsymbol{B}_v \dot{\boldsymbol{q}} \leq 0$ (pure dissipation of energy)
    - when $\boldsymbol{\tau} \neq \boldsymbol{0}$ and there is no friction, then $\dot{\mathcal{H}} = \dot{\boldsymbol{q}}^T \boldsymbol{\tau}$

Linearity in the Dynamic Parameters

- To consider both types of joints, a pivot point $\boldsymbol{p}_{pi}$ is conveniently defined as

$$\boldsymbol{p}_{pi} = \begin{cases} \boldsymbol{p}_i & \text{for a prismatic joint} \\ \boldsymbol{p}_{i-1} & \text{for a revolute joint} \end{cases}$$

- Velocity composition rule

$$\boldsymbol{v}_{ci} = \boldsymbol{v}_{pi} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{pi} = \boldsymbol{v}_{pi} - \boldsymbol{S}(\boldsymbol{r}_{pi})\boldsymbol{\omega}_i \qquad \text{with} \qquad \boldsymbol{r}_{pi} = \boldsymbol{p}_{ci} - \boldsymbol{p}_{pi}$$

- Constant vector both in the case of prismatic and revolute joints

$${}^{i}\boldsymbol{r}_{pi} = \begin{pmatrix} r_{pi,x} & r_{pi,y} & r_{pi,z} \end{pmatrix}^T = \boldsymbol{R}_i^T \boldsymbol{r}_{pi}$$

- The kinetic energy of link i can be rewritten using the pivot point as

$$\mathcal{T}_i = \frac{1}{2} m_i \boldsymbol{v}_{pi}^T \boldsymbol{v}_{pi} + \boldsymbol{\omega}_i^T \boldsymbol{R}_i \boldsymbol{S}(m_i {}^{i}\boldsymbol{r}_{pi}) \boldsymbol{R}_i^T \boldsymbol{v}_{pi} + \frac{1}{2} \boldsymbol{\omega}_i^T \boldsymbol{R}_i {}^{i}\bar{\boldsymbol{I}}_i \boldsymbol{R}_i^T \boldsymbol{\omega}_i$$

- Constant quantities in $\mathcal{T}_i$
  - the link mass m_i, a scalar known as the zero-th moment of inertia
  - the first moment of inertia

$$m_i\, {}^i\boldsymbol{r}_{pi} = \begin{pmatrix} m_i\, r_{pi,x} \\ m_i\, r_{pi,y} \\ m_i\, r_{pi,z} \end{pmatrix}$$

  - the inertia tensor ${}^i\bar{\boldsymbol{I}}_i$ referred to $\boldsymbol{p}_{pi}$
- According to Steiner theorem, the second moment of inertia can be expressed as

$$\begin{aligned}
{}^i\bar{\boldsymbol{I}}_i &= {}^i\boldsymbol{I}_i + m_i\, \boldsymbol{S}^T({}^i\boldsymbol{r}_{pi})\boldsymbol{S}({}^i\boldsymbol{r}_{pi}) \\
&= \begin{pmatrix} I_{ixx} + m_i(r_{di,y}^2 + r_{di,z}^2) & I_{ixy} - m_i r_{di,x} r_{di,y} & I_{ixz} - m_i r_{di,x} r_{di,z} \\ * & I_{iyy} + m_i(r_{di,x}^2 + r_{di,z}^2) & I_{iyz} - m_i r_{di,y} r_{di,z} \\ * & * & I_{izz} + m_i(r_{di,x}^2 + r_{di,y}^2) \end{pmatrix} \\
&= \begin{pmatrix} \bar{I}_{ixx} & \bar{I}_{ixy} & \bar{I}_{ixz} \\ * & \bar{I}_{iyy} & \bar{I}_{iyz} \\ * & * & \bar{I}_{izz} \end{pmatrix}
\end{aligned}$$

- Link kinetic energy $\mathcal{T}_i$ is linear in $n_\ell = 10$ standard dynamic parameters
- Link potential energy $\mathcal{U}_i$ is also linear in a subset of these parameters

$$\mathcal{U}_i = -\boldsymbol{g}_0^T \left(m_i\, \boldsymbol{p}_{pi} + \boldsymbol{R}_i\, m_i\, {}^i\boldsymbol{r}_{pi} \right)$$

- Contributions by motors to kinetic and potential energy keep the same linearity
- Lagrangian in linearly parametrized form

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \sum_{i=1}^{n} \left(\boldsymbol{\beta}_{\mathcal{T}_i}^T - \boldsymbol{\beta}_{\mathcal{U}_i}^T \right) \boldsymbol{\pi}_i$$

  where $\boldsymbol{\beta}_{\mathcal{T}_i}$ and $\boldsymbol{\beta}_{\mathcal{U}_i}$ are suitable vector functions of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$
- Vector of $n_\ell + 1 = 11$ dynamic parameters of link and motor i

$$\boldsymbol{\pi}_i = \begin{pmatrix} m_i & m_i r_{di,x} & m_i r_{di,y} & m_i r_{di,z} & \bar{I}_{ixx} & \bar{I}_{ixy} & \bar{I}_{ixz} & \bar{I}_{iyy} & \bar{I}_{iyz} & \bar{I}_{izz} & \bar{I}_{mi} \end{pmatrix}^T$$

  with $\bar{I}_{mi} = I_{mi} n_{ri}^2$

- The i-th dynamic equation can be written as

$$\sum_{j=1}^{n} \boldsymbol{y}_{ij}^T \boldsymbol{\pi}_j = \xi_i$$

$$\boldsymbol{y}_{ij} = \frac{d}{dt}\frac{\partial \boldsymbol{\beta}_{\mathcal{T}j}}{\partial \dot{q}_i} - \frac{\partial \boldsymbol{\beta}_{\mathcal{T}j}}{\partial q_i} + \frac{\partial \boldsymbol{\beta}_{\mathcal{U}j}}{\partial q_i}$$

- Since derivatives vanish for $j < i$, the following upper block-triangular structure is obtained

$$\begin{pmatrix} \boldsymbol{y}_{11}^T & \boldsymbol{y}_{12}^T & \dots & \boldsymbol{y}_{1n}^T \\ \mathbf{0}^T & \boldsymbol{y}_{22}^T & \dots & \boldsymbol{y}_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \dots & \boldsymbol{y}_{nn}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \\ \vdots \\ \boldsymbol{\pi}_n \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$

- When dissipative terms are included, check linearity in terms of additional parameters
- Summarizing, when link, motor, and dissipative coefficients are all included, the robot dynamic model can be rewritten as

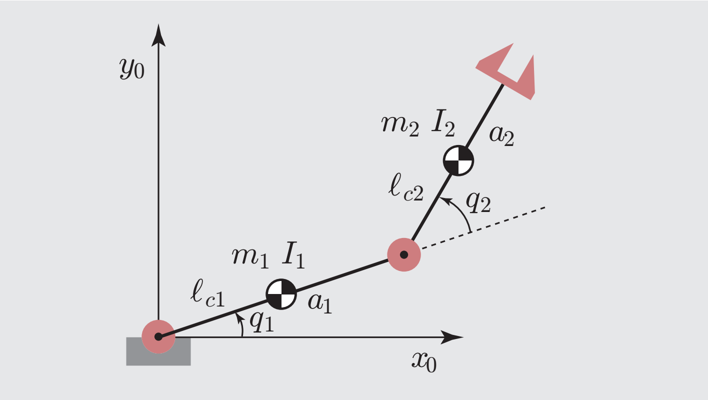
$$\boldsymbol{Y}_{\boldsymbol{\pi}}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right) \boldsymbol{\pi} = \boldsymbol{\tau}$$

- The dynamic model is linear with respect to a suitable set of parameters
- $\boldsymbol{\pi} \in \mathbb{R}^p$ collects all robot dynamic parameters
- $\boldsymbol{Y}_{\boldsymbol{\pi}}$ is an $n \times p$ matrix called the regressor of the robot dynamic model
- Some columns of the regressor matrix may be zero

Dynamic Model of Simple Manipulator Structures

- Three cases of derivation of the dynamic model via the EL formalism for simple manipulator structures having two DoFs
  - highlight the physical meaning of all dynamic terms, including inertial coupling
  - in last case, the example introduces also a general recursive computation of the kinetic energy
- For robots with more DoFs, it is advisable to derive the EL model with the aid of symbolic manipulation software

- 2R planar arm



- Derivation using partial Jacobians for the expression of kinetic energy

$$\boldsymbol{J}_p^1 = \begin{pmatrix} -\ell_{c1}s_1 & 0 \\ \ell_{c1}c_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{J}_p^2 = \begin{pmatrix} -a_1 s_1 - \ell_{c2}s_{12} & -\ell_{c2}s_{12} \\ a_1 c_1 + \ell_{c2}c_{12} & \ell_{c2}c_{12} \\ 0 & 0 \end{pmatrix}$$

$$\boldsymbol{J}_o^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{J}_o^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22} \end{pmatrix}$$

$$\begin{aligned} m_{11} &= I_1 + m_1\ell_{c1}^2 + I_{m1}n_{r1}^2 + I_2 + m_2(a_1^2 + \ell_{c2}^2) + 2m_2\ell_{c2}a_1c_2 \\ m_{12} &= m_{21} = I_2 + m_2\ell_{c2}^2 + m_2\ell_{c2}a_1c_2 \\ m_{22} &= I_2 + m_2\ell_{c2}^2 + I_{m2}n_{r2}^2 \end{aligned}$$

- Christoffel coefficients

$$
\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial m_{11}}{\partial q_1} = 0 \\
c_{112} = c_{121} &= \frac{1}{2}\frac{\partial m_{11}}{\partial q_2} = -m_2 \ell_{c2} a_1 s_2 \\
c_{122} &= \frac{\partial m_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial m_{22}}{\partial q_1} = -m_2 \ell_{c2} a_1 s_2 \\
c_{211} &= \frac{\partial m_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial m_{11}}{\partial q_2} = m_2 \ell_{c2} a_1 s_2 \\
c_{212} = c_{221} &= \frac{1}{2}\frac{\partial m_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial m_{22}}{\partial q_2} = 0
\end{aligned}
$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -m_2\ell_{c2}a_1s_2\begin{pmatrix}\dot{q}_2 & \dot{q}_1+\dot{q}_2\\ -\dot{q}_1 & 0\end{pmatrix}$$

$$\Downarrow$$

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\,\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\\ &= -m_2\ell_{c2}a_1s_2\begin{pmatrix}2\dot{q}_2 & \dot{q}_2\\ \dot{q}_2 & 0\end{pmatrix} + 2m_2\ell_{c2}a_1s_2\begin{pmatrix}\dot{q}_2 & \dot{q}_1+\dot{q}_2\\ -\dot{q}_1 & 0\end{pmatrix}\\ &= -m_2\ell_{c2}a_1s_2\begin{pmatrix}0 & -\left(2\dot{q}_1+\dot{q}_2\right)\\ 2\dot{q}_1+\dot{q}_2 & 0\end{pmatrix}\end{aligned}$$

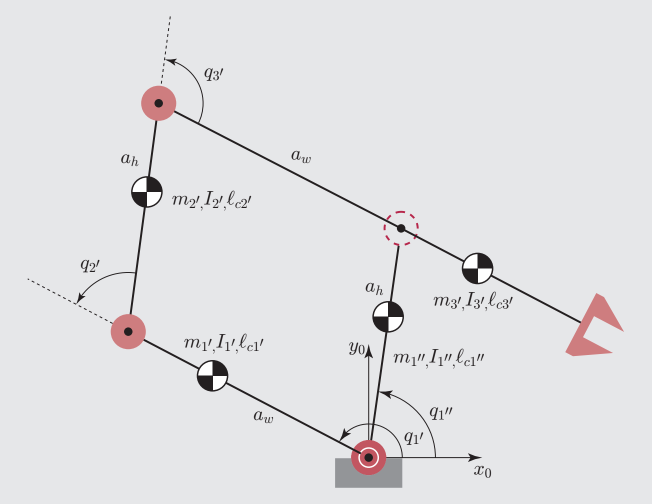
$$\begin{aligned}g_1 &= (m_1\ell_{c1}+m_2a_1)gc_1 + m_2\ell_{c2}gc_{12}\\ g_2 &= m_2\ell_{c2}gc_{12}\end{aligned}$$

- Resulting equations of motion

$$
\begin{aligned}
\left(I_1 + m_1\ell_{c1}^2 + I_{m1}n_{r1}^2 + I_2 + m_2(a_1^2 + \ell_{c2}^2) + 2m_2\ell_{c2}a_1c_2\right)\ddot{q}_1 & \\
+\left(I_2 + m_2\ell_{c2}^2 + m_2\ell_{c2}a_1c_2\right)\ddot{q}_2 - 2m_2\ell_{c2}a_1s_2\dot{q}_1\dot{q}_2 - m_2\ell_{c2}a_1s_2\dot{q}_2^2 & \\
+(m_1\ell_{c1} + m_2a_1)gc_1 + m_2\ell_{c2}gc_{12} &= \tau_1
\end{aligned}
$$

$$
\begin{aligned}
\left(I_2 + m_2\ell_{c2}^2 + m_2\ell_{c2}a_1c_2\right)\ddot{q}_1 + \left(I_2 + m_2\ell_{c2}^2 + I_{m2}n_{r2}^2\right)\ddot{q}_2 & \\
+m_2\ell_{c2}a_1s_2\dot{q}_1^2 + m_2\ell_{c2}gc_{12} &= \tau_2
\end{aligned}
$$

- Parallelogram arm



- Derivation using partial Jacobians for the expression of kinetic energy

$$\boldsymbol{J}_p^{1'} = \begin{pmatrix} -\ell_{c1'}s_{1'} & 0 & 0 \\ \ell_{c1'}c_{1'} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{J}_p^{2'} = \begin{pmatrix} -a_w s_{1'} - \ell_{c2'}s_{1'2'} & -\ell_{c2'}s_{1'2'} & 0 \\ a_w c_{1'} + \ell_{c2'}c_{1'2'} & \ell_{c2'}c_{1'2'} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{J}_p^{3'} = \begin{pmatrix} -a_w s_{1'} - a_h s_{1'2'} - \ell_{c3'}s_{1'2'3'} & -a_h s_{1'2'} - \ell_{c3'}s_{1'2'3'} & -\ell_{c3'}s_{1'2'3'} \\ a_w c_{1'} + a_h c_{1'2'} + \ell_{c3'}c_{1'2'3'} & a_h c_{1'2'} + \ell_{c3'}c_{1'2'3'} & \ell_{c3'}c_{1'2'3'} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{J}_p^{1''} = \begin{pmatrix} -\ell_{c1''}s_{1''} \\ \ell_{c1''}c_{1''} \\ 0 \end{pmatrix}$$

$$\boldsymbol{J}_o^{1'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \boldsymbol{J}_o^{2'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \boldsymbol{J}_o^{3'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \boldsymbol{J}_o^{1''} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
m_{1'1'} &= I_{1'} + m_{1'}\ell_{c1'}^2 + I_{m1'}n_{r1'}^2 + I_{2'} + m_{2'}(a_w^2 + \ell_{c2'}^2 + 2a_w\ell_{c2'}c_{2'}) + I_{3'} \\
&\quad + m_{3'}(a_w^2 + a_h^2 + \ell_{c3'}^2 + 2a_wa_hc_{2'} + 2a_w\ell_{c3'}c_{2'3'} + 2a_h\ell_{c3'}c_{3'}) \\
m_{1'2'} &= m_{2'1'} = I_{2'} + m_{2'}(\ell_{c2'}^2 + a_w\ell_{c2'}c_{2'}) + I_{3'} \\
&\quad + m_{3'}(a_h^2 + \ell_{c3'}^2 + a_wa_hc_{2'} + a_w\ell_{c3'}c_{2'3'} + 2a_h\ell_{c3'}c_{3'}) \\
m_{1'3'} &= m_{3'1'} = I_{3'} + m_{3'}(\ell_{c3'}^2 + a_w\ell_{c3'}c_{2'3'} + a_h\ell_{c3'}c_{3'}) \\
m_{2'2'} &= I_{2'} + m_{2'}\ell_{c2'}^2 + I_{3'} + m_{3'}(a_h^2 + \ell_{c3'}^2 + 2a_h\ell_{c3'}c_{3'}) \\
m_{2'3'} &= m_{3'2'} = I_{3'} + m_{3'}(\ell_{c3'}^2 + a_h\ell_{c3'}c_{3'}) \\
m_{3'3'} &= I_{3'} + m_{3'}\ell_{c3'}^2
\end{aligned}
$$

$$
m_{1''1''} = I_{1''} + m_{1''}\ell_{c1''}^2 + I_{m1''}n_{r1''}^2
$$

$$\boldsymbol{\Upsilon} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{\Upsilon}^T \boldsymbol{M}_o(\boldsymbol{q}) \boldsymbol{\Upsilon} + \boldsymbol{M}_m$$

$$\begin{aligned} m_{11} &= I_{1'} + m_{1'} \ell_{c1'}^2 + m_{2'} a_w^2 + I_{3'} + m_{3'} \bar{\ell}_{c3'}^2 + I_{m1'} n_{r1'}^2 \\ m_{12} &= m_{21} = \left(a_w m_{2'} \ell_{c2'} - a_h m_{3'} \bar{\ell}_{c3'}\right) \cos\left(q_{1''} - q_{1'}\right) \\ m_{22} &= I_{1'} + m_{1'} \ell_{c1'}^2 + I_{2'} + m_{2'} \ell_{c2'}^2 + m_{3'} a_h^2 + I_{m1''} n_{r1''}^2 \end{aligned}$$

- A configuration-independent decoupled inertia is obtained when

$$a_w m_{2'} \ell_{c2'} = a_h m_{3'} \bar{\ell}_{c3'}$$

$$
\begin{aligned}
g_{1'} &= (m_{1'}\ell_{c1'} + m_{2'}a_w + m_{3'}a_w)gc_{1'} + (m_{2'}\ell_{c2'} + m_{3'}a_h)gc_{1'2'} \\
&\quad + m_{3'}\ell_{c3'}gc_{1'2'3} \\
g_{2'} &= (m_{2'}\ell_{c2'} + m_{3'}a_h)gc_{1'2'} + m_{3'}\ell_{c3'}gc_{1'2'3} \\
g_{3'} &= m_{3'}\ell_{c3'}gc_{1'2'3}
\end{aligned}
$$

$$
g_{1''} = m_{1''}\ell_{c1''}gc_{1''}
$$

$$
\boldsymbol{g} = \boldsymbol{\Upsilon}^T \boldsymbol{g}_o = \begin{pmatrix} (m_{1'}\ell_{c1'} + m_{2'}a_w - m_{3'}\bar{\ell}_{c3'})gc_{1'} \\ (m_{1''}\ell_{c1''} + m_{2'}\ell_{c2'} + m_{3'}a_h)gc_{1''} \end{pmatrix}
$$

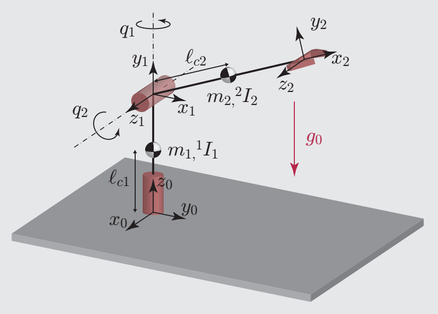
2R Polar Arm

- Kinetic energy

$$\mathcal{T}_i = \frac{1}{2} m_i \, {}^i\boldsymbol{v}_{ci}^T \, {}^i\boldsymbol{v}_{ci} + \frac{1}{2} \, {}^i\boldsymbol{\omega}_i^T \, {}^i\boldsymbol{I}_i \, {}^i\boldsymbol{\omega}_i$$

- Joint type

$$\sigma_i = \begin{cases} 1 & \text{if joint } i \text{ is prismatic} \\ 0 & \text{if joint } i \text{ is revolute} \end{cases}$$



Recursive Symbolic Computation of Velocities in Moving Frames

- Derivation using the recursion on velocities expressed in moving frames

$$
\begin{aligned}
{}^{i}\boldsymbol{\omega}_i &= {}^{i-1}\boldsymbol{R}_i^T(q_i)\left({}^{i-1}\boldsymbol{\omega}_{i-1} + (1-\sigma_i)\,\dot{q}_i\,{}^{i-1}\boldsymbol{z}_{i-1}\right) \\
{}^{i}\boldsymbol{v}_i &= {}^{i-1}\boldsymbol{R}_i^T(q_i)\left({}^{i-1}\boldsymbol{v}_{i-1} + {}^{i-1}\boldsymbol{\omega}_i \times {}^{i-1}\boldsymbol{r}_{i-1,i} + \sigma_i\dot{q}_i\,{}^{i-1}\boldsymbol{z}_{i-1}\right)
\end{aligned}
$$

$$\Downarrow$$

$$
{}^{i}\boldsymbol{v}_{ci} = {}^{i}\boldsymbol{v}_i + {}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{r}_{ci}
$$

- DH parameters

Link	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	0	a_2	0	q_2

Kinetic Energy and Inertia Matrix

$$
{}^{0}\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix} \qquad {}^{1}\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{1}\boldsymbol{r}_{0,1} = \begin{pmatrix} 0 \\ d_1 \\ 0 \end{pmatrix} \qquad {}^{2}\boldsymbol{r}_{1,2} = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}
$$

$$
{}^{1}\boldsymbol{r}_{c1} = \begin{pmatrix} 0 \\ -d_1 + \ell_{c1} \\ 0 \end{pmatrix} \quad {}^{2}\boldsymbol{r}_{c2} = \begin{pmatrix} -a_2 + \ell_{c2} \\ 0 \\ 0 \end{pmatrix} \quad {}^{2}\boldsymbol{I}_2 = \begin{pmatrix} I_{2xx} & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & I_{2zz} \end{pmatrix}
$$

$$
{}^{1}\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \qquad {}^{1}\boldsymbol{v}_1 = \mathbf{0} \qquad {}^{1}\boldsymbol{v}_{c1} = \mathbf{0}
$$

$$
{}^{2}\boldsymbol{\omega}_2 = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \qquad {}^{2}\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ a_2\,\dot{q}_2 \\ -a_2 c_2\,\dot{q}_1 \end{pmatrix} \qquad {}^{2}\boldsymbol{v}_{c2} = \begin{pmatrix} 0 \\ \ell_{c2}\,\dot{q}_2 \\ -\ell_{c2} c_2\,\dot{q}_1 \end{pmatrix}
$$

$$
\mathcal{T}_1 = \frac{1}{2}\,{}^{1}\boldsymbol{\omega}_1^{T}\,{}^{i}\boldsymbol{I}_i\,{}^{1}\boldsymbol{\omega}_1 = \frac{1}{2}\,I_{1,yy}\,\dot{q}_1^2
$$

$$
\mathcal{T}_2 = \frac{1}{2}\,m_2\ell_{c2}^2\left(c_2^2\,\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}\left(I_{2,xx}\,s_2^2 + I_{2,yy}\,c_2^2\right)\dot{q}_1^2 + \frac{1}{2}\,I_{2,zz}\,\dot{q}_2^2
$$

$$
\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\begin{pmatrix} \rho_1 + \rho_2\,c_2^2 & 0 \\ 0 & \rho_3 \end{pmatrix}\dot{\boldsymbol{q}}
$$

$$
\rho_1 = I_{1,yy} + I_{2,xx} \qquad \rho_2 = I_{2,yy} + m_2\ell_{c2}^2 - I_{2,xx} \qquad \rho_3 = I_{2,zz} + m_2\ell_{c2}^2
$$

Coriolis and Centrifugal Terms

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & -\rho_2\, s_2 c_2 \\ -\rho_2\, s_2 c_2 & 0 \end{pmatrix}$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} \rho_2\, s_2 c_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{aligned} \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \begin{pmatrix} -2\rho_2\, s_2 c_2\, \dot{q}_1 \dot{q}_2 \\ \rho_2\, s_2 c_2\, \dot{q}_1^2 \end{pmatrix} \\ &= \begin{pmatrix} -\rho_2\, s_2 c_2\, \dot{q}_2 & -\rho_2\, s_2 c_2\, \dot{q}_1 \\ \rho_2\, s_2 c_2\, \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} \end{aligned}$$

- $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is skew-symmetric

$$\mathcal{U}_1 = \text{constant}$$

$$\mathcal{U}_2 = g m_2 \ell_{c2}\, s_2 = \rho_4 g\, s_2$$

$$\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$$

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ \rho_4 g\, c_2 \end{pmatrix}$$

- Summing up yields

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Identification of Dynamic Parameters

- The use of the dynamic model for simulation and control purposes requires the knowledge of the numerical values of the dynamic parameters of the robot
- Obtaining such parameters from the design data of the mechanical structure may not be so easy for an end-user
- CAD modeling techniques allow the computation of inertial parameters from geometry and materials, but these estimates are typically inaccurate
- Other dynamic effects, such as joint friction, are not included in the CAD model

- Accurate estimates of the robot dynamic parameters can be obtained using identification techniques that exploit the linearity property
- These techniques compute the parameter vector $\boldsymbol{\pi}$ from the knowledge of the applied joint torques $\boldsymbol{\tau}$ and the regressor matrix $\boldsymbol{Y}_\pi$ along suitably chosen trajectories
- Commanded torques may differ from the torques actually realized by the servomotors, and thus it is preferable to use measurements
- When joint torque sensors are available, they directly measure the torques transferred to the robot links
- More often, torque is obtained from current measurements, provided the current-to-torque conversion factor is known
- With the kinematic parameters being known, only joint positions $\boldsymbol{q}$, velocities $\dot{\boldsymbol{q}}$ and accelerations $\ddot{\boldsymbol{q}}$ are required
    - joint positions are measured with good accuracy by encoders
    - velocities and accelerations are reconstructed numerically from recorded position data
    - offline processing allows the use of anti-causal filters incorporating future data samples

- The $n \times p$ regressor matrix $\boldsymbol{Y}_{\boldsymbol{\pi}}$ has several zero columns, corresponding to parameters that do not contribute to the equations of motion
  - removing them leaves $p_r < p$ columns
- If the p_r columns are structurally dependent, only some r linear combinations with fixed coefficients $\boldsymbol{\delta}_j \in I\!R^r$, $j = 1, \ldots, p_r - r$, can be identified

$$\boldsymbol{Y}_{\mathrm{dep}} = \boldsymbol{Y}_{\mathrm{ind}} \begin{pmatrix} \boldsymbol{\delta}_1 & \ldots & \boldsymbol{\delta}_{p_r-r} \end{pmatrix} = \boldsymbol{Y}_{\mathrm{ind}} \boldsymbol{\Delta}$$

$$\begin{aligned} \boldsymbol{Y}_{\boldsymbol{\pi}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{\pi} &= \begin{pmatrix} \boldsymbol{Y}_{\mathrm{ind}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) & \boldsymbol{Y}_{\mathrm{dep}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}_{\mathrm{ind}} \\ \boldsymbol{\pi}_{\mathrm{dep}} \end{pmatrix} \\ &= \boldsymbol{Y}_{\mathrm{ind}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) \left(\boldsymbol{\pi}_{\mathrm{ind}} + \boldsymbol{\Delta}\boldsymbol{\pi}_{\mathrm{dep}} \right) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{\rho}, \end{aligned}$$

- $\boldsymbol{\rho} = \boldsymbol{\pi}_{\mathrm{ind}} + \boldsymbol{\Delta}\boldsymbol{\pi}_{\mathrm{dep}}$ is the r-dimensional vector of dynamic coefficients

- Reduced parameters for the 2R planar arm

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \\ \pi_7 \end{pmatrix} = \begin{pmatrix} m_1 \ell_{c1} \\ I_1 + m_1 \ell_{c1}^2 \\ I_{m1} n_{r1}^2 \\ m_2 \\ I_2 + m_2 \ell_{c2}^2 \\ m_2 \ell_{c2} \\ I_{m2} n_{r2}^2 \end{pmatrix}$$

- Dynamic model linear in $\boldsymbol{\pi}$

$$\begin{aligned} (\pi_2 + \pi_3 + a_1^2 \pi_4 + \pi_5 + 2 a_1 c_2 \pi_6)\, \ddot{q}_1 + (\pi_5 + a_1 c_2 \pi_6)\, \ddot{q}_2 & \\ -2 a_1 s_2 \pi_6 \dot{q}_1 \dot{q}_2 - a_1 s_2 \pi_6 \dot{q}_2^2 + (\pi_1 + a_1 \pi_4)\, g c_1 + \pi_6 g c_{12} &= \tau_1 \\ (\pi_5 + a_1 c_2 \pi_6)\, \ddot{q}_1 + (\pi_5 + \pi_7)\, \ddot{q}_2 + a_1 s_2 \pi_6 \dot{q}_1^2 + \pi_6 g c_{12} &= \tau_2 \end{aligned}$$

$$\boldsymbol{Y}_{\boldsymbol{\pi}} = \begin{pmatrix} \boldsymbol{y}_{\boldsymbol{\pi}1} & \boldsymbol{y}_{\boldsymbol{\pi}2} & \boldsymbol{y}_{\boldsymbol{\pi}3} & \boldsymbol{y}_{\boldsymbol{\pi}4} & \boldsymbol{y}_{\boldsymbol{\pi}5} & \boldsymbol{y}_{\boldsymbol{\pi}6} & \boldsymbol{y}_{\boldsymbol{\pi}7} \end{pmatrix}$$

$$\boldsymbol{y}_{\boldsymbol{\pi}1} = \begin{pmatrix} gc_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{y}_{\boldsymbol{\pi}2} = \begin{pmatrix} \ddot{q}_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{y}_{\boldsymbol{\pi}3} = \begin{pmatrix} \ddot{q}_1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{y}_{\boldsymbol{\pi}4} = \begin{pmatrix} a_1^2\ddot{q}_1 + ga_1c_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{y}_{\boldsymbol{\pi}5} = \begin{pmatrix} \ddot{q}_1 + \ddot{q}_2 \\ \ddot{q}_1 + \ddot{q}_2 \end{pmatrix}$$

$$\boldsymbol{y}_{\boldsymbol{\pi}6} = \begin{pmatrix} 2a_1c_2\ddot{q}_1 + a_1c_2\ddot{q}_2 - 2a_1s_2\dot{q}_1\dot{q}_2 - a_1s_2\dot{q}_2^2 + gc_{12} \\ a_1c_2\ddot{q}_1 + a_1s_2\dot{q}_1^2 + gc_{12} \end{pmatrix} \qquad \boldsymbol{y}_{\boldsymbol{\pi}7} = \begin{pmatrix} 0 \\ \ddot{q}_2 \end{pmatrix}$$

$$\boldsymbol{y}_{\boldsymbol{\pi}3} = \boldsymbol{y}_{\boldsymbol{\pi}2}$$
$$\boldsymbol{y}_{\boldsymbol{\pi}4} = a_1 \boldsymbol{y}_{\boldsymbol{\pi}1} + a_1^2 \, \boldsymbol{y}_{\boldsymbol{\pi}2}$$

$$\boldsymbol{Y}_{\text{ind}} = \begin{pmatrix} \boldsymbol{y}_{\boldsymbol{\pi}1} & \boldsymbol{y}_{\boldsymbol{\pi}2} & \boldsymbol{y}_{\boldsymbol{\pi}5} & \boldsymbol{y}_{\boldsymbol{\pi}6} & \boldsymbol{y}_{\boldsymbol{\pi}7} \end{pmatrix} \qquad \boldsymbol{Y}_{\text{dep}} = \begin{pmatrix} \boldsymbol{y}_{\boldsymbol{\pi}3} & \boldsymbol{y}_{\boldsymbol{\pi}4} \end{pmatrix} = \boldsymbol{Y}_{\text{ind}} \, \boldsymbol{\Delta}$$

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & a_1 \\ 1 & a_1^2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\pi}_{\text{ind}} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_5 & \pi_6 & \pi_7 \end{pmatrix}^T \qquad \boldsymbol{\pi}_{\text{dep}} = \begin{pmatrix} \pi_3 & \pi_4 \end{pmatrix}^T$$

$$\boldsymbol{\rho} = \boldsymbol{\pi}_{\text{ind}} + \boldsymbol{\Delta\pi}_{\text{dep}} = \begin{pmatrix} m_1\ell_{c1} + m_2a_1 \\ I_1 + m_1\ell_{c1}^2 + I_{m1}n_{r1}^2 + m_2a_1^2 \\ I_2 + m_2\ell_{c2}^2 \\ m_2\ell_{c2} \\ I_{m2}n_{r2}^2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \end{pmatrix}$$

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{\rho} = \boldsymbol{\tau}$$

- New regressor matrix $\boldsymbol{Y}$ with structural independent columns

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \boldsymbol{y}_3 & \boldsymbol{y}_4 & \boldsymbol{y}_5 \end{pmatrix}$$

$$\boldsymbol{y}_1 = \begin{pmatrix} gc_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{y}_2 = \begin{pmatrix} \ddot{q}_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{y}_3 = \begin{pmatrix} \ddot{q}_1 + \ddot{q}_2 \\ \ddot{q}_1 + \ddot{q}_2 \end{pmatrix}$$

$$\boldsymbol{y}_4 = \begin{pmatrix} 2a_1c_2\ddot{q}_1 + a_1c_2\ddot{q}_2 - 2a_1s_2\dot{q}_1\dot{q}_2 - a_1s_2\dot{q}_2^2 + gc_{12} \\ a_1c_2\ddot{q}_1 + a_1s_2\dot{q}_1^2 + gc_{12} \end{pmatrix} \qquad \boldsymbol{y}_5 = \begin{pmatrix} 0 \\ \ddot{q}_2 \end{pmatrix}$$

- Joint data are collected at N time instants $t_1, \dots, t_N$ along a given motion trajectory
- The equations are stacked as

$$\bar{\boldsymbol{\tau}} = \begin{pmatrix} \boldsymbol{\tau}(t_1) \\ \vdots \\ \boldsymbol{\tau}(t_N) \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y}(t_1) \\ \vdots \\ \boldsymbol{Y}(t_N) \end{pmatrix} \boldsymbol{\rho} = \bar{\boldsymbol{Y}} \boldsymbol{\rho}$$

- With $\bar{\boldsymbol{Y}}$ full column rank

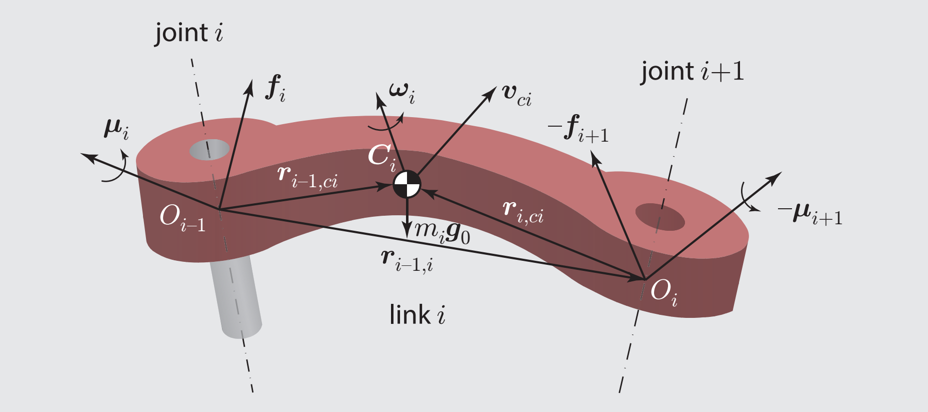
$$\hat{\boldsymbol{\rho}} = \bar{\boldsymbol{Y}}^{\dagger} \bar{\boldsymbol{\tau}} = (\bar{\boldsymbol{Y}}^T \bar{\boldsymbol{Y}})^{-1} \bar{\boldsymbol{Y}}^T \bar{\boldsymbol{\tau}}$$

- Choice of motions for identification
  - number of data should be large enough so to obtain a matrix $\bar{\boldsymbol{Y}}$ with full column rank
  - use of sufficiently rich periodic trajectories is advisable
  - trajectories should contain multiple harmonics at different frequencies and adequately cover the configuration space
  - a large range of velocities and accelerations is needed to obtain a suitable condition number κ for $\bar{\boldsymbol{Y}}$
  - trajectories should not excite unmodeled dynamic effects, such as joint elasticity
  - in the presence of an unknown payload at the end-effector, one may lump its mass and inertia into the standard parameter vector of the last link

Newton–Euler Formulation

- The EL formulation derives the equations of motion in a symbolic form suitable for studying model properties and for control design
- The NE formulation computes the dynamic model in numeric form, efficiently exploiting the serial nature of the kinematic chain
- The NE formulation proceeds from balance of forces and moments acting on each link
- The resulting NE equations lend themselves to a recursive computation
  - forward recursion propagates link velocities and accelerations from base to end-effector
  - backward recursion propagates forces and moments from end-effector to base
- From these forces and moments, joint torques can be computed

- Forces and moments acting on link i



Forward Recursion of Velocities and Accelerations

- In the following, forward recursive formulas are derived by exploiting the serial nature of the kinematic chain from link $i-1$ to link i
  - link velocities
  - link accelerations
- The corresponding formulas are mainly intended for numeric evaluation

- Composition of vectors in a serial chain

$$\boldsymbol{r}_{0,i} = \boldsymbol{r}_{0,i-1} + \boldsymbol{r}_{i-1,i} = \boldsymbol{r}_{0,i-1} + \boldsymbol{R}_{i-1}{}^{i-1}\boldsymbol{r}_{i-1,i}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{i-1} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i} + \boldsymbol{v}_{i-1,i}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}$$

- Relative velocities between successive links
  - if joint i is prismatic

$$\boldsymbol{\omega}_{i-1,i} = \mathbf{0} \qquad \boldsymbol{v}_{i-1,i} = \dot{q}_i\, \boldsymbol{z}_{i-1}$$

  - if joint i is revolute

$$\boldsymbol{\omega}_{i-1,i} = \dot{q}_i\, \boldsymbol{z}_{i-1} \qquad \boldsymbol{v}_{i-1,i} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,i} = \dot{q}_i\, \boldsymbol{z}_{i-1} \times \boldsymbol{r}_{i-1,i}$$

- Forward recursion on velocities (with $\sigma_i = 1$ for prismatic, $\sigma_i = 0$ for revolute joints)

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + (1 - \sigma_i)\, \dot{q}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i} + \sigma_i \dot{q}_i \boldsymbol{z}_{i-1}$$

Link Accelerations

- Forward recursion on angular accelerations

$$\dot{\boldsymbol{\omega}}_i = \dot{\boldsymbol{\omega}}_{i-1} + (1 - \sigma_i)\left(\ddot{q}_i \boldsymbol{z}_{i-1} + \dot{q}_i \dot{\boldsymbol{z}}_{i-1}\right)$$

$$\dot{\boldsymbol{z}}_{i-1} = \boldsymbol{\omega}_{i-1} \times \boldsymbol{z}_{i-1}$$

$$\dot{\boldsymbol{\omega}}_i = \dot{\boldsymbol{\omega}}_{i-1} + (1 - \sigma_i)\left(\ddot{q}_i \boldsymbol{z}_{i-1} + \boldsymbol{\omega}_{i-1} \times \dot{q}_i \boldsymbol{z}_{i-1}\right)$$

- Forward recursion on linear accelerations

$$\dot{\boldsymbol{v}}_i = \dot{\boldsymbol{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i-1,i} + \boldsymbol{\omega}_i \times \dot{\boldsymbol{r}}_{i-1,i} + \sigma_i(\ddot{q}_i \boldsymbol{z}_{i-1} + \dot{q}_i \dot{\boldsymbol{z}}_{i-1})$$

$$\dot{\boldsymbol{r}}_{i-1,i} = \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i} + \sigma_i \dot{q}_i \boldsymbol{z}_{i-1}$$

$$\dot{\boldsymbol{v}}_i = \dot{\boldsymbol{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i-1,i} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}) + \sigma_i(\ddot{q}_i \boldsymbol{z}_{i-1} + 2\boldsymbol{\omega}_i \times \dot{q}_i \boldsymbol{z}_{i-1})$$

Backward Recursion of Forces and Moments

- Newton and Euler equations

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} + m_i \boldsymbol{g}_0 = \frac{d}{dt}\left(m_i \boldsymbol{v}_{ci}\right) = m_i \dot{\boldsymbol{v}}_{ci}$$

$$\boldsymbol{\mu}_i + \boldsymbol{f}_i \times \boldsymbol{r}_{i-1,ci} - \boldsymbol{\mu}_{i+1} - \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,ci} = \frac{d}{dt}\left(\boldsymbol{I}_i \boldsymbol{\omega}_i\right)$$

$$\begin{aligned}
\frac{d}{dt}\left(\boldsymbol{I}_i \boldsymbol{\omega}_i\right) &= \dot{\boldsymbol{R}}_i {}^i\boldsymbol{I}_i \boldsymbol{R}_i^T \boldsymbol{\omega}_i + \boldsymbol{R}_i {}^i\boldsymbol{I}_i \dot{\boldsymbol{R}}_i^T \boldsymbol{\omega}_i + \boldsymbol{R}_i {}^i\boldsymbol{I}_i \boldsymbol{R}_i^T \dot{\boldsymbol{\omega}}_i \\
&= \boldsymbol{S}(\boldsymbol{\omega}_i) \boldsymbol{R}_i {}^i\boldsymbol{I}_i \boldsymbol{R}_i^T \boldsymbol{\omega}_i + \boldsymbol{R}_i {}^i\boldsymbol{I}_i \boldsymbol{R}_i^T \boldsymbol{S}^T(\boldsymbol{\omega}_i) \boldsymbol{\omega}_i + \boldsymbol{R}_i {}^i\boldsymbol{I}_i \boldsymbol{R}_i^T \dot{\boldsymbol{\omega}}_i \\
&= \boldsymbol{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i \boldsymbol{\omega}_i
\end{aligned}$$

- Linear acceleration of the link center of mass (CoM)

$$\dot{\boldsymbol{v}}_{ci} = \dot{\boldsymbol{v}}_i + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i,ci} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i,ci})$$

- Backward recursion on forces and moments

$$\boldsymbol{f}_i = \boldsymbol{f}_{i+1} + m_i(\dot{\boldsymbol{v}}_i - \boldsymbol{g}_0) + \dot{\boldsymbol{\omega}}_i \times m_i\boldsymbol{r}_{i,ci} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times m_i\boldsymbol{r}_{i,ci})$$

$$\boldsymbol{\mu}_i = -\boldsymbol{f}_i \times \boldsymbol{r}_{i-1,ci} + \boldsymbol{\mu}_{i+1} + \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,ci} + \boldsymbol{I}_i\,\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i\,\boldsymbol{\omega}_i$$

- Projection on joint i axis

$$\tau_{qi} = \begin{cases} \boldsymbol{\mu}_i^T \boldsymbol{z}_{i-1} & \text{for a revolute joint} \\ \boldsymbol{f}_i^T \boldsymbol{z}_{i-1} & \text{for a prismatic joint} \end{cases}$$

- Addition of motor inertia and joint dissipation to the actuation torque τ_i at joint i

$$\tau_{qi} + I_{mi} n_{ri}^2\, \ddot{q}_i = \tau_i + \tau_{Bi}$$

Recursive Algorithm in Moving Frames

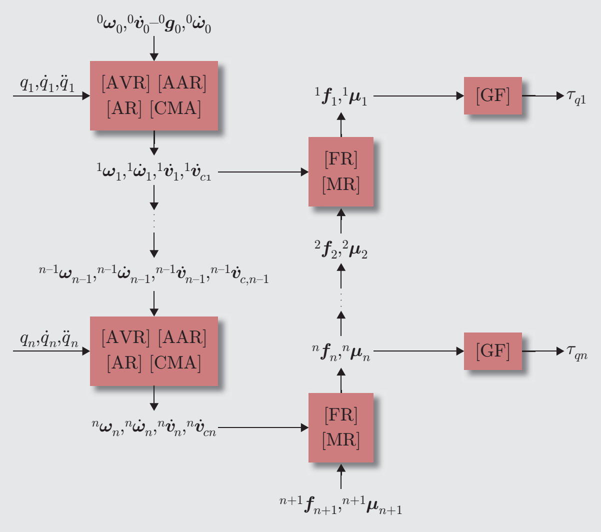
- Forward recursion in moving frames

$$
\begin{aligned}
[\text{AVR}] &= \text{Angular Velocity Recursion} \\
{}^{i}\boldsymbol{\omega}_i &= {}^{i-1}\boldsymbol{R}_i^T \left({}^{i-1}\boldsymbol{\omega}_{i-1} + (1-\sigma_i)\,\dot{\theta}_i \boldsymbol{z}_0\right) \\
[\text{AAR}] &= \text{Angular Acceleration Recursion} \\
{}^{i}\dot{\boldsymbol{\omega}}_i &= {}^{i-1}\boldsymbol{R}_i^T \left({}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + (1-\sigma_i)\left(\ddot{\theta}_i \boldsymbol{z}_0 + \dot{\theta}_i {}^{i-1}\boldsymbol{\omega}_{i-1} \times \boldsymbol{z}_0\right)\right) \\
[\text{AR}] &= \text{Acceleration Recursion} \\
{}^{i}\dot{\boldsymbol{v}}_i &= {}^{i-1}\boldsymbol{R}_i^T \left({}^{i-1}\dot{\boldsymbol{v}}_{i-1} + \sigma_i \ddot{d}_i \boldsymbol{z}_0\right) + 2\,\sigma_i \dot{d}_i {}^{i}\boldsymbol{\omega}_i \times {}^{i-1}\boldsymbol{R}_i^T \boldsymbol{z}_0 \\
&= + {}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\boldsymbol{r}_{i-1,i} + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{r}_{i-1,i}) \\
[\text{CMA}] &= \text{Center of Mass Acceleration} \\
{}^{i}\dot{\boldsymbol{v}}_{ci} &= {}^{i}\dot{\boldsymbol{v}}_i + {}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\boldsymbol{r}_{i,ci} + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{r}_{i,ci})
\end{aligned}
$$

- Backward recursion and generalized force in moving frames

$$
\begin{aligned}
[\text{FR}] &= \text{Force Recursion} \\
{}^{i}\boldsymbol{f}_i &= {}^{i}\boldsymbol{R}_{i+1}\,{}^{i+1}\boldsymbol{f}_{i+1} + m_i\,{}^{i}\dot{\boldsymbol{v}}_{ci} \\
[\text{MR}] &= \text{Moment Recursion} \\
{}^{i}\boldsymbol{\mu}_i &= -\,{}^{i}\boldsymbol{f}_i \times {}^{i}\boldsymbol{r}_{i-1,ci} + {}^{i}\boldsymbol{R}_{i+1}\,{}^{i+1}\boldsymbol{\mu}_{i+1} \\
&= +\,{}^{i}\boldsymbol{R}_{i+1}\,{}^{i+1}\boldsymbol{f}_{i+1} \times {}^{i}\boldsymbol{r}_{i,ci} + {}^{i}\boldsymbol{I}_i\,{}^{i}\dot{\boldsymbol{\omega}}_i + {}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{I}_i\,{}^{i}\boldsymbol{\omega}_i \\
[\text{GF}] &= \text{Generalized Force} \\
\tau_{qi} &= \begin{cases} {}^{i}\boldsymbol{\mu}_i^T\,{}^{i-1}\boldsymbol{R}_i^T \boldsymbol{z}_0 & \text{(revolute joint)} \\ {}^{i}\boldsymbol{f}_i^T\,{}^{i-1}\boldsymbol{R}_i^T \boldsymbol{z}_0 & \text{(prismatic joint)} \end{cases}
\end{aligned}
$$

- NE recursive algorithm for inverse dynamics



Closed Kinematic Chains

- Use forward recursion of velocities and accelerations to the links of the equivalent tree-structured open-chain arm
- Compute $\boldsymbol{q}_o$, $\dot{\boldsymbol{q}}_o$, $\ddot{\boldsymbol{q}}_o$ from $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$ using the constraint relations and their time derivatives
- Use backward recursion to the links of the equivalent tree-structured open-chain arm
- For a link connected to branches, replace $\boldsymbol{f}_{i+1}$ by $\boldsymbol{f}_{i+1'} + \boldsymbol{f}_{i+1''}$ and $\boldsymbol{\mu}_{i+1}$ by $\boldsymbol{\mu}_{i+1'} + \boldsymbol{\mu}_{i+1''}$
- Compute τ_{oi} as output of the NE algorithm, then compute τ_{qi} at the actuated joints (using virtual works)
- Include motor inertias (at the actuated joints) and friction (local to any joint) as in standard serial chains

Summary of Dynamic Formulations

- Comparison of the two presented formulations
  - both EL and NE formulations compute the relationship between joint torques and motion of the structure
  - each formulation has its pros and cons
- Computational issues illustrated using the NE algorithm
  - full inverse dynamics computation and beyond
  - use also for direct dynamics

Comparison of EL and NE

- EL formulation characteristics
  - looks at the multibody robot as a whole, based on kinetic and potential energy contributions
  - provides equations of motion in compact analytical form
  - explicit expressions of inertia matrix, centrifugal and Coriolis vector (and its factorization), gravitational vector and structural properties
  - advantageous for mechanical design assessment and for development and analysis of motion control laws
  - applicable to serial and parallel branches once generalized coordinates are identified
  - easy to add effects beyond rigidity, such as joint elasticity or link deformation
  - constraint forces at joints are automatically eliminated
  - lends itself naturally to symbolic manipulation languages

- NE formulation characteristics
    - derives dynamic equations from balance of forces and moments on each link or body
    - modular structure
    - recursive structure for serial and branched robots
    - computationally efficient numerical method for inverse and direct dynamics
    - allows explicit computation of constraint forces for joint stress evaluation
    - best tailored for real-time model-based control schemes
    - control torques can be computed numerically without the need of building explicitly the complete dynamic model

Computation of Dynamic Terms with the NE Algorithm

- Consider a serial manipulator with n joints and assume that
  - only joint torques $\boldsymbol{\tau} \in I\!R^n$ are applied
  - dissipative effects are neglected
  - no end-effector interaction wrenches are present (unless specified)
- NE algorithm evaluation can be represented as an algebraic function

$$\boldsymbol{u} = \mathsf{NEA}_{\boldsymbol{\alpha}}(\mathbf{arg}_1, \mathbf{arg}_2, \mathbf{arg}_3)$$

with 3 arguments and a parameter vector $\boldsymbol{\alpha} \in I\!R^3$ that accounts for gravity

  - $\boldsymbol{\alpha} = {}^0\boldsymbol{g}_0$ presence of gravity
  - $\boldsymbol{\alpha} = \mathbf{0}$ absence of gravity

- NEA needs
  - number n of joints
  - joint type information (e.g., indices $\sigma_i = \{0, 1\}$)
  - table of DH kinematic parameters
  - list of dynamic parameters of links and motors
- A call of NEA requires a forward and a backward recursion, each of n steps
- Computational complexity grows linearly with n, i.e., $O(n)$
- Addition of motor inertias and of friction terms is local to each joint and does not affect computational complexity
- Several examples of use of NEA are presented next

- Inverse dynamics by a single call of NEA

$$\boldsymbol{\tau}_d = \boldsymbol{M}(\boldsymbol{q}_d)\ddot{\boldsymbol{q}}_d + \boldsymbol{c}(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) + \boldsymbol{g}(\boldsymbol{q}_d) = \mathsf{NEA}_{\,^0\boldsymbol{g}_0}(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d)$$

  where the desired motion is specified by $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$, $\ddot{\boldsymbol{q}}_d$

- NE computation is typically more efficient than evaluating all symbolic terms in EL model, especially for large n

- Gravity vector

$$\boldsymbol{g}(\boldsymbol{q}) = \mathsf{NEA}_{\,^0\boldsymbol{g}_0}(\boldsymbol{q}, \boldsymbol{0}, \boldsymbol{0})$$

- Coriolis and centrifugal vector

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathsf{NEA}_{\boldsymbol{0}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{0})$$

- i-th column of inertia matrix

$$\boldsymbol{m}_i(\boldsymbol{q}) = \mathsf{NEA}_{\boldsymbol{0}}(\boldsymbol{q}, \boldsymbol{0}, \boldsymbol{e}_i)$$

  where $\boldsymbol{e}_i$ is the i-th column of the $n \times n$ identity matrix

- Generalized momentum

$$\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathsf{NEA}_{\mathbf{0}}(\boldsymbol{q}, \mathbf{0}, \dot{\boldsymbol{q}})$$

- Feedback linearizing control law

$$\boldsymbol{\tau}_{\mathrm{FBL}} = \mathsf{NEA}_{{}^0\boldsymbol{g}_0}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{a})$$

$$\boldsymbol{a} = \ddot{\boldsymbol{q}}_d + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}, \qquad \boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$$

- In case of end-effector interaction, joint torques to apply an end-effector wrench on the environment

$$\boldsymbol{\tau}_{\mathrm{env}} = -\boldsymbol{J}_e^T(\boldsymbol{q})\boldsymbol{w} = \mathsf{NEA}_{\mathbf{0}}(\boldsymbol{q}, \mathbf{0}, \mathbf{0})$$

where $\boldsymbol{w} = (\boldsymbol{f}, \boldsymbol{\mu})$ is used in the initialization of the backward NE recursion

- Mixed velocity term $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}'$ with $\dot{\boldsymbol{q}}' \neq \dot{\boldsymbol{q}}$
  - needed in some control laws, where $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ should be skew-symmetric
  - use the exchange identity valid for Christoffel factorization matrices

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}')(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}') = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}')\dot{\boldsymbol{q}}' + 2\,\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}'$$

- Three calls of NEA give

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}' = \frac{1}{2}\left(\mathrm{NEA}_{\mathbf{0}}(\boldsymbol{q}, \dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}', \mathbf{0}) - \mathrm{NEA}_{\mathbf{0}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mathbf{0}) - \mathrm{NEA}_{\mathbf{0}}(\boldsymbol{q}, \dot{\boldsymbol{q}}', \mathbf{0})\right)$$

- The i-th column $\boldsymbol{c}_i$ of $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ can be computed as

$$\boldsymbol{c}_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\left(\mathrm{NEA}_{\mathbf{0}}(\boldsymbol{q}, \dot{\boldsymbol{q}} + \boldsymbol{e}_i, \mathbf{0}) - \mathrm{NEA}_{\mathbf{0}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mathbf{0}) - \mathrm{NEA}_{\mathbf{0}}(\boldsymbol{q}, \boldsymbol{e}_i, \mathbf{0})\right)$$

- Direct dynamics at time t_k with state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and motor torque $\boldsymbol{\tau}$ as input
- EL formulation provides the acceleration as

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right)$$

  to be used in a numerical integration step Δt to obtain $\boldsymbol{q}(t_{k+1})$ and $\dot{\boldsymbol{q}}(t_{k+1})$
- Using instead NEA, evaluate quadratic velocity and gravity terms

$$\boldsymbol{n} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \text{NEA}_{\,{}^0\boldsymbol{g}_0}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{0})$$

- Evaluate the columns of inertia matrix $\boldsymbol{M}$ using n calls
  - overall complexity $O(n^2)$
- Invert numerical matrix $\boldsymbol{M}$ to get $\boldsymbol{M}^{-1}$
  - inversion has complexity $O(n^3)$, with low coefficient
- Compute $\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{n})$ —equivalently, solve $\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{\tau} - \boldsymbol{n}$ for $\ddot{\boldsymbol{q}}$
- NE formulation provides an $O(n^3)$ algorithm for direct dynamics
- Articulated-body algorithms compute $\ddot{\boldsymbol{q}}$ without building $\boldsymbol{M}$, with complexity $O(n)$

Dynamic Scaling of Trajectories

- A trajectory planned only from geometric or kinematic aspects may be dynamically unfeasible or overly conservative
- The torque profile $\boldsymbol{\tau}(t)$ needed to move the robot along a trajectory $\boldsymbol{q}(t)$, $t \in [0, T]$, planned in configuration space is computed using inverse dynamics

$$\boldsymbol{\tau}(t) = \boldsymbol{M}(\boldsymbol{q}(t))\,\ddot{\boldsymbol{q}}(t) + \boldsymbol{c}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) + \boldsymbol{g}(\boldsymbol{q}(t))$$

- The trajectory $\boldsymbol{q}(t)$ is dynamically feasible if the actuator torque limits are satisfied

$$|\tau_i(t)| \leq \tau_{i,\max} \qquad t \in [0, T] \quad i = 1, \ldots, n$$

- Apply now a uniform time scaling with $r = k\,t \in [0, T_s]$
  - k is a positive constant and $T_s = k\,T$
- The scaled trajectory is denoted by $\boldsymbol{q}_s(r)$ and its torque by $\boldsymbol{\tau}_s(r)$

$$\boldsymbol{q}(t) = \boldsymbol{q}_s(r) \qquad \dot{\boldsymbol{q}}(t) = k\,\dot{\boldsymbol{q}}_s(r) \qquad \ddot{\boldsymbol{q}}(t) = k^2\ddot{\boldsymbol{q}}(r)$$

- Substituting the time-warp relationships in inverse dynamics yields

$$\boldsymbol{\tau}(t) = k^2 \left(\boldsymbol{M}(\boldsymbol{q}_s(r))\, \ddot{\boldsymbol{q}}_s(r) + \boldsymbol{c}(\boldsymbol{q}_s(r), \dot{\boldsymbol{q}}_s(r))\right) + \boldsymbol{g}(\boldsymbol{q}(t))$$

  - the inertial and velocity terms scale with k^2
  - the gravitational term is unaffected since it depends only on the configuration

- Torque of the scaled trajectory

$$\boldsymbol{\tau}_s(r) = \boldsymbol{M}(\boldsymbol{q}_s(r))\, \ddot{\boldsymbol{q}}_s(r) + \boldsymbol{c}(\boldsymbol{q}_s(r), \dot{\boldsymbol{q}}_s(r)) + \boldsymbol{g}(\boldsymbol{q}_s(r))$$

- Comparing the two torque expressions gives the general relationship

$$\boldsymbol{\tau}(t) - \boldsymbol{g}(\boldsymbol{q}(t)) = k^2 \left(\boldsymbol{\tau}_s(r) - \boldsymbol{g}(\boldsymbol{q}_s(r))\right)$$

- Using $r = k\,t$, one obtains

$$\boldsymbol{\tau}_s(kt) = \frac{1}{k^2}\left(\boldsymbol{\tau}(t) - \boldsymbol{g}(\boldsymbol{q}(t))\right) + \boldsymbol{g}(\boldsymbol{q}(t))$$

- Uniform time scaling by k causes a scaling by $1/k^2$ of the torque associated with inertial and velocity terms only, while the gravitational torque is unchanged
- In practice, actuator limits dominate gravitational load at all configurations

$$\tau_{i,\max} > |g_i(\boldsymbol{q})| \qquad \text{for all } \boldsymbol{q} \quad i = 1, \ldots, n$$

- Under this assumption, feasibility can be recovered by sufficiently slowing down the motion, i.e., using a $k > 1$ that is large enough
- When $\boldsymbol{q}(t)$ is dynamically unfeasible, one is interested in the smallest $k^\star > 1$ such that $\boldsymbol{q}_s(k^\star t)$ is feasible

- Compute first, for $i = 1, \ldots, n$

$$k_i = \max\left(1, \max_{t\in[0,T]}\left(\frac{\tau_i(t) - g_i(\boldsymbol{q}(t))}{\tau_{i,\max} - g_i(\boldsymbol{q}(t))}, \frac{g_i(\boldsymbol{q}(t)) - \tau_i(t)}{\tau_{i,\max} + g_i(\boldsymbol{q}(t))}\right)\right) \geq 1$$

- The optimal factor is

$$k^\star = \max_{i=1,\ldots,n} \sqrt{k_i} > 1$$

- The scaled trajectory $\boldsymbol{q}_s(k^\star t)$ is dynamically feasible and (at least) one torque saturates its limit at some time instant
- When the trajectory $\boldsymbol{q}(t)$ is conservative, a similar procedure can be applied to uniformly speed up motion using the largest factor $k^\star < 1$ that preserves feasibility
- Uniform time scaling may be severely suboptimal, motivating nonuniform time scaling for minimum-time motion along a given geometric path under torque limits

Task-Space Dynamic Model

- The configuration-space dynamic model is naturally expressed in terms of the n-dimensional generalized coordinates $\boldsymbol{q}$
- An alternative is to write the equations of motion in task space, relating generalized forces to the m-dimensional task variables $\boldsymbol{y}$
- Redundant DoFs ($n > m$) and task singularities must be carefully handled
- A task-space model derived via EL represents a complete description only in the nonredundant case
- For redundant robots, internal motions are not reflected at the task level

- Consider for simplicity the configuration-space dynamic model without interaction and friction forces, and solve for the joint accelerations

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\,(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}))$$

- Second-order differential kinematics relates $\ddot{\boldsymbol{q}}$ to task acceleration

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Task acceleration in terms of joint torques

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\,(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- In the nonredundant case ($m = n$) and far from singularities, any joint torque can be written as

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{\gamma}$$

  a one-to-one mapping between $\boldsymbol{\tau}$ and the generalized forces $\boldsymbol{\gamma}$ performing work on $\boldsymbol{y}$

- Task-space inertia matrix

$$\boldsymbol{\Lambda}(\boldsymbol{q}) = (\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^{T}(\boldsymbol{q}))^{-1} = \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})$$

- Task-space dynamic model

$$\boldsymbol{\Lambda}(\boldsymbol{q})\ddot{\boldsymbol{y}} + \boldsymbol{\psi}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{\eta}(\boldsymbol{q}) = \boldsymbol{\gamma}$$

with velocity and gravity terms in task space

$$\begin{aligned}\boldsymbol{\psi}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{\Lambda}(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\\ \boldsymbol{\eta}(\boldsymbol{q}) &= \boldsymbol{\Lambda}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q})\end{aligned}$$

- For nonredundant robots, task variables $\boldsymbol{y}$ form a set of generalized coordinates far from singularities

- In the redundant case ($m < n$), infinitely many joint torques yield the same task-space generalized force γ
- Decompose a generic torque vector as

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\gamma + (\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{H}(\boldsymbol{q}))\boldsymbol{\tau}_N$$

  where $\boldsymbol{H}$ is a generalized inverse of $\boldsymbol{J}^T$, so that $\boldsymbol{J}^T\boldsymbol{H}\boldsymbol{J}^T = \boldsymbol{J}^T$, and $\boldsymbol{\tau}_N$ is an arbitrary n-vector
- With $\boldsymbol{J}(\boldsymbol{q})$ full row rank, substituting the decomposition yields

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{J}^T\gamma - \boldsymbol{c} - \boldsymbol{g}) + \boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{H})\boldsymbol{\tau}_N + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}$$

- Dynamic consistency requires that the $\boldsymbol{\tau}_N$ component does not contribute to $\ddot{\boldsymbol{y}}$

$$\boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{H}) = \boldsymbol{O}$$

- Dynamically consistent choice of $\boldsymbol{H}$

$$\boldsymbol{H} = (\boldsymbol{J}^T)^\dagger_{\boldsymbol{M}} = (\boldsymbol{J}^\dagger_{\boldsymbol{M}})^T = (\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{\Lambda}\boldsymbol{J}\boldsymbol{M}^{-1}$$

$$\boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{J}^\dagger_{\boldsymbol{M}})^T) = \boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{J}\boldsymbol{M}^{-1}) = \boldsymbol{O}$$

so that the torque decomposition becomes

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\gamma} + (\boldsymbol{I} - \boldsymbol{J}^\dagger_{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}))^T\boldsymbol{\tau}_N$$

- Setting $\boldsymbol{\tau}_N = \boldsymbol{0}$ gives $\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\gamma}$
- $\boldsymbol{J}^\dagger_{\boldsymbol{M}}$ is the dynamically consistent pseudoinverse of $\boldsymbol{J}$

$$\boldsymbol{\tau} = \boldsymbol{\tau} + \boldsymbol{J}^T(\boldsymbol{J}^\dagger_{\boldsymbol{M}})^T\boldsymbol{\tau} - \boldsymbol{J}^T(\boldsymbol{J}^\dagger_{\boldsymbol{M}})^T\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{J}^\dagger_{\boldsymbol{M}})^T\boldsymbol{\tau} + (\boldsymbol{I} - \boldsymbol{J}^\dagger_{\boldsymbol{M}}\boldsymbol{J})^T\boldsymbol{\tau}$$

$$\boldsymbol{\gamma} = (\boldsymbol{J}^\dagger_{\boldsymbol{M}}(\boldsymbol{q}))^T\boldsymbol{\tau}$$

- For nonredundant robots, $\boldsymbol{J}^\dagger_{\boldsymbol{M}} = \boldsymbol{J}^{-1}$ and this reduces to the inverse of $\boldsymbol{\tau} = \boldsymbol{J}^T\boldsymbol{\gamma}$

- Task-space model for nonredundant and redundant robots

$$\boldsymbol{\Lambda}\,\ddot{\boldsymbol{y}} = \boldsymbol{\gamma} - (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger})^T(\boldsymbol{c}+\boldsymbol{g}) + \boldsymbol{\Lambda}\dot{\boldsymbol{J}}\dot{\boldsymbol{q}}$$

- Reordering terms gives again the task-space dynamic model

$$\boldsymbol{\Lambda}(\boldsymbol{q})\ddot{\boldsymbol{y}} + \boldsymbol{\psi}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{\eta}(\boldsymbol{q}) = \boldsymbol{\gamma}$$

- In the task-space model the velocity terms can be factorized as $\boldsymbol{\psi}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{y}}$ so that $\dot{\boldsymbol{\Lambda}}(\boldsymbol{q}) - 2\boldsymbol{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is skew-symmetric
- A valid choice is

$$\boldsymbol{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}}) = (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q}))^T\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{J}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q}) - \boldsymbol{\Lambda}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\boldsymbol{J}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q})$$

- As task Jacobian one may choose the geometric Jacobian $\boldsymbol{J}_g$, yielding

$$\boldsymbol{\Lambda}_g(\boldsymbol{q})\dot{\boldsymbol{t}}_e + \boldsymbol{\Gamma}_g(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{t}_e + \boldsymbol{\eta}_g(\boldsymbol{q}) = \boldsymbol{w}_e$$

  with the control wrench $\boldsymbol{w}_e$ coming from the decomposition

$$\boldsymbol{\tau} = \boldsymbol{J}_g^T\boldsymbol{w}_e + (\boldsymbol{I} - \boldsymbol{J}_g^T\boldsymbol{H}_g)\boldsymbol{\tau}_N$$

- Task-space direct dynamics: compute $\ddot{\boldsymbol{y}}$ from joint torques $\boldsymbol{\tau}$ at state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$
  - nonredundant case: obtain $\boldsymbol{\gamma}$ by inverting $\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{\gamma}$
  - redundant case: obtain $\boldsymbol{\gamma}$ from $\boldsymbol{\tau}$ via $\boldsymbol{\gamma} = (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger})^T \boldsymbol{\tau}$
- For redundant robots, the task-space model does not capture internal motions for simulation purposes
- Task-space inverse dynamics: compute $\boldsymbol{\gamma}$ from desired $\ddot{\boldsymbol{y}}$, then compute joint torques
  - redundant case: all torques realizing $\boldsymbol{\gamma}$ are given by the dynamically consistent decomposition, with choices of $\boldsymbol{\tau}_N$ generating internal motions

Constrained Dynamic Model

- The robot in free motion has been considered as a fully actuated mechanical system described by the EL model in terms of generalized coordinates $\boldsymbol{q}$
- Suppose now that the robot motion is subject to a set of k physical constraints imposed on its mechanical parts
- Constraints on the end-effector pose variables $\boldsymbol{y}_e$ can be written as $\boldsymbol{f}(\boldsymbol{y}_e) = \mathbf{0}$ and, from the direct kinematics $\boldsymbol{y}_e = \boldsymbol{k}_e(\boldsymbol{q})$, as geometric constraints on $\boldsymbol{q}$

$$\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{k}_e(\boldsymbol{q})) = \mathbf{0}$$

- The constraint Jacobian is

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{q}} \in \mathbb{R}^{k \times n}$$

- Differentiating the constraint gives the kinematic constraints ($\boldsymbol{A}$ full row rank k)

$$\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathbf{0}$$

Example

- 2R planar arm with end-effector position $\boldsymbol{p}_e = (p_x, p_y) = \boldsymbol{k}_e(\boldsymbol{q})$ and Jacobian $\boldsymbol{J}(\boldsymbol{q})$
- End-effector position is constrained to move on a circle with center at (x_0, y_0) and radius R —a single scalar constraint ($k = 1$)

$$f(\boldsymbol{p}_e) = (p_x - x_0)^2 + (p_y - y_0)^2 - R^2 = 0$$

$$h(\boldsymbol{q}) = f(\boldsymbol{k}_e(\boldsymbol{q})) = (a_1 c_1 + a_2 c_{12} - x_0)^2 + (a_1 s_1 + a_2 s_{12} - y_0)^2 - R^2 = 0$$

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h}{\partial \boldsymbol{q}} = \frac{\partial f}{\partial \boldsymbol{p}_e}\frac{\partial \boldsymbol{p}_e}{\partial \boldsymbol{q}} &= \begin{pmatrix} 2(p_x - x_0) & 2(p_y - y_0) \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \\
&= \begin{pmatrix} 2(a_1 c_1 + a_2 c_{12} - x_0) & 2(a_1 s_1 + a_2 s_{12} - y_0) \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \\
&= \begin{pmatrix} 2x_0\,(a_1 s_1 + a_2 s_{12}) - 2y_0\,(a_1 c_1 + a_2 c_{12}) & 2a_2\,(x_0 s_{12} - y_0 c_{12}) - 2a_1 a_2 s_2 \end{pmatrix}
\end{aligned}$$

- The augmented Lagrangian is

$$\mathcal{L}_a(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\lambda}^T \boldsymbol{h}(\boldsymbol{q}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) + \boldsymbol{\lambda}^T \boldsymbol{h}(\boldsymbol{q})$$

  where $\boldsymbol{\lambda} \in \mathbb{R}^k$ collects the Lagrange multipliers associated with the constraints
- Constrained EL equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_a}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}_a}{\partial \boldsymbol{q}}\right)^T = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T - \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} = \boldsymbol{\tau}$$
$$\left(\frac{\partial \mathcal{L}_a}{\partial \boldsymbol{\lambda}}\right)^T = \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$$

  or mixed differential–algebraic equations containing the unknown multipliers

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda}$$
$$\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$$

Projected Robot Dynamics

- Differentiating $\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}$ yields

$$\boldsymbol{A}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{A}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}$$

- A closed-form expression for $\boldsymbol{\lambda}$ is obtained as

$$\boldsymbol{\lambda} = (\boldsymbol{A}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q}))^T (\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{\tau}) - (\boldsymbol{A}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{A}^T(\boldsymbol{q}))^{-1}\dot{\boldsymbol{A}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

  where $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$

- Projected dynamic model

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q})(\boldsymbol{A}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q}))^T) (\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{A}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- $\boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{A}_{\boldsymbol{M}}^{\dagger}(\boldsymbol{q}))^T$ is a (dynamic) projection matrix ensuring that, regardless of $\boldsymbol{\tau}$, the solution to these n 2nd-order differential equations complies with the constraints

Reduced Robot Dynamics

- Complete $\boldsymbol{A}(\boldsymbol{q})$ to a square, nonsingular matrix by adding a $(n-k) \times n$ matrix $\boldsymbol{D}(\boldsymbol{q})$

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{G}(\boldsymbol{q}) \end{pmatrix}$$

  with $\boldsymbol{E}(\boldsymbol{q}) \in \mathbb{R}^{n\times k}$, $\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^{n\times(n-k)}$
- The columns of $\boldsymbol{G}$ are a basis of the null space of $\boldsymbol{A}$
- Reduced velocities and accelerations

$$\boldsymbol{u} = \boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}} \quad \Rightarrow \quad \dot{\boldsymbol{u}} = \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{D}}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

- Inverse relationships (exploiting identities in the product of a matrix times its inverse)

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} \\ \ddot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{u}} + \dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{u} = \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{u}} - (\boldsymbol{E}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q}))\boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} \end{aligned}$$

Example

- 2R planar arm with end-effector constrained on a circle [reprise]

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} -\frac{1}{2}\left(p_y - y_0\right) & \frac{1}{2}\left(p_x - x_0\right) \end{pmatrix} \boldsymbol{J}(\boldsymbol{q})$$

$$\det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \det \left(\begin{pmatrix} 2(p_x - x_0) & 2(p_y - y_0) \\ -\frac{1}{2}\left(p_y - y_0\right) & \frac{1}{2}\left(p_x - x_0\right) \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \right) = R^2 \cdot \det \boldsymbol{J}(\boldsymbol{q})$$

- The reduced velocity $u = \boldsymbol{D}(\boldsymbol{q})\,\dot{\boldsymbol{q}} \in I\!R$ is simply the speed (with sign) of the end-effector along the tangent to the circle
- Admissible joint velocities

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\, u = \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{pmatrix} -2(p_y - y_0) \\ 2(p_x - x_0) \end{pmatrix} \frac{u}{R^2}$$

- Premultiplying the dynamics (being $\boldsymbol{A}\boldsymbol{E} = \boldsymbol{I}$ and $\boldsymbol{A}\boldsymbol{G} = \boldsymbol{O}$)

$$\boldsymbol{E}^T(\boldsymbol{q})\left(\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{\tau}\right) = \boldsymbol{\lambda}$$

$$\boldsymbol{G}^T(\boldsymbol{q})\left(\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right) = \boldsymbol{G}^T(\boldsymbol{q})\,\boldsymbol{\tau}$$

- Substituting in terms of reduced velocities and accelerations

$$\begin{aligned}\dot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}\\ \ddot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{u}} + \dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{u} = \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{u}} - \left(\boldsymbol{E}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q})\right)\boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}\end{aligned}$$

$$\Downarrow$$

$$\boldsymbol{M}_{\mathrm{red}}(\boldsymbol{q})\dot{\boldsymbol{u}} + \boldsymbol{n}_{\mathrm{red}}(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{G}^T(\boldsymbol{q})\,\boldsymbol{\tau}$$

$$\begin{aligned}\boldsymbol{M}_{\mathrm{red}}(\boldsymbol{q}) &= \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\\ \boldsymbol{n}_{\mathrm{red}}(\boldsymbol{q},\boldsymbol{u}) &= \boldsymbol{G}^T(\boldsymbol{q})\left(\boldsymbol{n}(\boldsymbol{q},\boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}) + \boldsymbol{M}(\boldsymbol{q})\,\dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{u}\right)\end{aligned}$$

$$\boldsymbol{\lambda} = \boldsymbol{E}^T(\boldsymbol{q})\Big(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{u}} + \boldsymbol{M}(\boldsymbol{q})\big(\boldsymbol{E}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q})\big)\dot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{\tau}\Big)$$

- The reduced dynamic model is lower-dimensional ($n-k$) than the projected dynamic model (n), but computationally more complex (the latter is preferred for simulation purposes)
- The reduced model is very useful for inverse dynamics computation as well as for control design based on feedback linearization

$$\boldsymbol{\tau} = \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\left(\boldsymbol{E}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{G}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q})\right)\dot{\boldsymbol{q}} + \boldsymbol{M}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\,\boldsymbol{a} - \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{f}$$

$$\Downarrow$$

$$\dot{\boldsymbol{u}} = \boldsymbol{a} \qquad \boldsymbol{\lambda} = \boldsymbol{f}$$

i.e., a desired reduced acceleration $\boldsymbol{a}$ and a desired constraint force $\boldsymbol{f}$ have been imposed to the closed-loop system