FOUNDATIONS OF ROBOTICS
Motion Control

B. Siciliano, L. Villani, G. Oriolo, A. De Luca

These slides are distributed as companion material to the textbook
Foundations of Robotics by B. Siciliano, L. Villani, G. Oriolo, A. De Luca
published by Springer Nature.
Modification is not allowed without permission

Outline

- The Motion Control Problem
- Single Joint Control
- Regulation in Joint Space
- Trajectory Tracking in Joint Space
- Control in Task Space

The Motion Control Problem

- The control problem for a robot consists in determining suitable input commands that allow the correct (or the best possible) execution of a planned task, both in nominal conditions and in the presence of model uncertainties or external disturbances
- Controlling a robot can be interpreted in multiple ways, according to the level of definition
  - at a high level, controlling a robot could mean to successfully complete a work program
  - at an intermediate level, it implies the capability of following portions of desired trajectories
  - at a low level, it requires the joint position error to be zeroed

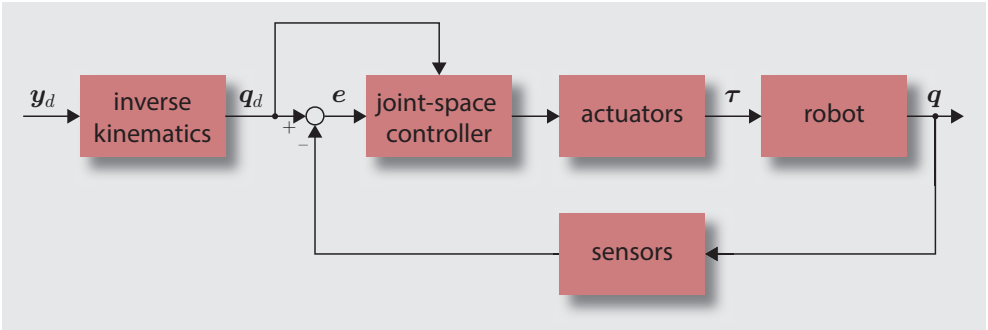
- The performance of a robot control system is evaluated from two points of view
  - nominal task execution, e.g., velocity/speed of motion completion, positioning accuracy, and energy requirements
  - robustness with respect to uncertainty or perturbations, including repeatability, parameter variation, or measurement noise
- Robustness mainly stems from the generalized application of feedback strategies, i.e., real-time use of sensor information, as well as extensive data collection and processing over repeated trials for learning purposes
- Control design is discussed under the assumption that the robot is not in contact with the environment, i.e., in the case of pure motion control
  - regulation, in which the robot must be driven to a desired setpoint in configuration space or in task space (e.g., a desired end-effector posture)
  - trajectory tracking, in which the robot must follow a desired trajectory in configuration space or in task space

- Robot dynamics is highly nonlinear and the motion of each link is generally coupled with the motion of the others
- A solution can only be found by a centralized approach which looks simultaneously at the whole robot
- Uniform performance is difficult to guarantee without resorting to some information about the dynamic model of the robot
- With motors connected to the joints by gears with large reduction ratios, each joint is approximately described by a linear and decoupled system
- This can be effectively controlled using a decentralized approach, treating neglected coupling terms as a disturbance to be rejected as much as possible

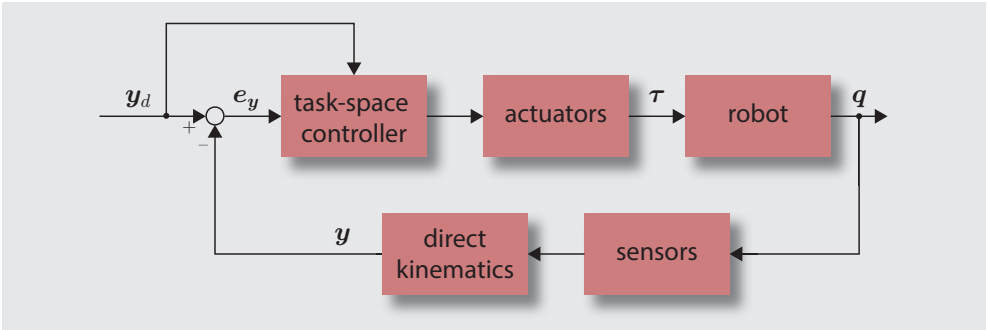
- Low-level actuator controllers are ubiquitous in all industrial robots (and beyond)
- This allows the implementation of kinematic control, in which purely kinematic commands are generated and sent as references to the actuator control loops
  - there are many situations in which the behavior of kinematic controllers is not satisfactory
  - this happens for highly dynamic motions, or when low-reduction gears or direct-drive actuation are adopted
- In all these cases, dynamic control laws are required and the actuator control design must be suitably modified to allow the execution of commands that are defined at the torque level

- The desired robot motion can be specified by a reference in joint space or in task space, denoted by $\boldsymbol{q}_d$ or $\boldsymbol{y}_d$
- Control actions are always defined in the joint (or actuator) space
- Two types of schemes are considered, namely, joint-space control and task-space control
- Control is obtained by the combination of a feedback action and a feedforward action
- In joint-space control, the error is $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$
- In task-space control, the error is $\boldsymbol{e}_{\boldsymbol{y}} = \boldsymbol{y}_d - \boldsymbol{y}$
- Both schemes produce a control signal that the actuators convert into a joint torque $\boldsymbol{\tau}$

- General scheme of joint-space control



- General scheme of task-space control



- Joint-space control is typically decomposed in two subproblems
  - inverse kinematics maps $\boldsymbol{y}_d$ to a corresponding desired motion $\boldsymbol{q}_d$
  - a control scheme is designed so that $\boldsymbol{q}$ reproduces $\boldsymbol{q}_d$ and the error $\boldsymbol{e}$ is zeroed asymptotically
  - with this approach, task-space variables $\boldsymbol{y}$ are essentially controlled in an open-loop fashion through the robot mechanical structure
- Task-space control acts directly on the task error, embedding inverse kinematics into the feedback control loop
  - this implies an increased computational load and requires the capability to handle singularities or redundancy in real time
- Sampling intervals in the order of 0.5 to 10 ms are considered satisfactory for most robot control problems

Single Joint Control

- Design of low-level actuator control laws for a single robot joint including actuator dynamics
- Electrical motors are the most common actuators in robotics
- Modeling and control techniques for the case of a DC motor

Modeling a DC Motor

- Consider a DC motor driven by the armature voltage v_a and denote its angular position by θ_m
- Electrical equation

$$L_a \frac{di_a}{dt} = v_a - R_a i_a - k_v \omega_m$$

  - R_a and L_a armature resistance and inductance
  - k_v back electromotive force constant
  - $\omega_m = \dot{\theta}_m$ motor angular velocity
- Torque relation

$$\tau_a = k_t i_a$$

  - k_t torque constant numerically equal to k_v in the SI unit system

- Mechanical equation

$$I_m\dot{\omega}_m + B_m\omega_m = \tau_a - \tau_l$$

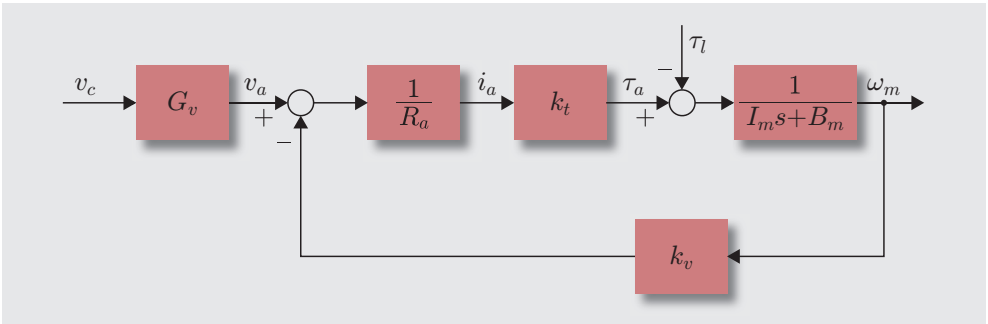
  - I_m motor moment of inertia
  - B_m viscous friction coefficient
  - τ_l load torque acting as a disturbance

- In the Laplace domain

$$(L_a s + R_a)\, i_a(s) = v_a(s) - k_v\omega_m(s)$$

$$(I_m s + B_m)\, \omega_m(s) = k_t i_a(s) - \tau_l(s)$$

- DC motor block diagram



- Simplified motor model
  - neglect the armature inductance dynamics and include an amplifier with gain $G_v > 0$
  - velocity response

$$\omega_m(s) = \frac{1}{I_m s + B'_m} \left(\frac{G_v k_t}{R_a} v_c(s) - \tau_l(s) \right)$$

  - modified viscous term

$$B'_m = B_m + \frac{k_v k_t}{R_a}$$

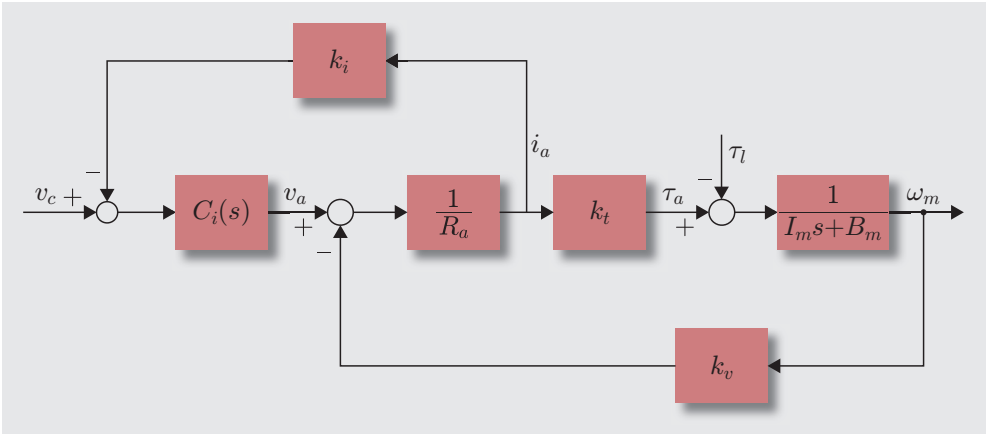
  - steady-state response to a constant voltage v_c

$$\omega_m \approx \frac{G_v}{k_v} v_c$$

  - assume B_m is negligible with respect to the other term in B'_m
  - in this case the motor behaves as a velocity generator

- DC motor with current feedback
  - armature current measurement via a transducer of constant k_i and a high gain controller $C_i(s)$
  - approximate torque relation
$$\tau_a = k_t\, i_a \approx \frac{k_t}{k_i}\, v_c$$
  - the last approximate equality is an equality if the current controller $C_i(s)$ contains an integral action
  - in this case the motor behaves as a torque generator
  - velocity response
$$\omega_m(s) = \frac{1}{I_m s + B_m}\left(\frac{k_t}{k_i}\, v_c(s) - \tau_l(s)\right)$$

- Block diagram of DC motor with current feedback



- Unified representation

$$\omega_m(s) = M(s)\left(\tau_m(s) - \tau_l(s)\right)$$

  - input relation

$$\tau_m = \alpha_m v_c$$

  - motor transfer function

$$M(s) = \frac{G_m}{1 + T_m s}$$

- For a motor used as velocity generator

$$G_m = \frac{1}{B'_m} \qquad T_m = \frac{I_m}{B'_m} \qquad \alpha_m = \frac{G_v k_t}{R_a}$$

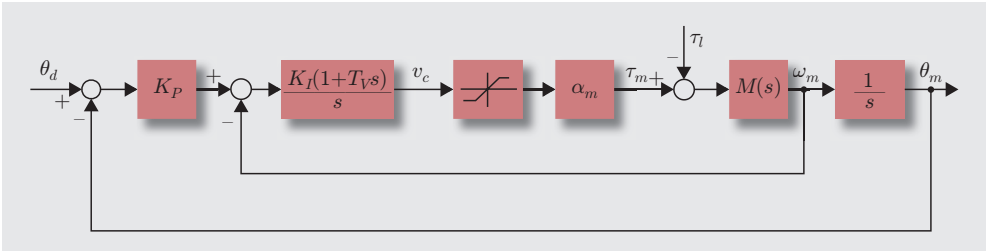
- For a motor used as torque generator

$$G_m = \frac{1}{B_m} \qquad T_m = \frac{I_m}{B_m} \qquad \alpha_m = \frac{k_t}{k_i}$$

Position Control

- Regulate the angular position θ_m of the motor to a constant reference value θ_d
- Cascaded design with two feedback loops
  - inner velocity loop
  - outer position loop
- Measurements from a position encoder with a numerical filter for velocity reconstruction
- Nonlinear saturation represents a limitation on the motor driving voltage and on the torque that can actually be delivered

- Block scheme of servomotor for position control



- Inner velocity loop PI controller transfer function

$$\frac{K_I}{s} + K_V = \frac{K_I(1 + T_V s)}{s}$$

  - $K_I > 0$
  - $T_V = K_V / K_I > 0$
  - integral action achieves steady-state rejection of constant load disturbances
  - outer position loop uses a proportional controller with gain K_P

- With $T_V = T_m$ the closed-loop transfer function becomes

$$W(s) = \frac{\theta_m(s)}{\theta_d(s)} = \frac{1}{1 + \dfrac{s}{K_P} + \dfrac{s^2}{\alpha_m G_m K_P K_I}}$$

- Canonical second-order transfer function

$$\frac{1}{1 + \dfrac{2\zeta s}{\omega_n} + \dfrac{s^2}{\omega_n^2}}$$

- Choice of K_P and K_I for given ω_n and ζ

$$K_P = \frac{\omega_n}{2\zeta} \qquad K_I = \frac{2\zeta\omega_n}{\alpha_m G_m}$$

- Disturbance transfer function from τ_l to θ_m

$$W_d(s) = \frac{\theta_m(s)}{\tau_l(s)} = -\frac{s}{\alpha_m K_P K_I (1 + T_m s)} \frac{1}{1 + \frac{s}{K_P} + \frac{s^2}{\alpha_m G_m K_P K_I}}$$

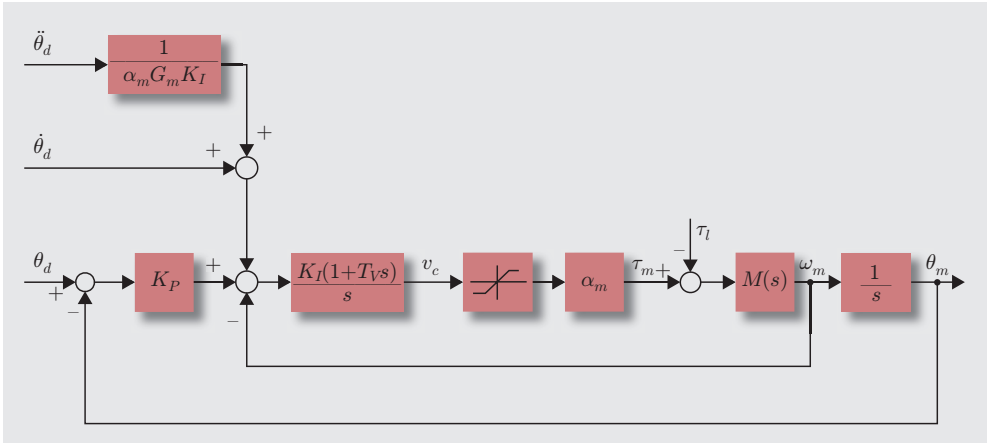
- The zero at the origin confirms that constant disturbances are rejected at steady state
- Output recovery time as dominant time constant

$$T_R = \max \left\{ T_m, \frac{2}{\alpha_m G_m K_I} \right\}$$

Including a Feedforward Action

- The cascaded scheme guarantees regulation for constant θ_d and constant load torque
- With a time-varying reference $\theta_d(t)$ tracking performance is degraded
- A feedforward action allows recovering error convergence

- Block diagram of control with feedforward action



- Modified reference signal

$$\theta'_d(s) = \frac{1}{W(s)}\,\theta_d(s) = \left(1 + \frac{s}{K_P} + \frac{s^2}{\alpha_m G_m K_P K_I}\right)\theta_d(s)$$

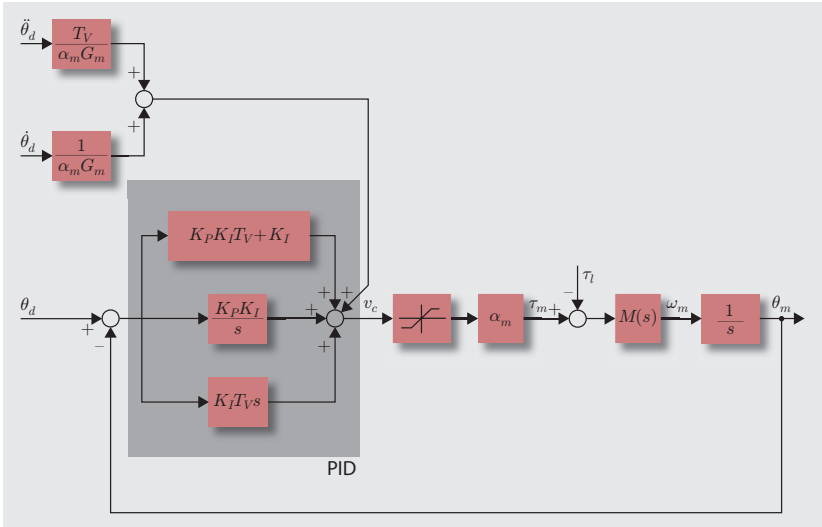
- The transfer function from θ_d to θ_m becomes unitary and stable
- Exact tracking at steady state for any reference time varying or not
- Desired velocity $\dot{\theta}_d$ and acceleration $\ddot{\theta}_d$ are computed from the desired trajectory
- Ideal performance assumes perfect knowledge of the model parameters

- Control voltage

$$
\begin{aligned}
v_c(s) &= \frac{K_I(1+T_V s)}{s}(K_P+s)(\theta_d(s)-\theta_m(s)) + \frac{1}{\alpha_m G_m}\left(1+T_V s\right)\dot{\theta}_d(s) \\
&= \left(K_P K_I T_V + K_I + \frac{K_P K_I}{s} + K_I T_V s\right)(\theta_d(s)-\theta_m(s)) \\
&\quad + \frac{1}{\alpha_m G_m}\,\dot{\theta}_d(s) + \frac{T_V}{\alpha_m G_m}\,\ddot{\theta}_d(s)
\end{aligned}
$$

- Equivalent to a proportional–integral–derivative (PID) control scheme with feedforward action
  - the PID scheme only requires position feedback

- Block diagram of equivalent PID control



Regulation in Joint Space

- Dynamic model in the absence of interaction and friction generalized forces

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

- Equilibrium points are all states $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_e, \boldsymbol{0})$
- Equilibria can be forced with $\boldsymbol{g}(\boldsymbol{q}_e) \neq \boldsymbol{0}$
- Equilibria can be unforced with $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$
- Regulation in joint space determines torques $\boldsymbol{\tau}$ that drive the robot toward the desired equilibrium $(\boldsymbol{q}_d, \boldsymbol{0})$

Independent PID Regulation

- The robot is treated as formed by n single-input single-output subsystems
- Each joint is controlled independently
- Inertial velocity and gravitational coupling among the joints are treated as disturbance inputs
- Motor dynamics for the i-th joint

$$I_{mi}\ddot{\theta}_{mi} + B_{mi}\dot{\theta}_{mi} = \tau_{mi} - \tau_{li}$$

- Link side dynamics

$$\boldsymbol{\tau}_q = \boldsymbol{M}_l(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$$

- i-th component of the dynamic model

$$\tau_{qi} = \left(m_{ii,a} + m_{ii,b}(\boldsymbol{q})\right) \ddot{q}_i + \sum_{j=1,\ldots,n}^{j \neq i} m_{ij}(\boldsymbol{q})\ddot{q}_j + c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) + g_i(\boldsymbol{q})$$

- Gear relation between link acceleration and motor acceleration

$$\ddot{q}_i = \frac{\ddot{\theta}_{mi}}{n_{ri}}$$

- Relation between joint torque and load torque

$$\tau_{qi} = n_{ri}\tau_{li}$$

- Power equality

$$\tau_{li}\,\dot{\theta}_{mi} = \tau_{qi}\,\dot{q}_i$$

- Load torque decomposition

$$\tau_{li} = \frac{m_{ii,a}}{n_{ri}^2}\,\ddot{\theta}_{mi} + d_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$$

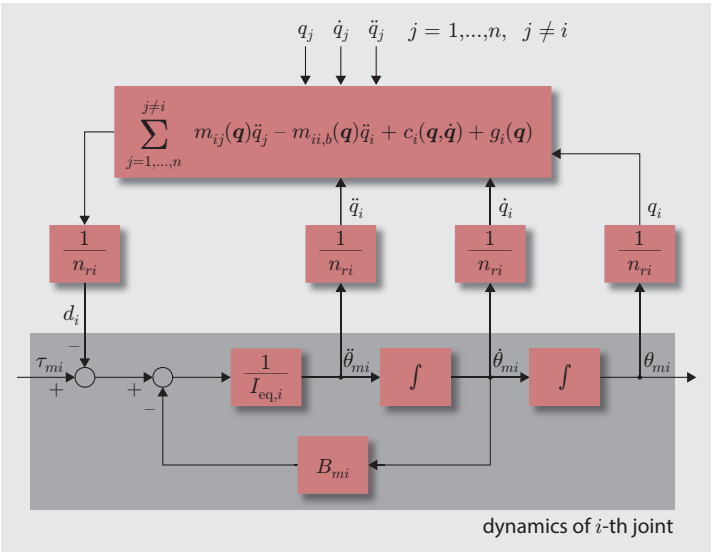
- Disturbance term

$$d_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \frac{1}{n_{ri}} \Big(\sum_{j=1,\ldots,n}^{j\neq i} m_{ij}(\boldsymbol{q})\ddot{q}_j - m_{ii,b}(\boldsymbol{q})\ddot{q}_i + c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) + g_i(\boldsymbol{q}) \Big)$$

- Motor side dynamics with equivalent inertia and disturbance

$$\left(I_{mi} + \frac{m_{ii,a}}{n_{ri}^2} \right) \ddot{\theta}_{mi} + B_{mi}\dot{\theta}_{mi} = \tau_{mi} - d_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$$

- Block diagram of robot dynamics for independent joint control



- Regulation problem for independent joint control to be addressed using the cascaded control scheme, which can be rewritten as a PID controller
- Convergence to $(q_{di}, 0)$ is no longer guaranteed because d_i is time-varying, yet steady state can be reached thanks to integral action (local result)
- Resulting controller is decentralized: each PID action based on the corresponding joint error
- Knowledge of system dynamic parameters is not needed, yet they affect the performance of the closed-loop system

PD Control with Gravity Cancellation

- Joint-space position error

$$\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$$

- Candidate Lyapunov function

$$V = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\,\boldsymbol{e}^T \boldsymbol{K}_P \boldsymbol{e}$$

- Time derivative of V

$$\dot{V} = \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\,\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{K}_P \boldsymbol{e}$$

- Using factorization $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$

$$\dot{V} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) \dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T \left(\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}_P \boldsymbol{e}\right)$$

- PD control law with gravity cancellation

$$\boldsymbol{\tau} = \boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

- Resulting Lyapunov derivative

$$\dot{V} = -\dot{\boldsymbol{q}}^T \boldsymbol{K}_D \, \dot{\boldsymbol{q}}$$

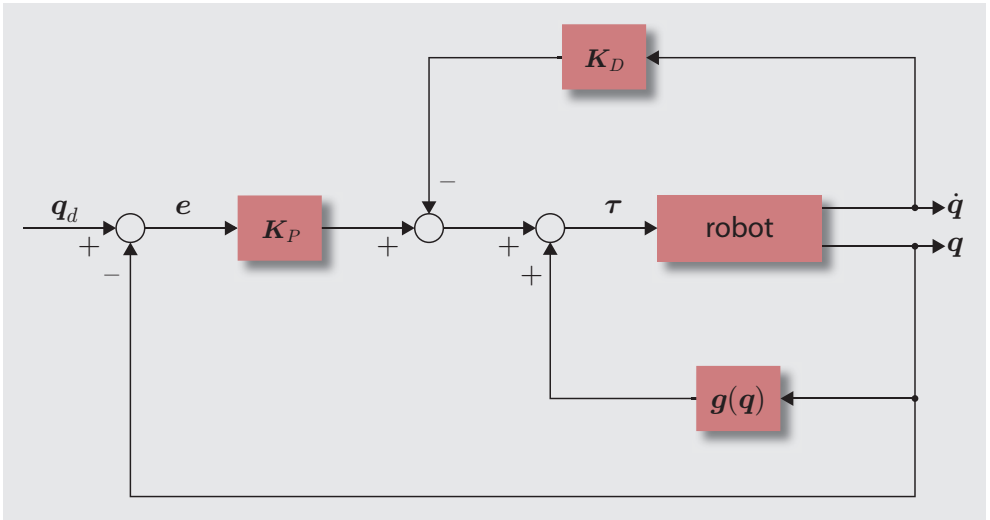
- Closed-loop relation

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{K}_P \boldsymbol{e}$$

$$\Downarrow$$

- Global asymptotic stability of $(\boldsymbol{q}_d, \boldsymbol{0})$ is guaranteed

- Block diagram of PD control with gravity cancellation



PD Control with Gravity Compensation

- Compensation of gravity at desired equilibrium

$$\boldsymbol{\tau} = \boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d)$$

- Matrix norm

$$\|\boldsymbol{A}\| = \sqrt{\lambda_{\max}(\boldsymbol{A}^T \boldsymbol{A})} = A_M$$

- Minimum singular value

$$A_m = \sqrt{\lambda_{\min}(\boldsymbol{A}^T \boldsymbol{A})} \geq 0$$

- Bounded Hessian condition

$$\left\| \frac{\partial^2 \mathcal{U}}{\partial \boldsymbol{q}^2} \right\| = \left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right\| \leq \alpha \quad \text{for all } \boldsymbol{q}$$

- Lipschitz bound on gravity

$$\|\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q}_d)\| \leq \alpha \|\boldsymbol{q} - \boldsymbol{q}_d\| \quad \text{for all } \boldsymbol{q}, \boldsymbol{q}_d$$

- Condition on proportional gain

$$K_{P,m} > \alpha$$

- Gain dominance inequality

$$\|\boldsymbol{K}_P \boldsymbol{e}\| \geq K_{P,m} \|\boldsymbol{e}\| > \alpha \|\boldsymbol{e}\| \geq \|\boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})\|$$

- Static balance relation at equilibrium

$$\boldsymbol{K}_P \boldsymbol{e} = \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q}_d)$$

- Potential energy approximation around $\boldsymbol{q}_d$

$$\mathcal{U}(\boldsymbol{q}) \approx \mathcal{U}(\boldsymbol{q}_d) + \left.\frac{\partial \mathcal{U}}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{q}_d} (\boldsymbol{q} - \boldsymbol{q}_d) = \mathcal{U}(\boldsymbol{q}_d) - \boldsymbol{g}^T(\boldsymbol{q}_d)\, \boldsymbol{e}$$

- Candidate Lyapunov function

$$V = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} + \frac{1}{2}\, \boldsymbol{e}^T \boldsymbol{K}_P \boldsymbol{e} + \mathcal{U}(\boldsymbol{q}) - \mathcal{U}(\boldsymbol{q}_d) + \boldsymbol{e}^T \boldsymbol{g}(\boldsymbol{q}_d)$$

- Time derivative of V

$$\dot{V} = \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \frac{1}{2}\, \dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q}) \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{K}_P \boldsymbol{e} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{q}}\, \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}_d)$$

- Resulting $\dot{V}$

$$\begin{aligned} \dot{V} &= \dot{\boldsymbol{q}}^T (\boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})) - \dot{\boldsymbol{q}}^T \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{g}^T(\boldsymbol{q}) \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}_d) \\ &= -\dot{\boldsymbol{q}}^T \boldsymbol{K}_D\, \dot{\boldsymbol{q}} \end{aligned}$$

- Closed-loop dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{g}(\boldsymbol{q}_d)$$

- Equivalent acceleration expression

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q}))$$

$$\Downarrow$$

- Global asymptotic stability of $(\boldsymbol{q}_d, \boldsymbol{0})$ is guaranteed
- Alternative scheme with integral action

$$\boldsymbol{\tau} = \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I \int_0^t \boldsymbol{e}(s)ds - \boldsymbol{K}_D\dot{\boldsymbol{q}}$$

Trajectory Tracking in Joint Space

- Regulation laws can track slowly time-varying references especially with high reduction ratios
- Tracking performance degrades for large velocities and accelerations due to nonlinear and coupling terms
- Model-based control uses the full robot dynamic model
- First assume perfect knowledge of the model parameters, then relax this hypothesis

Independent PID Tracking

- Desired joint trajectory is $\boldsymbol{q}_d(t)$
- Independent controllers are designed for each joint
- Neglected inertial velocity and gravitational terms collected as d_i act as persistent time-varying disturbances
- Add a feedforward torque $\tau_{di}(t)$ at joint i
  - $\tau_{di}(t)$ is the i-th component of $\boldsymbol{\tau}_d(t) = \boldsymbol{\tau}_d(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t), \ddot{\boldsymbol{q}}_d(t))$
- $\boldsymbol{\tau}_d(t)$ is computed along $\boldsymbol{q}_d(t)$ through inverse dynamics of the robot
- The feedforward term can be scaled by α_{mi} and added to the PID output
- Feedback acts locally on the trajectory error $q_{di} - q_i$
- Feedforward is centralized since $\tau_{di}(t)$ includes the full robot dynamics
- Tracking performance is satisfactory for smooth trajectories of limited speed and acceleration especially with large reduction gears

Control Based on Feedback Linearization

- Desired joint trajectory is $\boldsymbol{q}_d(t)$
- Robot dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

- Nonlinear terms

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$$

- Exact linearizing control law

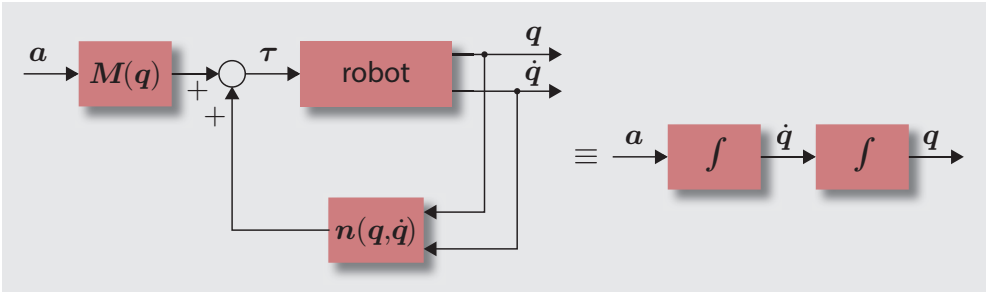
$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Closed-loop dynamics

$$\ddot{\boldsymbol{q}} = \boldsymbol{a}$$

and the system becomes n decoupled chains of double integrators

- Linearizing effect of control law



- New acceleration input

$$\boldsymbol{a} = \ddot{\boldsymbol{q}}_d + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}}$$

  where $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$ is the tracking error
- Error dynamics

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_D \dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} = \boldsymbol{0}$$

- The origin is globally exponentially stable
- Control torque expression

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})(\ddot{\boldsymbol{q}}_d + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}}) + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

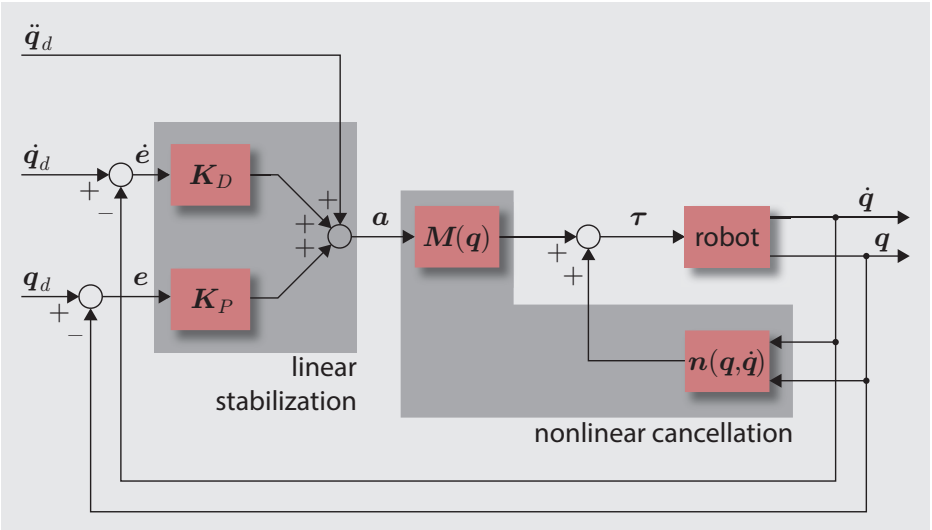
- Gain choice for independent second-order error transients uses diagonal $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$

$$\boldsymbol{K}_P = \operatorname{diag}\{\omega_{n1}^2, \ldots, \omega_{nn}^2\} \qquad \boldsymbol{K}_D = \operatorname{diag}\{2\zeta_1\omega_{n1}, \ldots, 2\zeta_n\omega_{nn}\}$$

- Regulation case with constant $\boldsymbol{q}_d$

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})(\boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{q}}) + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Block diagram of joint-space trajectory tracking via feedback linearization control



- State and input

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix} \qquad \boldsymbol{u} = \boldsymbol{\tau}$$

- Nonlinear affine dynamics

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x}_2 \\ -\boldsymbol{M}^{-1}(\boldsymbol{x}_1)\boldsymbol{n}(\boldsymbol{x}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{M}^{-1}(\boldsymbol{x}_1) \end{pmatrix} \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$$

- Inverse inertia matrix notation

$$\boldsymbol{H}(\boldsymbol{x}_1) = \begin{pmatrix} \boldsymbol{h}_1(\boldsymbol{x}_1) & \dots & \boldsymbol{h}_n(\boldsymbol{x}_1) \end{pmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{x}_1)$$

- First distribution

$$\Delta_{L0} = \text{span} \left\{ \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_1(\boldsymbol{x}_1) \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_n(\boldsymbol{x}_1) \end{pmatrix} \right\}$$

- Lie brackets for all i and j

$$\begin{aligned} \left[\boldsymbol{g}_i, \boldsymbol{g}_j\right] &= \left[\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_i(\boldsymbol{x}_1) \end{pmatrix}, \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_j(\boldsymbol{x}_1) \end{pmatrix} \right] \\ &= \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \dfrac{\partial \boldsymbol{h}_j}{\partial \boldsymbol{x}_1} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_i(\boldsymbol{x}_1) \end{pmatrix} - \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \dfrac{\partial \boldsymbol{h}_i}{\partial \boldsymbol{x}_1} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_j(\boldsymbol{x}_1) \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \end{aligned}$$

- Computation of $[\boldsymbol{f}, \boldsymbol{g}_j]$

$$[\boldsymbol{f}, \boldsymbol{g}_j] = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \dfrac{\partial \boldsymbol{h}_j}{\partial \boldsymbol{x}_1} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_2 \\ -\boldsymbol{H}(\boldsymbol{x}_1)\boldsymbol{n}(\boldsymbol{x}) \end{pmatrix} - \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ * & * \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_j(\boldsymbol{x}_1) \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}_j(\boldsymbol{x}_1) \\ * \end{pmatrix}$$

- Involutive distribution

$$\Delta_{L1} = \text{span} \left\{ \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_1(\boldsymbol{x}_1) \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{h}_n(\boldsymbol{x}_1), \end{pmatrix}, \begin{pmatrix} \boldsymbol{h}_1(\boldsymbol{x}_1) \\ * \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{h}_n(\boldsymbol{x}_1) \\ * \end{pmatrix} \right\}$$

- Output selection

$$\boldsymbol{y} = \boldsymbol{x}_1 = \boldsymbol{q}$$

- Output derivatives

$$\dot{\boldsymbol{y}} = \boldsymbol{x}_2$$

$$\ddot{\boldsymbol{y}} = -\boldsymbol{H}(\boldsymbol{x}_1)\boldsymbol{n}(\boldsymbol{x}) + \boldsymbol{H}(\boldsymbol{x}_1)\boldsymbol{u}$$

- Input transformation

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Linearized input–output relation

$$\ddot{\boldsymbol{y}} = \boldsymbol{a}$$

Control Based on Inverse Dynamics

- Evaluate model-based terms along $\boldsymbol{q}_d(t)$ to reduce real-time computational load
- First alternative version with feedforward torque along reference trajectory (computed torque)

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{q}_d)(\ddot{\boldsymbol{q}}_d + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}) + \boldsymbol{n}(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) \\ &= \boldsymbol{\tau}_d + \boldsymbol{M}(\boldsymbol{q}_d)(\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}})\end{aligned}$$

- Second alternative version

$$\boldsymbol{\tau} = \boldsymbol{\tau}_d + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}$$

- Third alternative version

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}$$

Adaptive Control

- Model uncertainty can be structured or unstructured
- Define reference velocity and filtered error

$$\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Sigma} \boldsymbol{e}$$

$$\boldsymbol{\sigma} = \dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}} = \dot{\boldsymbol{e}} + \boldsymbol{\Sigma} \boldsymbol{e}$$

- Controller with perfect model (passivity-based control)

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}}_r + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}_r + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_D \boldsymbol{\sigma}$$

- Closed-loop error dynamics

$$\boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{\sigma}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \boldsymbol{\sigma} + \boldsymbol{K}_D \boldsymbol{\sigma} = \boldsymbol{0}$$

- Candidate function

$$V = \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{M}(\boldsymbol{q})\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{e}^T \boldsymbol{Q}\boldsymbol{e}$$

- Time derivative

$$\dot{V} = \boldsymbol{\sigma}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\sigma}} + \frac{1}{2}\boldsymbol{\sigma}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{\sigma} + \boldsymbol{e}^T \boldsymbol{Q}\dot{\boldsymbol{e}} = -\boldsymbol{\sigma}^T \boldsymbol{K}_D \boldsymbol{\sigma} + \boldsymbol{e}^T \boldsymbol{Q}\dot{\boldsymbol{e}}$$

- With $\boldsymbol{Q} = 2\boldsymbol{\Sigma}\boldsymbol{K}_D$ one obtains

$$\dot{V} = -\dot{\boldsymbol{e}}^T \boldsymbol{K}_D \dot{\boldsymbol{e}} - \boldsymbol{e}^T \boldsymbol{\Sigma}\boldsymbol{K}_D \boldsymbol{\Sigma}\boldsymbol{e}$$

- Closed-loop system is time-varying ($\boldsymbol{\sigma}$ depends on $\boldsymbol{q}_d(t)$) $\Rightarrow$ Global asymptotic stability by applying Barbalat lemma: V is lower bounded by zero, $\dot{V}$ is nonpositive, and $\ddot{V}$ is bounded

$$\|\boldsymbol{M}^{-1}(\boldsymbol{q})\| \leq \beta_M \qquad \|\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\| \leq \beta_C \|\dot{\boldsymbol{q}}\|$$

- Robot dynamics linear in parameters

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\rho}$$

- Control with estimated parameters

$$\boldsymbol{\tau} = \hat{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \boldsymbol{K}_D\boldsymbol{\sigma} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{\rho}} + \boldsymbol{K}_D\boldsymbol{\sigma}$$

- Parameter error

$$\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho} - \hat{\boldsymbol{\rho}}$$

- Resulting dynamics

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\sigma}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{\sigma} + \boldsymbol{K}_D\boldsymbol{\sigma} = \tilde{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \tilde{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \tilde{\boldsymbol{g}}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\tilde{\boldsymbol{\rho}}$$

- Lyapunov function

$$V = \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{M}(\boldsymbol{q})\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{e}^T \boldsymbol{Q}\boldsymbol{e} + \frac{1}{2}\tilde{\boldsymbol{\rho}}^T \boldsymbol{K}_{\boldsymbol{\rho}}^{-1}\tilde{\boldsymbol{\rho}}$$

- Time derivative

$$\dot{V} = -\dot{\boldsymbol{e}}^T \boldsymbol{K}_D \dot{\boldsymbol{e}} - \boldsymbol{e}^T \boldsymbol{\Sigma}\boldsymbol{K}_D\boldsymbol{\Sigma}\boldsymbol{e} + \tilde{\boldsymbol{\rho}}^T(\boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\boldsymbol{\sigma} - \boldsymbol{K}_{\boldsymbol{\rho}}^{-1}\dot{\hat{\boldsymbol{\rho}}})$$

- Adaptation law

$$\dot{\hat{\boldsymbol{\rho}}} = \boldsymbol{K}_{\boldsymbol{\rho}}\boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\boldsymbol{\sigma}$$

- At steady state

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}) = \boldsymbol{0}$$

  - $\hat{\boldsymbol{\rho}}$ is ultimately bounded but does not converge to $\boldsymbol{\rho}$
  - convergence depends on the structure of $\boldsymbol{Y}$ (persistency of excitation)

- Lyapunov derivative

$$\dot{V} = -\dot{\boldsymbol{e}}^T \boldsymbol{K}_D \dot{\boldsymbol{e}} - \boldsymbol{e}^T \boldsymbol{\Sigma} \boldsymbol{K}_D \boldsymbol{\Sigma} \boldsymbol{e}$$

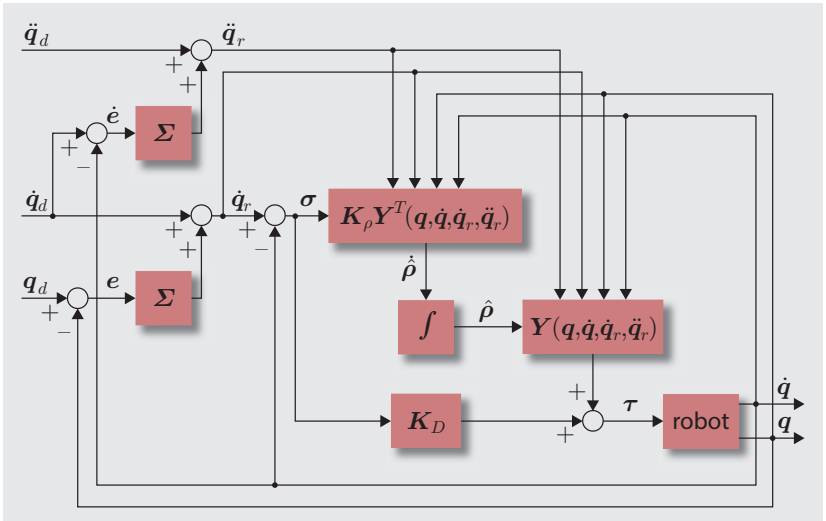
- Implemented controller form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{\rho}} + \boldsymbol{K}_D(\dot{\boldsymbol{e}} + \boldsymbol{\Sigma}\boldsymbol{e})$$

- Implemented adaptation law form

$$\dot{\hat{\boldsymbol{\rho}}} = \boldsymbol{K}_{\boldsymbol{\rho}} \boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)(\dot{\boldsymbol{e}} + \boldsymbol{\Sigma}\boldsymbol{e})$$

- Block diagram of joint-space tracking via adaptive control



- In the regulation case (constant $\boldsymbol{q}_d$) the adaptive control becomes

$$\begin{aligned}\boldsymbol{\tau} &= -\hat{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{\Sigma}\dot{\boldsymbol{q}} + \hat{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{\Sigma}\boldsymbol{e} + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \boldsymbol{K}_D\left(\boldsymbol{\Sigma}\boldsymbol{e} - \dot{\boldsymbol{q}}\right) \\ &= \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\Sigma}\boldsymbol{e},-\boldsymbol{\Sigma}\dot{\boldsymbol{q}})\hat{\boldsymbol{\rho}} + \boldsymbol{K}_D\left(\boldsymbol{\Sigma}\boldsymbol{e} - \dot{\boldsymbol{q}}\right) \\ \dot{\hat{\boldsymbol{\rho}}} &= \boldsymbol{K}_{\boldsymbol{\rho}}\,\boldsymbol{Y}^T(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\Sigma}\boldsymbol{e},-\boldsymbol{\Sigma}\dot{\boldsymbol{q}})\left(\boldsymbol{\Sigma}\boldsymbol{e} - \dot{\boldsymbol{q}}\right)\end{aligned}$$

- In the presence of unmodeled dynamics (simplified model or external disturbances) performance of adaptive control is degraded

Robust Control

- Uncertainty on dynamic coefficients

$$\|\tilde{\boldsymbol{\rho}}\| \leq \bar{\rho}$$

- Robust control law

$$\boldsymbol{\tau} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\,(\hat{\boldsymbol{\rho}} + \boldsymbol{u}) + \boldsymbol{K}_D\boldsymbol{\sigma}$$

- Define

$$\boldsymbol{z} = \boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\,\boldsymbol{\sigma}$$

- Lyapunov derivative term

$$\dot{V} = -\dot{\boldsymbol{e}}^T\boldsymbol{K}_D\dot{\boldsymbol{e}} - \boldsymbol{e}^T\boldsymbol{\Sigma}\boldsymbol{K}_D\boldsymbol{\Sigma}\boldsymbol{e} - \boldsymbol{z}^T(\tilde{\boldsymbol{\rho}} + \boldsymbol{u})$$

- Discontinuous robust control law

$$\boldsymbol{u} = \begin{cases} \dfrac{\bar{\rho}}{\|\boldsymbol{z}\|}\,\boldsymbol{z} & \text{for } \|\boldsymbol{z}\| \neq 0 \\ \mathbf{0} & \text{for } \|\boldsymbol{z}\| = 0 \end{cases}$$

- Component-wise bounded uncertainty

$$|\tilde{\rho}_i| \leq \bar{\rho}_i \qquad i = 1, \ldots, r$$

- Inequality

$$\begin{aligned} -\boldsymbol{z}^T(\tilde{\boldsymbol{\rho}} + \boldsymbol{u}) &= -\boldsymbol{z}^T\tilde{\boldsymbol{\rho}} - \frac{\bar{\rho}}{\|\boldsymbol{z}\|}\,\boldsymbol{z}^T\boldsymbol{z} \\ &\leq \|\boldsymbol{z}\|\|\tilde{\boldsymbol{\rho}}\| - \bar{\rho}\,\|\boldsymbol{z}\| = -\|\boldsymbol{z}\|\,(\bar{\rho} - \|\tilde{\boldsymbol{\rho}}\|) \leq 0 \end{aligned}$$

- Continuous approximation with boundary layer ϵ

$$\boldsymbol{u} = \begin{cases} \dfrac{\bar{\rho}}{\|\boldsymbol{z}\|}\,\boldsymbol{z} & \text{for } \|\boldsymbol{z}\| > \epsilon \\ \dfrac{\bar{\rho}}{\epsilon}\,\boldsymbol{z} & \text{for } \|\boldsymbol{z}\| \leq \epsilon \end{cases}$$

- Resulting bound

$$-\boldsymbol{z}^T(\tilde{\boldsymbol{\rho}} + \boldsymbol{u}) = -\boldsymbol{z}^T\tilde{\boldsymbol{\rho}} - \frac{\bar{\rho}}{\epsilon}\,\boldsymbol{z}^T\boldsymbol{z} \leq \bar{\rho}\,\|\boldsymbol{z}\| - \frac{\bar{\rho}}{\epsilon}\,\|\boldsymbol{z}\|^2$$

- When $\|\boldsymbol{z}\| = \epsilon/2$, the bound attains its maximum value $\epsilon\,\bar{\rho}/4$

$$\Downarrow$$

- Lyapunov bound

$$\dot{V} \leq -\dot{\boldsymbol{e}}^T\boldsymbol{K}_D\dot{\boldsymbol{e}} - \boldsymbol{e}^T\boldsymbol{\Sigma}\boldsymbol{K}_D\boldsymbol{\Sigma}\boldsymbol{e} + \frac{1}{4}\,\epsilon\,\bar{\rho}$$

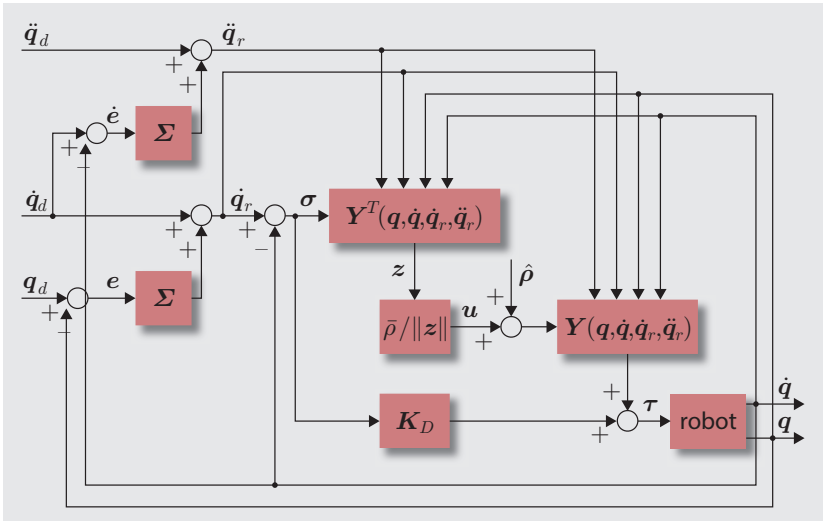
- $\dot{V}$ is negative outside a region M containing the equilibrium, whose extent depends on the product of the uncertainty bound $\bar{\rho}$ by the constant $\epsilon \Rightarrow$
  $\boldsymbol{e}$ will remain bounded and the evolution of the closed-loop system converges to M in finite time; the size of M can be tuned by choosing ϵ
- Different uncertainty bound for each coefficient

$$|\tilde{\rho}_i| \leq \bar{\rho}_i \qquad i = 1, \ldots, r$$

- Component-wise law

$$u_i = \begin{cases} \dfrac{\bar{\rho}_i}{|z_i|}\, z_i & \text{for } |z_i| \geq \epsilon_i \\ \dfrac{\bar{\rho}_i}{\epsilon_i}\, z_i & \text{for } |z_i| < \epsilon_i \end{cases}$$

- Block diagram of joint-space tracking via robust control



Control in Task Space

- Previous schemes assume desired motion specified in joint space or converted to joint space
- Task-space schemes operate directly in task space
- Task-space schemes are computationally more demanding
- Task-space schemes are useful when the task is updated in real time using exteroceptive sensors

PD Control with Gravity Cancellation

- Regulate task variables $\boldsymbol{y}$ to a desired constant value $\boldsymbol{y}_d$
- Define the task error

$$\boldsymbol{e_y} = \boldsymbol{y}_d - \boldsymbol{y}$$

- Define the Lyapunov candidate

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{e}_{\boldsymbol{y}}^T \boldsymbol{K}_P \boldsymbol{e_y}$$

- Time derivative of the Lyapunov candidate

$$\dot{V} = \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_P \boldsymbol{e_y}$$

- Equivalent expression

$$\dot{V} = \dot{\boldsymbol{q}}^T (\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_P \boldsymbol{e_y})$$

- PD control law with gravity cancellation

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_P\boldsymbol{e_y} - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

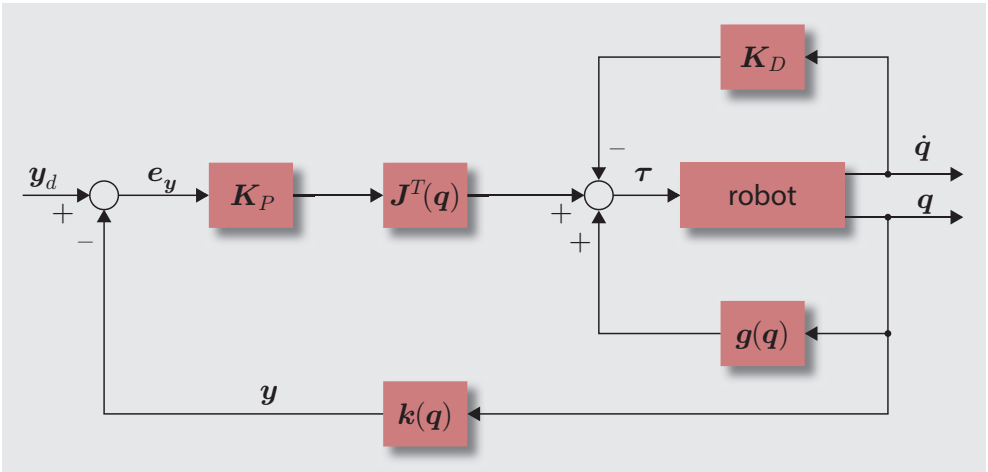
- Resulting Lyapunov derivative

$$\dot{V} = -\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\dot{\boldsymbol{q}}$$

- Alternative form

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})(\boldsymbol{K}_P\boldsymbol{e_y} - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$$

- Block diagram of task-space regulation via PD control with gravity cancellation



- Task error partition

$$\boldsymbol{e_y} = \begin{pmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_o \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_d - \boldsymbol{p}_e \\ \boldsymbol{\phi}_{ed} \end{pmatrix}$$

- Task error derivative

$$\dot{\boldsymbol{e}}_{\boldsymbol{y}} = -\boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- End-effector analytic Jacobian

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}_E^{-1}(\boldsymbol{\phi}_{ed}) \end{pmatrix} \boldsymbol{J}_g(\boldsymbol{q})$$

Control Based on Feedback Linearization

- Use the task acceleration relation

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Choose the control torque

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})(\boldsymbol{a_y} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Obtain the linear relation

$$\ddot{\boldsymbol{y}} = \boldsymbol{a_y}$$

- Choose the task-space acceleration input

$$\boldsymbol{a_y} = \ddot{\boldsymbol{y}}_d + \boldsymbol{K}_P \boldsymbol{e_y} + \boldsymbol{K}_D \dot{\boldsymbol{e}}_{\boldsymbol{y}}$$

- Obtain the error dynamics

$$\ddot{\boldsymbol{e}}_{\boldsymbol{y}} = \ddot{\boldsymbol{y}}_d - \ddot{\boldsymbol{y}} = \ddot{\boldsymbol{y}}_d - \boldsymbol{a_y} = -\boldsymbol{K}_P \boldsymbol{e_y} - \boldsymbol{K}_D \dot{\boldsymbol{e}}_{\boldsymbol{y}}$$

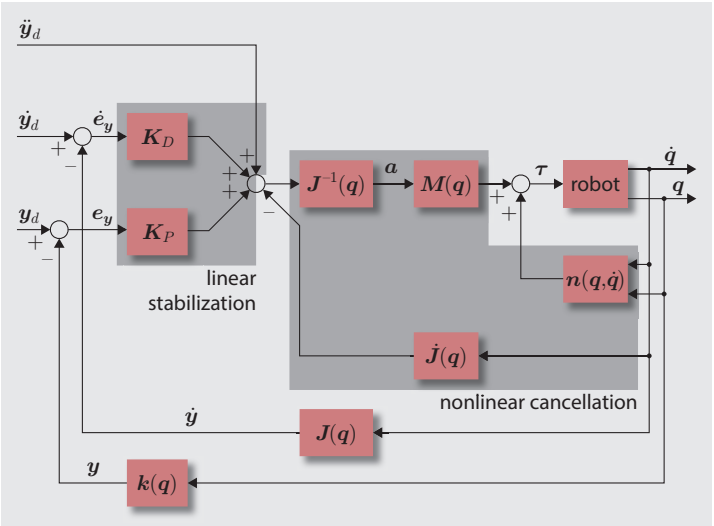
- Final torque expression

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})(\ddot{\boldsymbol{y}}_d + \boldsymbol{K}_P \boldsymbol{e_y} + \boldsymbol{K}_D \dot{\boldsymbol{e}}_{\boldsymbol{y}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Regulation variant

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})(\boldsymbol{K}_P \boldsymbol{e_y} - \boldsymbol{K}_D \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Block diagram of task-space trajectory tracking via feedback linearization control



- Task-space dynamic model

$$\boldsymbol{\Lambda}(\boldsymbol{q})\ddot{\boldsymbol{y}} + \boldsymbol{\psi}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\eta}(\boldsymbol{q}) = \boldsymbol{\gamma}$$

- Equivalent expression

$$\boldsymbol{\gamma} = \boldsymbol{\Lambda}(\boldsymbol{q})\boldsymbol{a}_{\boldsymbol{y}} + \boldsymbol{\psi}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\eta}(\boldsymbol{q})$$

- Tracking expression

$$\boldsymbol{\gamma} = \boldsymbol{\Lambda}(\boldsymbol{q})(\ddot{\boldsymbol{y}}_d + \boldsymbol{K}_P\boldsymbol{e}_{\boldsymbol{y}} + \boldsymbol{K}_D\dot{\boldsymbol{e}}_{\boldsymbol{y}}) + \boldsymbol{\psi}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\eta}(\boldsymbol{q})$$

- End-effector twist derivative

$$\dot{\boldsymbol{t}}_e = \begin{pmatrix} \dot{\boldsymbol{v}}_e \\ \dot{\boldsymbol{\omega}}_e \end{pmatrix} = \boldsymbol{J}_g(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_g(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Alternative expression

$$\dot{\boldsymbol{t}}_e = \boldsymbol{J}_g(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})) + \dot{\boldsymbol{J}}_g(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Control torque

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}_g^{-1}(\boldsymbol{q})(\boldsymbol{a}_e - \dot{\boldsymbol{J}}_g(\boldsymbol{q})\dot{\boldsymbol{q}}) + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- Linear relation

$$\dot{\boldsymbol{t}}_e = \boldsymbol{a}_e \quad \text{or} \quad \begin{pmatrix} \dot{\boldsymbol{v}}_e \\ \dot{\boldsymbol{\omega}}_e \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_{e,p} \\ \boldsymbol{a}_{e,o} \end{pmatrix}$$

- Position component

$$\boldsymbol{a}_{e,p} = \dot{\boldsymbol{v}}_d + \boldsymbol{K}_{P,p}\boldsymbol{e}_p + \boldsymbol{K}_{D,p}\dot{\boldsymbol{e}}_p$$

- Position error dynamics

$$\ddot{\boldsymbol{e}}_p = -\boldsymbol{K}_{P,p}\boldsymbol{e}_p - \boldsymbol{K}_{D,p}\dot{\boldsymbol{e}}_p$$

- Angular velocity relation

$$\dot{\boldsymbol{\omega}}_e = \dot{\boldsymbol{\omega}}_d - \boldsymbol{T}_E(\boldsymbol{\phi}_{ed})\ddot{\boldsymbol{\phi}}_{ed} - \dot{\boldsymbol{T}}_E(\boldsymbol{\phi}_{ed})\dot{\boldsymbol{\phi}}_{ed}$$

- Orientation component

$$\boldsymbol{a}_{e,o} = \dot{\boldsymbol{\omega}}_d - \dot{\boldsymbol{T}}_E(\boldsymbol{\phi}_{ed})\dot{\boldsymbol{\phi}}_{ed} + \boldsymbol{T}_E(\boldsymbol{\phi}_{ed})(\boldsymbol{K}_{P,o}\boldsymbol{\phi}_{ed} + \boldsymbol{K}_{D,o}\dot{\boldsymbol{\phi}}_{ed})$$

- Orientation error dynamics

$$\ddot{\boldsymbol{\phi}}_{ed} = -\boldsymbol{K}_{P,o}\boldsymbol{\phi}_{ed} - \boldsymbol{K}_{D,o}\dot{\boldsymbol{\phi}}_{ed}$$

The Case of Redundant Robots

- Joint-space dynamic model and direct kinematics

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}))$$
$$\boldsymbol{y} = \boldsymbol{k}(\boldsymbol{q})$$

- Task acceleration relation

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}$$

- General torque expression

$$\boldsymbol{\tau} = (\boldsymbol{J}\boldsymbol{M}^{-1})^{\dagger}_{\boldsymbol{W}}(\boldsymbol{a_y} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + (\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{M}^{-1})^{\dagger}_{\boldsymbol{W}}\boldsymbol{J}\boldsymbol{M}^{-1})\boldsymbol{\tau}_N + \boldsymbol{n}$$

- Weighted pseudoinverse expression ($\boldsymbol{W} = \boldsymbol{M}^{-1}$)

$$(\boldsymbol{J}\boldsymbol{M}^{-1})^{\dagger}_{\boldsymbol{M}^{-1}} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1} = \boldsymbol{J}^T\boldsymbol{\Lambda}$$

- Null-space projector

$$\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{M}^{-1})^{\dagger}_{\boldsymbol{M}^{-1}}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{\Lambda}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{J}^{\dagger}_{\boldsymbol{M}})^T$$

- Equivalent form

$$\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{\Lambda} \left(\boldsymbol{a_y} - \dot{\boldsymbol{J}} \dot{\boldsymbol{q}} \right) + (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger} \boldsymbol{J})^T \boldsymbol{\tau}_N + \boldsymbol{n}$$

- Decomposition identity

$$\boldsymbol{n} = \boldsymbol{n} + \boldsymbol{J}^T (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger})^T \boldsymbol{n} - \boldsymbol{J}^T (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger})^T \boldsymbol{n} = \boldsymbol{J}^T (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger})^T \boldsymbol{n} + (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger} \boldsymbol{J})^T \boldsymbol{n}$$

- Torque expression

$$\boldsymbol{\tau} = \boldsymbol{J}^T (\boldsymbol{\Lambda} \left(\boldsymbol{a_y} - \dot{\boldsymbol{J}} \dot{\boldsymbol{q}} \right) + (\boldsymbol{J}_{\boldsymbol{M}}^{\dagger})^T \boldsymbol{n}) + (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger} \boldsymbol{J})^T (\boldsymbol{\tau}_N + \boldsymbol{n})$$

- Alternative pseudoinverse with $\boldsymbol{W} = \boldsymbol{M}^{-2}$

$$(\boldsymbol{J}\boldsymbol{M}^{-1})^{\dagger}_{\boldsymbol{M}^{-2}} = \boldsymbol{M}\boldsymbol{J}^{T}(\boldsymbol{J}\boldsymbol{J}^{T})^{-1} = \boldsymbol{M}\boldsymbol{J}^{\dagger}$$

- Associated projector

$$\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{M}^{-1})^{\dagger}_{\boldsymbol{M}^{-2}}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{J}^{\dagger}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{M}(\boldsymbol{I} - \boldsymbol{J}^{\dagger}\boldsymbol{J})\boldsymbol{M}^{-1}$$

- Resulting torque ($\boldsymbol{M}$ not involved in pseudoinversion)

$$\boldsymbol{\tau} = \boldsymbol{M}\boldsymbol{J}^{\dagger}(\boldsymbol{a_y} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + \boldsymbol{M}(\boldsymbol{I} - \boldsymbol{J}^{\dagger}\boldsymbol{J})\boldsymbol{M}^{-1}\boldsymbol{\tau}_N + \boldsymbol{n}$$

- Acceleration decomposition

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}_{\boldsymbol{V}}^{\dagger}(\boldsymbol{a}_{\boldsymbol{y}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{V}}^{\dagger}\boldsymbol{J})\ddot{\boldsymbol{q}}_N$$

- Resulting torque

$$\boldsymbol{\tau} = \boldsymbol{M}\boldsymbol{J}_{\boldsymbol{V}}^{\dagger}(\boldsymbol{a}_{\boldsymbol{y}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + \boldsymbol{M}(\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{V}}^{\dagger}\boldsymbol{J})\ddot{\boldsymbol{q}}_N + \boldsymbol{n}$$

- Relation with $\boldsymbol{V} = \boldsymbol{M}$

$$\boldsymbol{M}\boldsymbol{J}_{\boldsymbol{M}}^{\dagger} = \boldsymbol{M}\boldsymbol{M}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1} = \boldsymbol{J}^T\boldsymbol{\Lambda}$$

- Resulting torque with $\ddot{\boldsymbol{q}}_N = \boldsymbol{M}^{-1}\boldsymbol{\tau}_N$

$$\boldsymbol{\tau} = \boldsymbol{J}^T\boldsymbol{\Lambda}\,(\boldsymbol{a}_{\boldsymbol{y}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + \boldsymbol{M}(\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger}\boldsymbol{J})\boldsymbol{M}^{-1}\boldsymbol{\tau}_N + \boldsymbol{n}$$

- Identical to previous linearizing torque $\boldsymbol{\tau} = \boldsymbol{J}^T\boldsymbol{\Lambda}\,(\boldsymbol{a}_{\boldsymbol{y}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger}\boldsymbol{J})^T\boldsymbol{\tau}_N + \boldsymbol{n}$ since

$$\boldsymbol{M}(\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger}\boldsymbol{J})\boldsymbol{M}^{-1} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{J}_{\boldsymbol{M}}^{\dagger}\boldsymbol{J}\boldsymbol{M}^{-1} = (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\dagger}\boldsymbol{J})^T$$