Robots with Flexible Joints

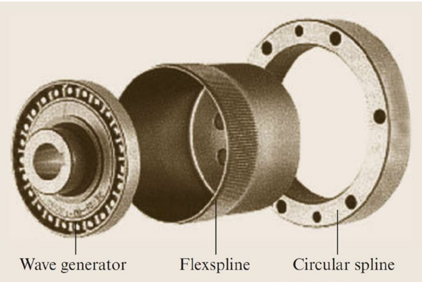
- Standard assumption underlying robot kinematics, dynamics, and control design: manipulators consisting of *rigid bodies* (links and joints), ok for slow motion and small interacting forces
- Mechanical flexibility
    - Compliant transmission elements
    - Use of lightweight materials and slender design

- Static and dynamic deflections
- Performance degradation

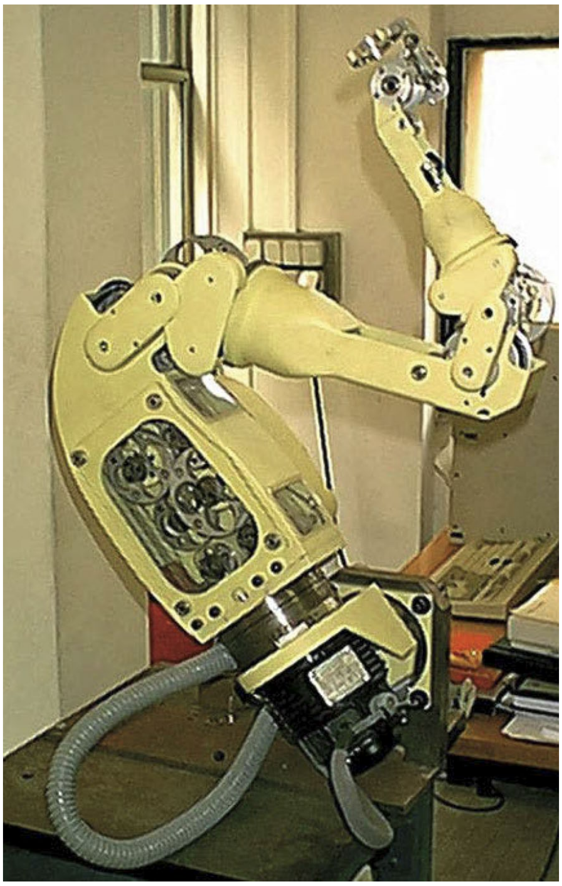
- *Flexible joints* (concentrated)
- Flexible links (distributed)

Joint Flexibility

- Common in current industrial robots when motion transmission/reduction elements are used
  - Belts
  - Long shafts
  - Cables
  - Harmonic drives
  - Cycloidal gears



- Intrinsic flexibility
  - Time-varying displacement between position of actuator and that of driven link
  - Oscillatory behavior (small magnitude, high frequency)
  - Possible instability when in contact with environment



Dynamic Modeling

- Robot with flexible joints ≡ Open kinematic chain having $N + 1$ rigid bodies (base + N links), interconnected by N (revolute or prismatic) joints undergoing deflection, and actuated by N electrical drives

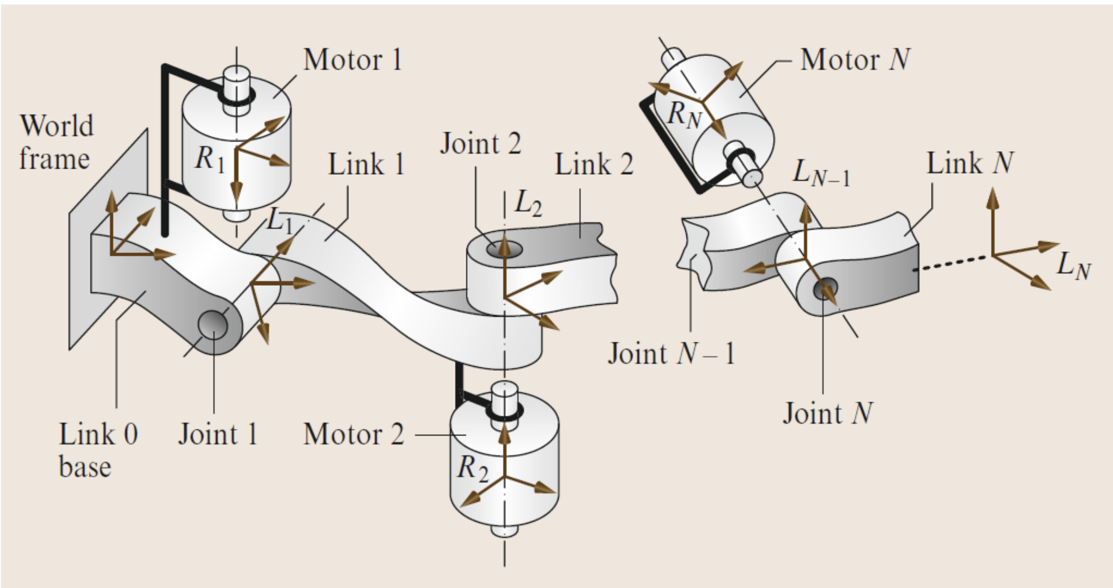
Assumptions

A1. Joint deflections are small, so that flexibility effects are limited to the domain of linear elasticity

A2. Actuators' rotors are modeled as uniform bodies having their center of mass on the rotation axis

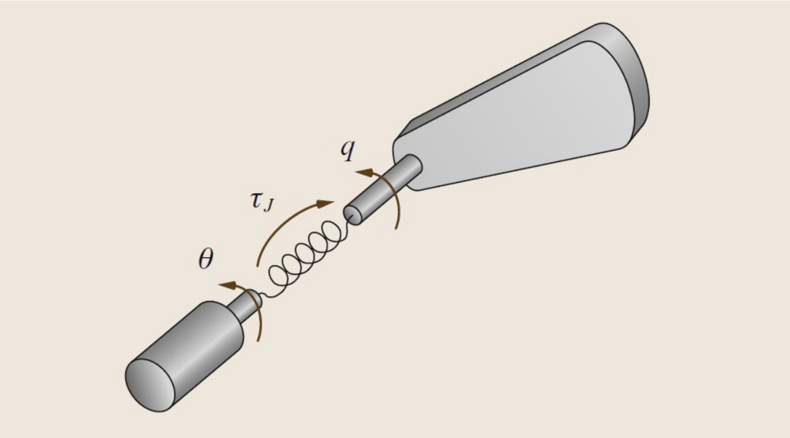
A3. Each motor is located on the robot arm in a position preceding the driven link

- $2N$ frames attached to $2N$ moving rigid bodies
  - N link frames L_i
  - N motor frames R_i



- 2*N* generalized coordinates

  $\boldsymbol{\Theta} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{\theta} \end{pmatrix} \in \mathbb{R}^{2N}$



  - Model independent of reduction ratios
  - Position variables with similar dynamic range
  - Robot kinematics only a function of link variables $\boldsymbol{q}$

- Motor directly placed on *i*-th joint axis

  $\dot{\theta}_{m,i} = n_i \, \dot{\theta}_i$
- Deflection at *i*-th joint

  $\delta_i = q_i - \theta_i$
- Torque transmitted to *i*-th link

  $\tau_{J,i} = K_i(\theta_i - q_i)$

Lagrangian approach

$$\mathcal{L} = \mathcal{T}\left(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}\right) - \mathcal{U}(\boldsymbol{\Theta})$$

Potential energy

$$\mathcal{U}(\boldsymbol{\Theta}) = \mathcal{U}_{\text{grav}}(\boldsymbol{q}) + \mathcal{U}_{\text{elas}}(\boldsymbol{q} - \boldsymbol{\theta})$$

- Gravity (independent of $\boldsymbol{\theta}$, see **A2**)

  $$\mathcal{U}_{\text{grav}} = \mathcal{U}_{\text{grav,link}}(\boldsymbol{q}) + \mathcal{U}_{\text{grav,motor}}(\boldsymbol{q})$$

- Joint elasticity (see **A1**)

  $$\mathcal{U}_{\text{elas}} = \frac{1}{2}(\boldsymbol{q} - \boldsymbol{\theta})^{\text{T}}\mathbf{K}(\boldsymbol{q} - \boldsymbol{\theta})$$

  $$\mathbf{K} = \text{diag}(K_1, \dots, K_N)$$

Kinetic energy

- Links

$$\mathcal{T}_{\text{link}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^{\text{T}}\mathbf{M}_{\text{L}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Rotors

$$\mathcal{T}_{\text{rotor}} = \sum_{i=1}^{N}\mathcal{T}_{\text{rotor}_i} = \sum_{i=1}^{N}\left(\frac{1}{2}m_{r_i}\boldsymbol{v}_{r_i}^{\text{T}}\boldsymbol{v}_{r_i} + \frac{1}{2}\,{}^{R_i}\boldsymbol{\omega}_{r_i}^{\text{T}}\,{}^{R_i}\mathbf{I}_{r_i}\,{}^{R_i}\boldsymbol{\omega}_{r_i}\right)$$

  - Rotor inertia matrix (see **A2**)

    $${}^{R_i}\mathbf{I}_{r_i} = \text{diag}\left(I_{r_{i_{xx}}}, I_{r_{i_{yy}}}, I_{r_{i_{zz}}}\right)$$

  - Angular velocity (see **A3**)

    $${}^{R_i}\boldsymbol{\omega}_{r_i} = \sum_{j=1}^{i-1}\mathbf{J}_{r_{i,j}}(\boldsymbol{q})\,\dot{q}_j + \begin{pmatrix}0\\0\\\dot{\theta}_{m,i}\end{pmatrix}$$

$$\Downarrow$$

$$\mathcal{T}_{\text{rotor}} = \frac{1}{2}\dot{\boldsymbol{q}}^{\text{T}}[\mathbf{M}_{\text{R}}(\boldsymbol{q}) + \mathbf{S}(\boldsymbol{q})\mathbf{B}^{-1}\mathbf{S}^{\text{T}}(\boldsymbol{q})]\,\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^{\text{T}}\mathbf{S}(\boldsymbol{q})\dot{\boldsymbol{\theta}} + \frac{1}{2}\dot{\boldsymbol{\theta}}^{\text{T}}\mathbf{B}\dot{\boldsymbol{\theta}}$$

  - $\mathbf{B}$: constant diagonal inertia matrix collecting rotors inertial components $I_{r_{i_{zz}}}$ around their spinning axes
  - $\mathbf{M}_{\text{R}}(\boldsymbol{q})$: rotor masses (and, possibly, rotor inertial components along the other principal axes)
  - $\mathbf{S}(\boldsymbol{q})$: inertial couplings between rotors and previous links

Planar robot with two revolute flexible joints and motors mounted directly on joint axes

- Kinetic energy

$$\mathcal{T}_{\text{rotor}_1} = \frac{1}{2} I_{r_{1zz}} \dot{\theta}_{m,1}^2 = \frac{1}{2} I_{r_{1zz}} n_1^2 \dot{\theta}_1^2$$

$$\mathcal{T}_{\text{rotor}_2} = \frac{1}{2} m_{r_2} l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{r_{2zz}} \left(\dot{q}_1 + \dot{\theta}_{m,2}\right)^2$$

$$= \frac{1}{2} m_{r_2} l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{r_{2zz}} \left(\dot{q}_1^2 + 2 n_2 \dot{q}_1 \dot{\theta}_2 + n_2^2 \dot{\theta}_2^2\right)$$

$$\mathbf{B} = \begin{pmatrix} I_{r_{1zz}} n_1^2 & 0 \\ 0 & I_{r_{2zz}} n_2^2 \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} 0 & I_{r_{2zz}} n_2 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{R}} = \begin{pmatrix} m_{r_2} l_1^2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{S}\mathbf{B}^{-1}\mathbf{S}^{\mathrm{T}} = \begin{pmatrix} I_{r_{2zz}} & 0 \\ 0 & 0 \end{pmatrix}$$

  - $\mathbf{S}$ and $\mathbf{M}_{\mathrm{R}}$ constant
  - If second motor mounted remotely on first joint, or close to second joint but with spinning axis orthogonal to joint axis, then $\mathbf{S} = \mathbf{O}$

- General expression of **S** (see **A3**)

$$\mathbf{S}(\boldsymbol{q}) = \begin{pmatrix} 0 & S_{12} & S_{13}(q_2) & S_{14}(q_2,q_3) & \cdots & \cdots & S_{1N}(q_2,\dots,q_{N-1}) \\ 0 & 0 & S_{23} & S_{24}(q_3) & \cdots & \cdots & S_{2N}(q_3,\dots,q_{N-1}) \\ 0 & 0 & 0 & S_{34} & \cdots & \cdots & S_{3N}(q_4,\dots,q_{N-1}) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & S_{N-2,N-1} & S_{N-2,N}(q_{N-1}) \\ 0 & 0 & 0 & \dots & 0 & 0 & S_{N-1,N} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

- Total kinetic energy

$$\mathcal{T} = \frac{1}{2}\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\mathcal{M}(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} = \frac{1}{2}\begin{pmatrix}\dot{\boldsymbol{q}}^{\mathrm{T}} & \dot{\boldsymbol{\theta}}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\mathbf{M}(\boldsymbol{q}) & \mathbf{S}(\boldsymbol{q}) \\ \mathbf{S}^{\mathrm{T}}(\boldsymbol{q}) & \mathbf{B}\end{pmatrix}\begin{pmatrix}\dot{\boldsymbol{q}} \\ \dot{\boldsymbol{\theta}}\end{pmatrix}$$

$$\mathbf{M}(\boldsymbol{q}) = \mathbf{M}_{\mathrm{L}}(\boldsymbol{q}) + \mathbf{M}_{\mathrm{R}}(\boldsymbol{q}) + \mathbf{S}(\boldsymbol{q})\mathbf{B}^{-1}\mathbf{S}^{\mathrm{T}}(\boldsymbol{q})$$

  - $\mathcal{M}$ depends only on $\boldsymbol{q}$

Complete dynamic model (N link eqs + N motor eqs)

$$\begin{pmatrix} \mathbf{M}(\boldsymbol{q}) & \mathbf{S}(\boldsymbol{q}) \\ \mathbf{S}^{\mathrm{T}}(\boldsymbol{q}) & \mathbf{B} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}} \\ \ddot{\boldsymbol{\theta}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{c}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{\theta}}) \\ \boldsymbol{c}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{g}(\boldsymbol{q}) + \mathbf{K}(\boldsymbol{q} - \boldsymbol{\theta}) \\ \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\tau} \end{pmatrix} \qquad \boldsymbol{\tau}_{\mathrm{J}} = \boldsymbol{K}(\boldsymbol{\theta} - \boldsymbol{q})$$

Additional terms for energy-dissipating effects on right-hand side

$$\begin{pmatrix} -\mathbf{F}_q \dot{\boldsymbol{q}} - \boldsymbol{D}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{\theta}}) \\ -\mathbf{F}_\theta \dot{\boldsymbol{\theta}} - \boldsymbol{D}(\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{q}}) \end{pmatrix}$$

- In case of contact with environment, additional term for *N* link eqs $\tau_{\mathrm{ext}} = \mathbf{J}^{\mathrm{T}}(\boldsymbol{q})\mathbf{F}$

Model properties

- All elements in the velocity-dependent terms are independent of motor positions, to be computed via Christoffel symbols

$$c_{\mathrm{tot},i}(\boldsymbol{\Theta},\dot{\boldsymbol{\Theta}}) = \frac{1}{2}\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\left[\frac{\partial \mathcal{M}_i}{\partial \boldsymbol{\Theta}} + \left(\frac{\partial \mathcal{M}_i}{\partial \boldsymbol{\Theta}}\right)^{\mathrm{T}} - \frac{\partial \mathcal{M}_i}{\partial \boldsymbol{\Theta}_i}\right]\dot{\boldsymbol{\Theta}}$$

  - $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ arise only in the presence of configuration-dependent $\mathbf{S}(\boldsymbol{q})$
  - $\boldsymbol{c}_1$ does not contain quadratic velocity terms in $\dot{\boldsymbol{q}}$ or $\dot{\boldsymbol{\theta}}$, but only mixed quadratic terms $\dot{\theta}_i \dot{q}_j$
- Same properties as for rigid case
  - Linearity in terms of suitable set of dynamic parameters, including joint stiffnesses and motor inertias (useful for model identification and adaptive control)
  - Coriolis and centrifugal terms can be factorized as $\boldsymbol{c}_{tot}(\boldsymbol{\Theta},\dot{\boldsymbol{\Theta}}) = \boldsymbol{\mathcal{C}}(\boldsymbol{\Theta},\dot{\boldsymbol{\Theta}})\dot{\boldsymbol{\Theta}}$ so that $\dot{\boldsymbol{\mathcal{M}}} - 2\boldsymbol{\mathcal{C}}$ is skew-symmetric (useful for control)
  - For robots having only revolute joints, the gradient of $\boldsymbol{g}(\boldsymbol{q})$ is globally bounded in norm by a constant
- If $\mathbf{K} \rightarrow \infty$, then $\boldsymbol{\theta} \rightarrow \boldsymbol{q}$ while $\boldsymbol{\tau}_{\mathrm{J}} \rightarrow \boldsymbol{\tau}$ (collapsing into standard model of fully rigid robots, including links and motors)

Reduced model

- In case of large reduction ratios (n_i ~ 100–150), energy contributions due to inertial couplings between motors and links can be neglected

A4. Angular velocity of rotors due only to their own spinning

$$^{R_i}\boldsymbol{\omega}_{r_i} = \begin{pmatrix} 0 & 0 & \dot{\theta}_{m,i} \end{pmatrix}^{\mathrm{T}} \qquad i = 1, \dots, N$$

$$\mathbf{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \mathbf{K}(\boldsymbol{q} - \boldsymbol{\theta}) = \mathbf{0}$$

$$\mathbf{B}\ddot{\boldsymbol{\theta}} + \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q}) = \boldsymbol{\tau}$$

$$\mathbf{M}(\boldsymbol{q}) = \mathbf{M}_L(\boldsymbol{q}) + \mathbf{M}_R(\boldsymbol{q})$$

- The link and motor equations are dynamically coupled through the elastic torque $\boldsymbol{\tau}_{\mathrm{J}}$
- The motor equations are fully linear

Singular perturbation model

- Large but finite joint stiffness → *two-time-scale* dynamic behavior

$$\mathbf{K} = \frac{1}{\epsilon^2}\widehat{\mathbf{K}} = \frac{1}{\epsilon^2}\mathrm{diag}\big(\widehat{K}_1, \ldots, \widehat{K}_\mathrm{N}\big) \qquad \frac{1}{\epsilon^2} \gg 1$$

  - *Slow* subsystem

$$\mathbf{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}_\mathrm{J}$$

  - *Fast* subsystem (differentiating joint torque twice)

$$\epsilon^2\ddot{\boldsymbol{\tau}}_\mathrm{J} = \widehat{\mathbf{K}}\left\{\mathbf{B}^{-1}\boldsymbol{\tau} - [\mathbf{B}^{-1} + \mathbf{M}^{-1}(\boldsymbol{q})]\boldsymbol{\tau}_\mathrm{J} + \mathbf{M}^{-1}(\boldsymbol{q})[\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})]\right\}$$

$$\epsilon^2\ddot{\boldsymbol{\tau}}_\mathrm{J} = \epsilon^2\frac{\mathrm{d}^2\boldsymbol{\tau}_\mathrm{J}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{\tau}_\mathrm{J}}{\mathrm{d}\sigma^2} \qquad \sigma = t/\epsilon$$

- Composite control

$$\boldsymbol{\tau} = \boldsymbol{\tau}_s(\boldsymbol{q},\dot{\boldsymbol{q}},t) + \epsilon\boldsymbol{\tau}_\mathrm{f}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\tau}_J,\dot{\boldsymbol{\tau}}_J)$$

  - Slow action $\boldsymbol{\tau}_s$ designed when neglecting joint elasticity
  - Fast action $\boldsymbol{\tau}_\mathrm{f}$ for locally stabilizing fast flexible dynamics around suitable manifold in state space

- If $\epsilon = 0$ → equivalent rigid robot model

$$[\mathbf{M}(\boldsymbol{q}) + \mathbf{B}]\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}_s$$

Computed Torque

- Rigid robots: straightforward algebraic computation by replacing desired motion of generalized coordinates in the dynamic model
    - Planned motion with continuously differentiable desired velocity
- Robots with flexible joints: desired motion of link variables available from kinematic inversion of desired motion of end-effector pose
    - Additional derivatives are needed

Reduced model

- Link equations for desired link motion

$$\mathbf{M}(\boldsymbol{q}_\mathrm{d})\ddot{\boldsymbol{q}}_\mathrm{d} + \boldsymbol{n}(\boldsymbol{q}_\mathrm{d}, \dot{\boldsymbol{q}}_\mathrm{d}) + \mathbf{K}\boldsymbol{q}_\mathrm{d} = \mathbf{K}\boldsymbol{\theta}_\mathrm{d}$$

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$$

    - Desired motor variables can be computed
- Time differentiation …

$$\mathbf{M}(\boldsymbol{q}_\mathrm{d})\boldsymbol{q}_\mathrm{d}^{[3]} + \dot{\mathbf{M}}(\boldsymbol{q}_\mathrm{d})\ddot{\boldsymbol{q}}_\mathrm{d} + \dot{\boldsymbol{n}}(\boldsymbol{q}_\mathrm{d}, \dot{\boldsymbol{q}}_\mathrm{d}) + \boldsymbol{K}\dot{\boldsymbol{q}}_\mathrm{d} = \boldsymbol{K}\dot{\boldsymbol{\theta}}_\mathrm{d}$$

    - Desired motor velocities can be computed
- Time differentiation …

$$\mathbf{M}(\boldsymbol{q}_\mathrm{d})\boldsymbol{q}_\mathrm{d}^{[4]} + 2\dot{\mathbf{M}}(\boldsymbol{q}_\mathrm{d})\boldsymbol{q}_\mathrm{d}^{[3]} + \ddot{\boldsymbol{n}}(\boldsymbol{q}_\mathrm{d}, \dot{\boldsymbol{q}}_\mathrm{d}) + \left[\ddot{\mathbf{M}}(\boldsymbol{q}_\mathrm{d}) + \boldsymbol{K}\right]\ddot{\boldsymbol{q}}_\mathrm{d} = \boldsymbol{K}\ddot{\boldsymbol{\theta}}_\mathrm{d}$$

    - Desired motor accelerations can be computed

- Nominal torque

$$\begin{aligned}\boldsymbol{\tau}_{\mathrm{d}} &= [\mathbf{M}(\boldsymbol{q}_{\mathrm{d}}) + \mathbf{B}]\ddot{\boldsymbol{q}}_{\mathrm{d}} + \boldsymbol{n}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}) \\ &\quad + \mathbf{B}\mathbf{K}^{-1}[\mathbf{M}(\boldsymbol{q}_{\mathrm{d}})\boldsymbol{q}_{\mathrm{d}}^{[4]} + 2\dot{\mathbf{M}}(\boldsymbol{q}_{\mathrm{d}})\boldsymbol{q}_{\mathrm{d}}^{[3]} \\ &\qquad + \ddot{\mathbf{M}}(\boldsymbol{q}_{\mathrm{d}})\ddot{\boldsymbol{q}}_{\mathrm{d}} + \ddot{\boldsymbol{n}}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}})]\end{aligned}$$

$$\dot{\mathbf{M}}[\boldsymbol{q}_{\mathrm{d}}(t)] = \sum_{i=1}^{N} \left.\frac{\partial \mathbf{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{q}_{\mathrm{d}}(t)} \dot{\boldsymbol{q}}_{\mathrm{d}}(t)\boldsymbol{e}_i^{\mathrm{T}}$$

$\boldsymbol{e}_i$: *i*-th unit vector

$\mathbf{M}_i$: *i*-th column of $\mathbf{M}(\boldsymbol{q})$

  - $\boldsymbol{q}_{\mathrm{d}}(t)$ admits continuously differentiable jerk

- Recursive numerical Newton-Euler algorithm
  - Forward recursion of motion variables up to 4$^{\text{th}}$ differential order
  - Backward recursion of second time derivatives of forces and moments

Complete model

- Link equations for desired link motion (constant $\mathbf{S}$)

  $\mathbf{M}(\boldsymbol{q}_{\rm d})\ddot{\boldsymbol{q}}_{\rm d} + \mathbf{S}\ddot{\boldsymbol{\theta}}_{\rm d} + \boldsymbol{n}(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}) + \mathbf{K}\boldsymbol{q}_{\rm d} = \mathbf{K}\boldsymbol{\theta}_{\rm d}$

  - Desired motor variables cannot be directly computed

- Exploiting upper triangular structure of $\mathbf{S}$ …

  - *N*-th equation is independent of $\ddot{\theta}_{\rm d}$

    $\mathbf{M}_N^{\rm T}(\boldsymbol{q}_{\rm d})\ddot{\boldsymbol{q}}_{\rm d} + \mathbf{0}^{\rm T}\ddot{\boldsymbol{\theta}}_{\rm d} + n_N(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}) + K_N q_{{\rm d},N} = K_N\theta_{{\rm d},N}$

    $\theta_{{\rm d},N} = f_N(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}, \ddot{\boldsymbol{q}}_{\rm d})$

    After double differentiation

    $\ddot{\theta}_{{\rm d},N} = f_N''\left(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}, \ldots, \boldsymbol{q}_{\rm d}^{[4]}\right)$

  - (*N*–1)-th equation …

    $\mathbf{M}_{N-1}^{\rm T}(\boldsymbol{q}_{\rm d})\ddot{\boldsymbol{q}}_{\rm d} + S_{N-1,N}\ddot{\theta}_{{\rm d},N}$

    $+ n_{N-1}(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}) + K_{N-1}q_{{\rm d},N-1} = K_{N-1}\theta_{{\rm d},N-1}$

    $\theta_{{\rm d},N-1} = f_{N-1}\left(q_{\rm d}, \dot{q}_{\rm d}, \ldots, q_{\rm d}^{[4]}\right)$

    After double differentiation

    $\ddot{\theta}_{{\rm d},N-1} = f_{N-1}''\left(q_{\rm d}, \dot{q}_{\rm d}, \ldots, q_{\rm d}^{[6]}\right)$

  - Proceeding backward …

    $\theta_{{\rm d},1} = f_1\left(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}, \ldots, \boldsymbol{q}_{\rm d}^{[2N]}\right)$

    $\ddot{\theta}_{{\rm d},1} = f_1''\left(\boldsymbol{q}_{\rm d}, \dot{\boldsymbol{q}}_{\rm d}, \ldots, \boldsymbol{q}_{\rm d}^{[2(N+1)]}\right)$

- Nominal torque

$$\boldsymbol{\tau}_{\mathrm{d}} = [\mathbf{M}(\boldsymbol{q}_{\mathrm{d}}) + \mathbf{S}^{\mathrm{T}}]\ddot{\boldsymbol{q}}_{\mathrm{d}} + \boldsymbol{n}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}) + (\mathbf{B} + \mathbf{S})\ddot{\boldsymbol{\theta}}_{\mathrm{d}}\left(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}, \dots, \boldsymbol{q}_{\mathrm{d}}^{[2(N+1)]}\right)$$

  - $\boldsymbol{q}_{\mathrm{d}}(t)$ admits continuously differentiable (2*N*+1)-th derivative

Presence of dissipative terms

- Inclusion of spring damping in reduced model

  $$\mathbf{M}(\boldsymbol{q}_{\mathrm{d}})\ddot{\boldsymbol{q}}_{\mathrm{d}} + \boldsymbol{n}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}) + \left(\mathbf{D} + \mathbf{F}_q\right)\dot{\boldsymbol{q}}_{\mathrm{d}} + \mathbf{K}\boldsymbol{q}_{\mathrm{d}} = \mathbf{D}\dot{\boldsymbol{\theta}}_{\mathrm{d}} + \mathbf{K}\boldsymbol{\theta}_{\mathrm{d}}$$

- Time differentiation …

  $$\boldsymbol{D}\ddot{\boldsymbol{\theta}}_{\mathrm{d}} + \mathbf{K}\dot{\boldsymbol{\theta}}_{d} = \boldsymbol{w}_{\mathrm{d}}$$

  $$\boldsymbol{w}_{\mathrm{d}} = \mathbf{M}(\boldsymbol{q}_{\mathrm{d}})\boldsymbol{q}_{\mathrm{d}}^{[3]} + \left[\dot{\mathbf{M}}(\boldsymbol{q}_{\mathrm{d}}) + \mathbf{D} + \mathbf{F}_{\mathrm{q}}\right]\ddot{\boldsymbol{q}}_{\mathrm{d}} + \dot{\boldsymbol{n}}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}) + \mathbf{K}\dot{\boldsymbol{q}}_{\mathrm{d}}$$

  - First-order linear *asymptotically stable* dynamical system (internal dynamics) with state $\dot{\boldsymbol{\theta}}_{\mathrm{d}}$ and forcing signal $\boldsymbol{w}_{\mathrm{d}}(t)$, to be solved for given initial condition $\dot{\boldsymbol{\theta}}_{\mathrm{d}}(0)$
- Nominal torque

  $$\boldsymbol{\tau}_{\mathrm{d}} = \mathbf{M}(\boldsymbol{q}_{\mathrm{d}})\ddot{\boldsymbol{q}}_{\mathrm{d}} + \boldsymbol{n}(\boldsymbol{q}_{\mathrm{d}}, \dot{\boldsymbol{q}}_{\mathrm{d}}) + \boldsymbol{F}_{\mathrm{q}}\dot{\boldsymbol{q}}_{\mathrm{d}} + \mathbf{B}\ddot{\boldsymbol{\theta}}_{\mathrm{d}} + \boldsymbol{F}_{\theta}\dot{\boldsymbol{\theta}}_{\mathrm{d}}$$

  - $\boldsymbol{q}_{\mathrm{d}}$(t) admits continuously differentiable acceleration
- Similar procedure for complete model with spring damping
  - Smoothness requirement on $\boldsymbol{q}_{\mathrm{d}}(t)$ dramatically reduced

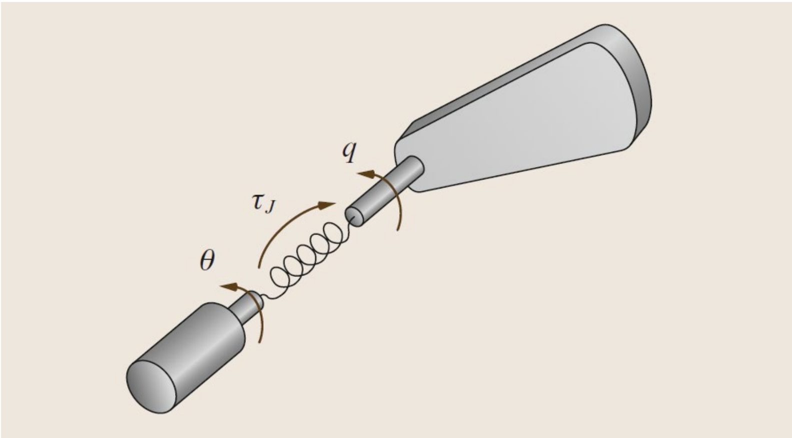
Regulation Control

- Controlling motion of robot with flexible joints to *constant* $\boldsymbol{q}_{\mathrm{d}}$

  $\boldsymbol{\theta}_{\mathrm{d}} = \boldsymbol{q}_{\mathrm{d}} + \mathbf{K}^{-1}\boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}})$

  $\boldsymbol{\tau}_{\mathrm{d}} = \boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}})$

Single flexible joint example



- Dynamic model with viscous friction on motor and link side + spring damping

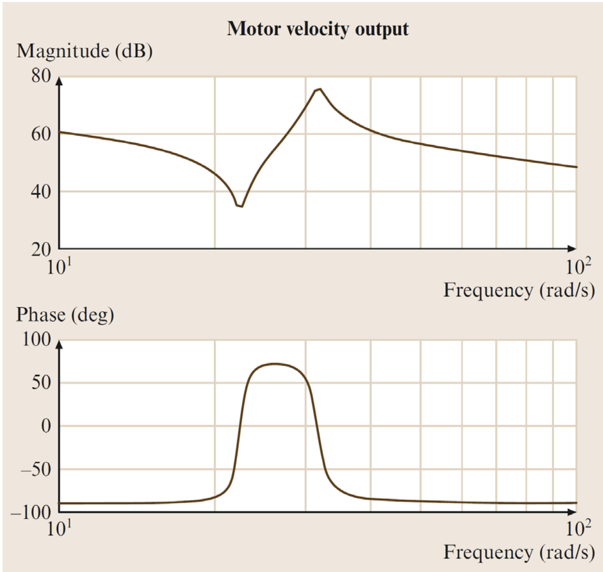
  $M\ddot{q} + D(\dot{q} - \dot{\theta}) + K(q - \theta) + F_q\dot{q} = 0$

  $B\ddot{\theta} + D(\dot{\theta} - \dot{q}) + K(\theta - q) + F_\theta\dot{\theta} = \tau$

- Laplace transforms

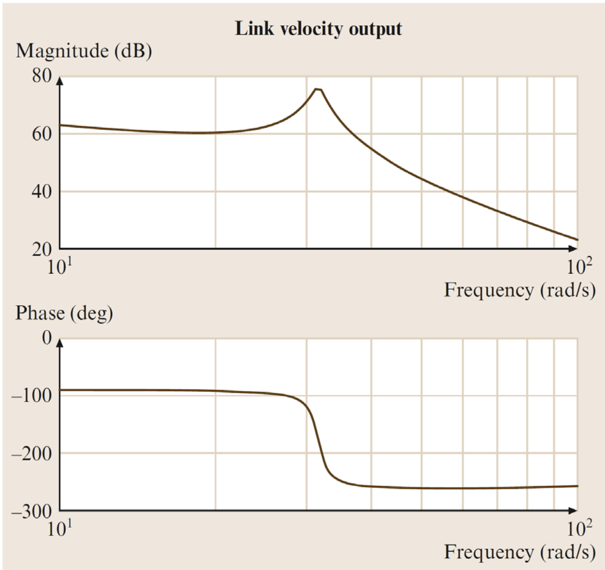
$$\frac{\theta(s)}{\tau(s)} = \frac{Ms^2 + (D + F_q)s + K}{\text{den}(s)}$$

  - Presence of antiresonance/ resonance



$$\frac{q(s)}{\tau(s)} = \frac{Ds + K}{\text{den}(s)}$$

  - Presence of resonance
  - High-frequency lag of 270°



$$\begin{aligned}\text{den}(s) = \ &\{MBs^3 + [M(D + F_\theta) + B(D + F_q)]\, s^2 \\ &+ [(M + B)K + (F_q + F_\theta)D + F_qF_\theta]\, s \\ &+ (F_q + F_\theta)K\}\, s\end{aligned}$$

- Neglecting all dissipative effects ($D = F_q = F_\theta = 0$, i.e. worst case)

$$\left.\frac{\theta(s)}{\tau(s)}\right|_{\text{no diss}} = \frac{Ms^2 + K}{[MBs^2 + (M + B)K]s^2}$$

  - Double pole at origin
  - Pair of imaginary poles
  - Pair of imaginary zeros at locked frequency ($\theta \equiv 0$)

    $$\omega_1 = \sqrt{\frac{K}{M}}$$

    lower than that of pole pair
  - To achieve enough damping in closed-loop system, bandwidth shall be limited to one third of ω_l

$$\left.\frac{q(s)}{\tau(s)}\right|_{\text{no diss}} = \frac{K}{[MBs^2 + (M + B)K]s^2}$$

  - No zeros

- Feedback control using link position and link velocity
  $\tau = u_q - \left(K_{P,q}q + K_{D,q}\dot{q}\right)$ $\quad u_q = K_{P,q}q_{\mathrm{d}}$
  - Closed-loop poles unstable no matter how gains are chosen
- Feedback control using motor position and link velocity … unstable!
- Feedback control using link position (optical encoder on load shaft) and motor velocity (tachometer integrated in DC motor)
  $\tau = u_q - \left(K_{P,q}q + K_{D,m}\dot{\theta}\right)$
  - Closed-loop characteristic equation
    $BMs^4 + MK_{D,m}s^3 + (B + M)Ks^2 + KK_{D,m}s + KK_{P,q} = 0$

    Asymptotic stability iff $K_{D,m} > 0$ and $0 < K_{P,q} < K$ (proportional gain should not *override* spring stiffness)
- Feedback control using motor position and motor velocity
  $\tau = u_\theta - \left(K_{P,m}\theta + K_{D,m}\dot{\theta}\right)$ $\quad u_\theta = K_{P,m}\theta_{\mathrm{d}} = K_{P,m}q_{\mathrm{d}}$
  - Asymptotic stability iff $K_{P,\mathrm{m}} > 0$ and $K_{D,\mathrm{m}} > 0$
- Other partial state feedback combinations …
  - Strain gauge on transmission shaft → direct measure of elastic torque $\tau_{\mathrm{J}} = K(\theta - q)$ for control use

PD control using only motor variables

- General multilink case in absence of gravity ($\boldsymbol{\theta}_{\mathrm{d}} = \boldsymbol{q}_{\mathrm{d}}$)

  $$\boldsymbol{\tau} = \boldsymbol{K}_P(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) - \boldsymbol{K}_D\dot{\boldsymbol{\theta}}$$

- Lyapunov argument

  $$V = \frac{1}{2}\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\boldsymbol{\mathcal{M}}(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} + \frac{1}{2}(\boldsymbol{q}-\boldsymbol{\theta})^{\mathrm{T}}\mathbf{K}(\boldsymbol{q}-\boldsymbol{\theta}) + \frac{1}{2}(\boldsymbol{\theta}_{\mathrm{d}}-\boldsymbol{\theta})^{\mathrm{T}}\mathbf{K}_P(\boldsymbol{\theta}_{\mathrm{d}}-\boldsymbol{\theta}) \geq 0$$

  - Time derivative along trajectories of closed-loop system

    $$\dot{V} = -\dot{\boldsymbol{\theta}}^{\mathrm{T}}\mathbf{K}_D\dot{\boldsymbol{\theta}} \leq 0$$

  - La Salle's theorem is applied
  - Inclusion of dissipative terms (viscous friction and spring damping) would render $\dot{V}$ even more negative semi-definite

PD control with constant gravity compensation

- In view of **A2**, for robots with revolute joints (flexible or not)

  $$\left\|\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right\| \leq \alpha \qquad \forall \boldsymbol{q} \in \mathbb{R}^N$$

  $$\|\boldsymbol{g}(\boldsymbol{q}_1) - \boldsymbol{g}(\boldsymbol{q}_2)\| \leq \alpha\|\boldsymbol{q}_1 - \boldsymbol{q}_2\| \qquad \forall \boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{R}^N$$

A5. The lowest joint stiffness is larger than the upper bound on the gradient of gravity forces

$$\min_{i=1,\dots,N} K_i > \alpha$$

- Addition of constant gravity compensation

  $$\boldsymbol{\tau} = \boldsymbol{K}_P(\boldsymbol{\theta}_\mathrm{d} - \boldsymbol{\theta}) - \boldsymbol{K}_D\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{q}_\mathrm{d}) \qquad \boldsymbol{\theta}_\mathrm{d} = \boldsymbol{q}_d + \mathbf{K}^{-1}\boldsymbol{g}(\boldsymbol{q}_\mathrm{d})$$

- Sufficient condition for global asymptotic stability

  $$(\boldsymbol{q} = \boldsymbol{q}_\mathrm{d}, \boldsymbol{\theta} = \boldsymbol{\theta}_\mathrm{d}, \dot{\boldsymbol{q}} = \dot{\boldsymbol{\theta}} = \mathbf{0})$$

  $$\lambda_{\min}\left[\begin{pmatrix} \mathbf{K} & -\mathbf{K} \\ -\mathbf{K} & \mathbf{K} + \mathbf{K}_P \end{pmatrix}\right] > \alpha$$

  - Fulfilled by increasing smallest proportional gain

- $(\boldsymbol{q}_\mathrm{d}, \boldsymbol{\theta}_\mathrm{d})$ satisfies equilibrium

  $$\mathbf{K}(\boldsymbol{q} - \boldsymbol{\theta}) + \boldsymbol{g}(\boldsymbol{q}) = \mathbf{0}$$

  $$\mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q}) - \mathbf{K}_P(\boldsymbol{\theta}_\mathrm{d} - \boldsymbol{\theta}) - \boldsymbol{g}(\boldsymbol{q}_\mathrm{d}) = \mathbf{0}$$

  and is the unique solution

  $$\mathbf{K}(\boldsymbol{q} - \boldsymbol{q}_\mathrm{d}) - \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{\theta}_\mathrm{d}) = \boldsymbol{g}(\boldsymbol{q}_\mathrm{d}) - \boldsymbol{g}(\boldsymbol{q})$$

  $$-\mathbf{K}(\boldsymbol{q} - \boldsymbol{q}_\mathrm{d}) + (\mathbf{K} + \mathbf{K}_P)(\boldsymbol{\theta} - \boldsymbol{\theta}_\mathrm{d}) = \mathbf{0}$$

- Lyapunov argument

  $$V_{\mathrm{g}1} = V + \mathcal{U}_{\mathrm{grav}}(\boldsymbol{q}) - \mathcal{U}_{\mathrm{grav}}(\boldsymbol{q}_\mathrm{d}) - (\boldsymbol{q} - \boldsymbol{q}_\mathrm{d})^\mathrm{T}\boldsymbol{g}(\boldsymbol{q}_\mathrm{d}) - \frac{1}{2}\boldsymbol{g}^\mathrm{T}(\boldsymbol{q}_\mathrm{d})\mathbf{K}^{-1}\boldsymbol{g}(\boldsymbol{q}_\mathrm{d}) \geq 0$$

  $$\dot{V}_{\mathrm{g}1} = -\dot{\boldsymbol{\theta}}^\mathrm{T}\mathbf{K}_D\dot{\boldsymbol{\theta}} \leq 0$$

- The better $\widehat{\mathbf{K}}$ and $\widehat{\boldsymbol{g}}(\boldsymbol{q}_\mathrm{d})$, the closer the equilibrium to desired one

PD control with online gravity compensation

- *Gravity-biased* modification of measured motor position $\boldsymbol{\theta}$ (approximate cancellation of gravity during motion)
  $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \mathbf{K}^{-1}\boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}})$
- Feedback control using only motor variables
  $\boldsymbol{\tau} = \mathbf{K}_P(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) - \mathbf{K}_D\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\tilde{\boldsymbol{\theta}})$
  leading to correct gravity compensation at steady state
  $\tilde{\boldsymbol{\theta}}_{\mathrm{d}} \coloneqq \boldsymbol{\theta}_{\mathrm{d}} - \mathbf{K}^{-1}\boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}}) = \boldsymbol{q}_{\mathrm{d}} \qquad \boldsymbol{g}(\tilde{\boldsymbol{\theta}}_{\mathrm{d}}) = \boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}})$
- Lyapunov argument
  $$V_{\mathrm{g2}} = V + \mathcal{U}_{\mathrm{grav}}(\boldsymbol{q}) - \mathcal{U}_{\mathrm{grav}}(\tilde{\boldsymbol{\theta}}) - \frac{1}{2}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{d}})\mathbf{K}^{-1}\boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}}) \geq 0$$
- Smoother time course and noticeable reduction of positional transient errors, with no additional control effort in terms of peak and average torques
- Possible refinement of control, using quasi-static estimate $\bar{\boldsymbol{q}}(\boldsymbol{\theta})$ of measured $\boldsymbol{q}$
  $\boldsymbol{\tau} = \mathbf{K}_P(\boldsymbol{\theta}_d - \boldsymbol{\theta}) - \mathbf{K}_D\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\bar{\boldsymbol{q}}(\boldsymbol{\theta}))$

- These control laws realize a *compliance control* in the joint space with only motor measurements
  - Can be extended to operational space control via Jacobian transpose

Full-state feedback

- If joint torque sensors are available, a convenient control design for reduced model including spring damping can be derived
- Motor equation using $\boldsymbol{\tau}_{\mathrm{J}} = \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q})$
  $$\mathbf{B}\ddot{\boldsymbol{\theta}} + \boldsymbol{\tau}_{\mathrm{J}} + \mathbf{D}\mathbf{K}^{-1}\dot{\boldsymbol{\tau}}_{\mathrm{J}} = \boldsymbol{\tau}$$
- Feedback control
  $$\boldsymbol{\tau} = \mathbf{B}\mathbf{B}_{\theta}^{-1}\boldsymbol{u} + \left(\boldsymbol{I} - \mathbf{B}\mathbf{B}_{\theta}^{-1}\right)\left(\boldsymbol{\tau}_{\mathrm{J}} + \mathbf{D}\mathbf{K}^{-1}\dot{\boldsymbol{\tau}}_{\mathrm{J}}\right)$$
  gives
  $$\mathbf{B}_{\theta}\ddot{\boldsymbol{\theta}} + \boldsymbol{\tau}_{\mathrm{J}} + \mathbf{D}\mathbf{K}^{-1}\dot{\boldsymbol{\tau}}_{\mathrm{J}} = \boldsymbol{u}$$
  - The apparent motor inertia can be reduced to desired, arbitrary small value $\mathbf{B}_{\theta}$, with clear benefits in terms of vibration damping
- Choice of auxiliary input
  $$\boldsymbol{u} = \mathbf{K}_{P,\theta}(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) - \mathbf{K}_{D,\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}})$$
  leads to state feedback control
  $$\boldsymbol{\tau} = \mathbf{K}_{P}(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) - \mathbf{K}_{D}\dot{\boldsymbol{\theta}} + \mathbf{K}_{T}\left[\boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}}) - \boldsymbol{\tau}_{\mathrm{J}}\right] - \mathbf{K}_{S}\dot{\boldsymbol{\tau}}_{\mathrm{J}} + \boldsymbol{g}(\boldsymbol{q}_{\mathrm{d}})$$

$$\mathbf{K}_{P} = \mathbf{B}\mathbf{B}_{\theta}^{-1}\mathbf{K}_{P,\theta}$$
$$\mathbf{K}_{D} = \mathbf{B}\mathbf{B}_{\theta}^{-1}\mathbf{K}_{D,\theta}$$
$$\mathbf{K}_{T} = \mathbf{B}\mathbf{B}_{\theta}^{-1} - \mathbf{I}$$
$$\mathbf{K}_{s} = \left(\mathbf{B}\mathbf{B}_{\theta}^{-1} - \mathbf{I}\right)\mathbf{D}\mathbf{K}^{-1}$$

Trajectory Tracking

- Controlling motion of robot with flexible joints to smooth *reference trajectory* $\boldsymbol{q}_{\mathrm{d}}(t)$

Feedback linearization

- Link equation of reduced model
  $\mathbf{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{K}(\boldsymbol{q} - \boldsymbol{\theta}) = \mathbf{0}$
- Time differentiation …
  $\mathbf{M}(\boldsymbol{q})\boldsymbol{q}^{[3]} + \dot{\mathbf{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{n}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{K}\big(\dot{\boldsymbol{q}} - \dot{\boldsymbol{\theta}}\big) = \mathbf{0}$
- Time differentiation …
  $\mathbf{M}(\boldsymbol{q})\boldsymbol{q}^{[4]} + \mathbf{2}\dot{\mathbf{M}}(\boldsymbol{q})\boldsymbol{q}^{[3]} + \ddot{\mathbf{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{n}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{K}\big(\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{\theta}}\big) = \mathbf{0}$
- Motor equation of reduced model
  $\mathbf{B}\ddot{\boldsymbol{\theta}} + \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q}) = \boldsymbol{\tau}$

$$\mathbf{M}(\boldsymbol{q})\boldsymbol{q}^{[4]} + 2\dot{\mathbf{M}}(\boldsymbol{q})\boldsymbol{q}^{[3]} + \ddot{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{n}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{K}\ddot{\boldsymbol{q}} =$$
$$= \mathbf{KB}^{-\mathbf{1}}[\boldsymbol{\tau} - \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q})]$$

  - Last term $\mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q})$ can be replaced by $\mathbf{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$

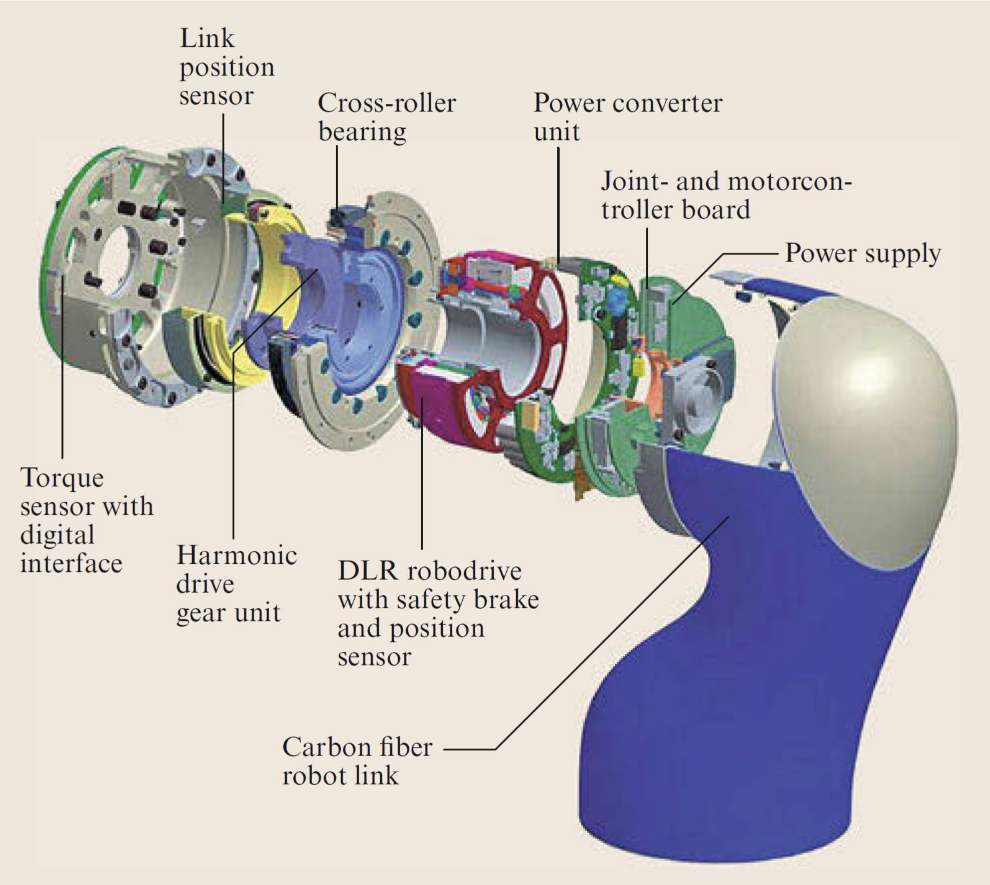
- Decoupling matrix $\mathbf{A}(\boldsymbol{q}) = \mathbf{M}^{-1}(\boldsymbol{q})\mathbf{KB}^{-1}$ is always nonsingular
  - Feedback linearizing control (relative degree 4*N*)

$$\boldsymbol{\tau} = \mathbf{BK}^{-1}\big[\mathbf{M}(\boldsymbol{q})\boldsymbol{v} + \boldsymbol{\alpha}\big(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}^{[3]}\big)\big] +$$
$$+ [\mathbf{M}(\boldsymbol{q}) + \mathbf{B}]\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

$$\boldsymbol{\alpha}\big(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}^{[3]}\big) = \ddot{\mathbf{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}} + 2\dot{\mathbf{M}}(\boldsymbol{q})\boldsymbol{q}^{[3]} + \ddot{\boldsymbol{n}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

$\boldsymbol{q}^{[4]} = \boldsymbol{v}$ (chains of 4 input–output integrators from each new input v_i to each link position output q_i)

- Measures needed to implement feedback linearizing control
  - Direct measures of link acceleration $\ddot{\boldsymbol{q}}$ and jerk $\boldsymbol{q}^{[3]}$ are impossible to obtain with currently available sensors … multiple numerical differentiation of position measures in real time causes noise
- Latest technology with joint torque sensors
  - Measures of motor position $\boldsymbol{\theta}$ (and possibly its velocity $\dot{\boldsymbol{\theta}}$), joint torque $\boldsymbol{\tau}_{\mathrm{J}} = \mathbf{K}(\boldsymbol{\theta} - \boldsymbol{q})$ and link position $\boldsymbol{q}$



- Equivalent state variables for robots with flexible joints

$$(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}^{[3]}) \qquad (\boldsymbol{q}, \boldsymbol{\theta}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{\theta}}) \qquad (\boldsymbol{q}, \boldsymbol{\tau}_{\mathrm{J}}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{\tau}}_{\mathrm{J}})$$

- Instead of measuring link acceleration and jerk, compute them as
$$\ddot{\boldsymbol{q}} = \mathbf{M}^{-1}(\boldsymbol{q})[\mathbf{K}(\boldsymbol{\theta}-\boldsymbol{q})-\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})]$$
$$= \mathbf{M}^{-1}(\boldsymbol{q})\left[\boldsymbol{\tau}_{\mathrm{J}}-\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right]$$
$$\boldsymbol{q}^{[3]} = \mathbf{M}^{-1}\left[\mathbf{K}\left(\dot{\boldsymbol{\theta}}-\dot{\boldsymbol{q}}\right)-\dot{\mathbf{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}-\dot{\boldsymbol{n}}(\boldsymbol{q},\dot{\boldsymbol{q}})\right]$$
$$= \mathbf{M}^{-1}(\boldsymbol{q})\left[\dot{\boldsymbol{\tau}}_{\mathrm{J}}-\dot{\mathbf{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}-\dot{\boldsymbol{n}}(\boldsymbol{q},\dot{\boldsymbol{q}})\right]$$
- Feedback linearizing control in terms of static state feedback law
$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q},\boldsymbol{\theta},\dot{\boldsymbol{q}},\dot{\boldsymbol{\theta}},\boldsymbol{v}) \quad \text{or} \quad \boldsymbol{\tau} = \boldsymbol{\tau}\left(\boldsymbol{q},\boldsymbol{\tau}_{\mathrm{J}},\dot{\boldsymbol{q}},\dot{\boldsymbol{\tau}}_{\mathrm{J}},\boldsymbol{v}\right)$$
- Choice of new input
$$\boldsymbol{v} = \boldsymbol{q}_{\mathrm{d}}^{[4]} + \mathbf{K}_3\left(\boldsymbol{q}_{\mathrm{d}}^{[3]}-\boldsymbol{q}^{[3]}\right) + \mathbf{K}_2(\ddot{\boldsymbol{q}}_{\mathrm{d}}-\ddot{\boldsymbol{q}})$$
$$+ \mathbf{K}_1(\dot{\boldsymbol{q}}_{\mathrm{d}}-\dot{\boldsymbol{q}}) + \mathbf{K}_0(\boldsymbol{q}_{\mathrm{d}}-\boldsymbol{q})$$
  - Reference trajectory $\boldsymbol{q}_{\mathrm{d}}(t)$ at least three times continuously differentiable
  - Diagonal matrices $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ have scalar elements $K_{\cdot,i}$ such that
  $$s^4 + K_{3,i}s^3 + K_{2,i}s^2 + K_{1,i}s + K_{0,i}, \qquad i = 1, \ldots, N$$
  are Hurwitz polynomials → $e_i(t)$ converges to zero in a global exponential way for any initial state
- Compared to inverse dynamics for rigid robots, feedback linearizing control requires inversion of inertia matrix $\mathbf{M}(\boldsymbol{q})$ and additional evaluation of derivatives of inertia matrix and other terms in dynamic model
- In case of friction at motor or link side, third-order decoupled differential relation between new input $\boldsymbol{v}$ and $\boldsymbol{q}$ is obtained, leaving *N*-dimensional unobservable (asymptotically stable) dynamics in closed loop → only input–output (not full-state) linearization and decoupling is achieved
- For complete dynamic model … dynamic state feedback control is to be designed

Linear control

- Given sufficiently smooth reference link trajectory $\boldsymbol{q}_{\mathrm{d}}(t)$, with computed torque method it is always possible to associate:
  - Nominal torque $\boldsymbol{\tau}_{\mathrm{d}}(t)$ needed for its exact reproduction
  - Reference evolution for all other state variables $\boldsymbol{\theta}_{\mathrm{d}}(t)$ or $\boldsymbol{\tau}_{\mathrm{J,d}}(t)$

  defining a sort of steady-state (though, time-varying) behavior for the system

- Combination of model-based feedforward term with linear feedback term using trajectory errors (locally valid)

$$\begin{aligned}\boldsymbol{\tau} = \; & \boldsymbol{\tau}_{\mathrm{d}} + \mathbf{K}_{P,\theta}(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) + \mathbf{K}_{D,\theta}(\dot{\boldsymbol{\theta}}_{\mathrm{d}} - \dot{\boldsymbol{\theta}}) \\ & + \mathbf{K}_{P,\theta}(\boldsymbol{q}_{\mathrm{d}} - \boldsymbol{q}) + \mathbf{K}_{D,\theta}(\dot{\boldsymbol{q}}_{\mathrm{d}} - \dot{\boldsymbol{q}})\end{aligned}$$

$$\begin{aligned}\boldsymbol{\tau} = \; & \boldsymbol{\tau}_{\mathrm{d}} + \mathbf{K}_{P,\theta}(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) + \mathbf{K}_{D,\theta}(\dot{\boldsymbol{\theta}}_{\mathrm{d}} - \dot{\boldsymbol{\theta}}) \\ & + \mathbf{K}_{P,J}(\boldsymbol{\tau}_{\mathrm{J,d}} - \boldsymbol{\tau}_{\mathrm{J}}) + \mathbf{K}_{D,J}(\dot{\boldsymbol{\tau}}_{\mathrm{J,d}} - \dot{\boldsymbol{\tau}}_{\mathrm{J}})\end{aligned}$$

  - In absence of full-state measures, they can be combined with observer of unmeasurable quantities

- Even simpler realization

  $\boldsymbol{\tau} = \boldsymbol{\tau}_{\mathrm{d}} + \mathbf{K}_P(\boldsymbol{\theta}_{\mathrm{d}} - \boldsymbol{\theta}) + \mathbf{K}_D(\dot{\boldsymbol{\theta}}_{\mathrm{d}} - \dot{\boldsymbol{\theta}})$

  using only motor measures and relying on results obtained for regulation case