

AN INTEGRAL MANIFOLD APPROACH TO CONTROL OF A ONE LINK FLEXIBLE ARM

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ABSTRACT

The problem of controlling a one link flexible arm is considered in this paper. An assumed mode method is adopted to derive the dynamic equations of motion; the system is then transformed to singularly perturbed form. An integral manifold approach is proposed leading to the derivation of a reduced order system which incorporates the effects of the flexibility distributed along the structure. An approximate technique is finally presented which allows the synthesis of a feedback linearizing control.

INTRODUCTION

The performance of today's manipulator arms is limited by their rigidity. Lower arm cost, higher motion speed, better energy efficiency, safer operation and improved mobility are all benefits which are potentially achievable with lighter arms [1]. The price to pay, however, is the much more complex dynamics, due to the flexibility distributed along a lightweight mechanical structure.

This issue hardly complicates the control problem and very little literature exists in the field of flexible link arm control. First research efforts are described in [2,3,4]. The same idea which is behind [4] is followed in this paper. The system is transformed to singular perturbation form to achieve a reduced order system which could allow the synthesis of a feedback linearizing control, in the same manner as it is possible for rigid arms. To this goal an integral manifold approach is pursued [5]. The solution on the manifold is then expanded in powers of the perturbation parameter so as to obtain an approximate computational means to synthesize the linearizing control. Control implementation issues are finally discussed.

THE MODEL

The one link flexible arm of fig. 1 is considered. A solution to the flexible motion of the link can be obtained through modal analysis, under the assumption of small deflections of the link,

$$y(\eta, \tau) = \sum_{i=1}^m \delta_i(\tau) \phi_i(\eta) \quad (1)$$

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where ϕ_i is the eigenfunction expressing the displacement of assumed mode i of link deflection, δ_i is the time-varying amplitude of mode i of the link and m is the number of modes used to describe the deflection of the link.

For a clamped-free vibrating beam the orthonormal modal eigenfunctions in (1) are given by

$$\phi_i(\xi) = \sin(\beta_i \xi) - \sinh(\beta_i \xi) + \quad (2)$$

$$v_i (\cos(\beta_i \xi) - \cosh(\beta_i \xi))$$

$$v_i = \frac{\sin \beta_i + \sinh \beta_i}{\cos \beta_i + \cosh \beta_i} \quad i = 1, \dots, m$$

$$\beta_i^4 = \frac{\rho A (2\pi f_i)^2 L^4}{EI}$$

$$\xi = \eta/L$$

where:

- L = beam length
- A = beam cross area
- E = Young's modulus
- I = beam area inertia
- ρ = density
- f_i = frequency of the i th mode.

The dynamic equations of motion for the one link flexible arm can be written in the following form [4]

$$M(\theta, \delta) \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} f_1(\dot{\theta}, \delta, \dot{\delta}) \\ f_2(\dot{\theta}, \delta) \end{bmatrix} + \begin{bmatrix} 0 \\ K\delta \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix} \quad (3)$$

where

θ is the joint variable,
 $\delta = (\delta_1 \dots \delta_m)^T$ is the vector of deflections,
 u is the control torque at joint location,
 M is the inertia matrix

$$M = \begin{bmatrix} m_{11} & \dots & m_{1p} & \dots & \dots & m_{1,m+1} \\ m_{1p} & \dots & m_{pp} & \dots & m_{pq} & \dots \\ \vdots & \dots & m_{pq} & \dots & \dots & \dots \\ m_{1,m+1} & \dots & \dots & \dots & \dots & m_{m+1,m+1} \end{bmatrix} \quad (4)$$

$$m_{11} = J_0 + M_L L^2 + I_0 + M_L (\phi_{-e}^T \delta)^2$$

$$\begin{aligned}
m_{lp} &= M_L L \phi_{p-1,e} + w_{p-1} & p &= 2, \dots, m+1 \\
m_{pp} &= m_b + M_L \phi_{p-1,e}^2 + J_p \phi_{p-1,e}^2 \\
m_{pq} &= M_L \phi_{p-1,e} \phi_{q-1,e} + J_p \phi_{p-1,e} \phi_{q-1,e} & q &= p+1, \dots, m+1
\end{aligned}$$

with

$$\begin{aligned}
\phi_{-e}^T &= (\phi_{1e} \dots \phi_{me}), & \phi_{ie} &= \phi_i(\xi) \Big|_{\xi=1} \\
\phi_{-e}'^T &= (\phi_{1e}' \dots \phi_{me}'), & \phi_{ie}' &= \frac{d\phi_i(\xi)}{d\xi} \Big|_{\xi=1} \\
w_i &= \rho A L^2 \int_0^1 \phi_i(\xi) \xi d\xi & i &= 1, \dots, m
\end{aligned}$$

where:

$$\begin{aligned}
m_b &= \text{beam mass} \\
M_L^b &= \text{payload mass} \\
I_{-e}^0 &= \text{joint inertia} \\
J_{-e}^0 &= \text{beam inertia relative to joint} \\
J_p &= \text{payload inertia,}
\end{aligned}$$

f_1 and f_2 are nonlinear terms

$$f_1 = 2M_L \dot{\theta} (\phi_{-e}^T \delta) (\phi_{-e}^T \dot{\delta}) \quad (5)$$

$$f_2 = -M_L \dot{\theta}^2 (\phi_{-e}^T \delta) \delta, \quad (6)$$

K is an equivalent spring constant matrix

$$K = \text{diag}(k_1 \dots k_m) \quad (7)$$

$$k_i = \frac{EI}{L^3} \int_0^1 \frac{[d^2 \phi_i(\xi)]^2}{d\xi^2} d\xi.$$

Since the clamped-free assumption has been made for the vibrating beam, there is no displacement at joint location and then no control force in the lower eqs. of (3).

Being the inertia matrix positive definite, it can be inverted and denoted by H which can be partitioned as follows:

$$M^{-1} = H = \begin{bmatrix} h_{11} & | & h_{12}^T [1 \times m] \\ \hline h_{12} [m \times 1] & | & H_{22} [m \times n] \end{bmatrix}. \quad (8)$$

Eqs. (3) then become

$$\ddot{\theta} = -h_{11} f_1 - h_{12}^T f_2 - h_{12}^T K \delta + h_{11} u \quad (9)$$

$$\ddot{\delta} = -h_{12} f_1 - H_{22} f_2 - H_{22} K \delta + h_{12} u. \quad (10)$$

In order to put the system (9) and (10) in singularly perturbed form, the ratio EI/L^3 in (7) can be regarded as the inverse of the perturbation parameter, i.e. $\mu = L^3/EI$. As a matter of fact the longer the arm the bigger μ , and similarly for EI , i.e. in the limit $\mu \rightarrow 0$ when $E \rightarrow \infty$ (rigid arm). Consequently the matrix K in (7) can be factored as $K = \tilde{K}/\mu$. Defining

$$z := \frac{1}{\mu} \tilde{K} \delta \quad (11)$$

yields the eqs. of the system in singularly perturbed form, i.e.

$$\ddot{\theta} = -h_{11}(\mu z) f_1(\dot{\theta}, \mu z, \mu \dot{z}) - h_{12}^T(\mu z) f_2(\dot{\theta}, \mu z) + h_{11}^T(\mu z) z + h_{11}(\mu z) u \quad (12)$$

$$\mu \ddot{z} = -h_{12}^T(\mu z) f_1(\dot{\theta}, \mu z, \mu \dot{z}) - H_{22}^T(\mu z) f_2(\dot{\theta}, \mu z) + h_{22}^T(\mu z) z + h_{12}(\mu z) u \quad (13)$$

where the prime ' indicates the fact that the terms on the right side of (13) have been scaled by \tilde{K} , by virtue of the definition (11). In the following, however, the primes will be dropped without loss of generality.

AN INTEGRAL MANIFOLD APPROACH

Although most standard results in singular perturbation theory have been derived for systems in state space, for the purpose of this work, the second order Lagrangian formulation (12) and (13) will be considered in the following.

It can be first observed that setting $\mu = 0$ yields

$$\ddot{\theta} = -h_{11}(0) f_1(\dot{\theta}, 0, 0) - h_{12}^T(0) f_2(\dot{\theta}, 0) + h_{11}^T(0) h_0 + h_{11}(0) u_0 \quad (14)$$

$$0 = -h_{12}(0) f_1(\dot{\theta}, 0, 0) - H_{22}(0) f_2(\dot{\theta}, 0) + h_{22}(0) h_0 + h_{12}(0) u_0 \quad (15)$$

where $h_0 = z(\mu=0)$ and $u_0 = u(\mu=0)$. It is seen from (5) that $f_1(\dot{\theta}, 0, 0) = 0$, and from (6) that $f_2(\dot{\theta}, 0) = 0$. Since $H_{22}(0)$ is invertible, eqs. (15) can be solved for h_0 as

$$h_0 = H_{22}^{-1}(0) h_{12}(0) u_0 \quad (16)$$

which, when substituted into (14), yields the reduced order system ($\mu = 0$)

$$\ddot{\theta} = (h_{11}(0) - h_{12}^T(0) H_{22}^{-1}(0) h_{12}(0)) u_0. \quad (17)$$

It can be checked that the system (17) is right the rigid system, i.e.

$$\ddot{\theta} = \frac{1}{m_{11}(0)} u_0. \quad (18)$$

In the following an integral manifold approach is pursued with the goal of accounting for the flexibility distributed along the structure in the reduced order system. From [5] a $2m$ -dimensional manifold Σ_μ defined by the equations

$$z = \underline{h}(\theta, \dot{\theta}, u, \mu) \quad (19)$$

is said to be an integral manifold for the system (12) and (13) if it is invariant under solutions of (12) and (13). In other words, if the system lies at $t = t_0$ on the manifold Σ_μ then the solution trajectory remains on the manifold Σ_μ for $t > t_0$. It follows from [6] that Σ_μ actually exists for (12) and (13) since H_{22} is nonsingular, being positive definite. (The vector h defined in (19) is not to be confused with the blocks of the matrix H in (8)).

If the flexible dynamics is asymptotically stable, the solution of (12) and (13) will rapidly approach Σ_μ on a fast manifold $\Phi_{\theta, \dot{\theta}}$ and then flow along the integral manifold Σ_μ [7]. As $\mu \rightarrow 0$, of course, $\Sigma_\mu \rightarrow \Sigma_0$ which is the slow manifold identified by h_0 in (16).

The function h defining Σ_μ must be a solution of (13), i.e.

$$\mu \ddot{\underline{h}}(\theta, \dot{\theta}, u, \mu) = \underline{g}(\theta, \dot{\theta}, u, \mu) \quad (20)$$

$$\begin{aligned} \underline{g} = & -\underline{h}_{12}(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) \underline{f}_1(\dot{\theta}, \mu \underline{h}(\theta, \dot{\theta}, u, \mu), \mu \dot{\underline{h}}(\theta, \dot{\theta}, u, \mu)) + \\ & -\underline{H}_{22}(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) \underline{f}_2(\dot{\theta}, \mu \underline{h}(\theta, \dot{\theta}, u, \mu)) + \\ & -\underline{H}_{22}(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) \underline{h}(\theta, \dot{\theta}, u, \mu) + \\ & \underline{h}_{12}(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) u \end{aligned}$$

where it is understood that $\dot{\underline{h}}$ and $\ddot{\underline{h}}$ are total derivatives along the solutions of (12) and (13).

Once \underline{h} is determined from the manifold condition (20), the desired reduced order system is defined by combining (12) and (19) as

$$\begin{aligned} \ddot{\theta} = & -\underline{h}_{11}(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) \underline{f}_1(\dot{\theta}, \mu \underline{h}(\theta, \dot{\theta}, u, \mu), \mu \dot{\underline{h}}(\theta, \dot{\theta}, u, \mu)) + \\ & -\underline{h}_{12}^T(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) \underline{f}_2(\dot{\theta}, \mu \underline{h}(\theta, \dot{\theta}, u, \mu)) + \\ & -\underline{h}_{12}^T(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) \underline{h}(\theta, \dot{\theta}, u, \mu) + \\ & \underline{h}_{11}(\mu \underline{h}(\theta, \dot{\theta}, u, \mu)) u \end{aligned} \quad (21)$$

This system is of the same dimension as the rigid system (18), but it incorporates the effects of the flexibility through the integral manifold defined by (19). This point is helpful since, in the following section, it will be shown that an approximate linearizing control for (21) can be synthesized, provided that the functions \underline{h} and u are expanded to any order in μ .

APPROXIMATE FEEDBACK LINEARIZING CONTROL

The computation of a linearizing control $u(\theta, \dot{\theta}, v, \mu)$ for (21), where v is a new input to the system [8], is complicated by the need to solve the manifold condition (20) for \underline{h} . A practical computational approach is based on expanding the function \underline{h} in (19) as [7]

$$\underline{h}(\theta, \dot{\theta}, u, \mu) = \underline{h}_0(\theta, \dot{\theta}, u) + \mu \underline{h}_1(\theta, \dot{\theta}, u) + \dots \quad (22)$$

and correspondingly the control u as

$$u(\theta, \dot{\theta}, v, \mu) = u_0(\theta, \dot{\theta}, v) + \mu u_1(\theta, \dot{\theta}, v) + \dots \quad (23)$$

where it can be recognized that \underline{h}_0 and u_0 are the functions introduced in (14)-(16). The expansions (22) and (23) shall be substituted in (20) to yield a set of eqs. in which the like powers of μ on both sides are to be equated. This process is usually very tedious, but it can be performed using a symbolic manipulation language. For the system (12) and (13) all the following expressions have been obtained using REDUCE:

$$\mu^0: \underline{0} = -\bar{\underline{H}}_{22}(\underline{0}) \underline{h}_0 + \bar{\underline{h}}_{12}(\underline{0}) u_0 \quad (24)$$

$$\mu^1: \Delta_0 \ddot{\underline{h}}_0 = \underline{M}_L \dot{\theta}^2 (\underline{\phi}_e \underline{\phi}_e^T) \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_0 - \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_1 + \bar{\underline{h}}_{12}(\underline{0}) u_1$$

$$\begin{aligned} \mu^2: \Delta_0 \ddot{\underline{h}}_1 = & -2 \underline{M}_L \dot{\theta} \bar{\underline{h}}_{12}(\underline{0}) (\underline{\phi}_e^T \underline{h}_0) (\underline{\phi}_e^T \underline{h}_0) + \\ & \underline{M}_L \dot{\theta}^2 (\underline{\phi}_e \underline{\phi}_e^T) \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_1 - \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_2 + \bar{\underline{h}}_{12}(\underline{0}) u_2 \end{aligned}$$

etc., with $\Delta_0 = \det(\underline{M}(\underline{0}))$ and the bars over $\bar{\underline{h}}_{12}$ and $\bar{\underline{H}}_{22}$ indicate that the terms have been scaled by Δ_0 ; this position is necessary since the mass matrix is function of μ . The first line of (24) can be solved for \underline{h}_0 as in (16) (Δ_0 cancels out)

$$\underline{h}_0 = \bar{\underline{H}}_{22}^{-1}(\underline{0}) \bar{\underline{h}}_{12}(\underline{0}) u_0 \quad (16')$$

and, after obtaining the rigid system (18) (neglecting a term $O(\mu)$), \underline{u}_0 can be designed. Knowing u_0 , \underline{h}_0 is also known and \underline{h}_0 can be explicitly computed as

$$\ddot{\underline{h}}_0 = \bar{\underline{H}}_{22}^{-1}(\underline{0}) \bar{\underline{h}}_{12}(\underline{0}) \ddot{u}_0. \quad (25)$$

The second line of (24) then can be solved for \underline{h}_1 as

$$\begin{aligned} \underline{h}_1 = & -\Delta_0 \bar{\underline{H}}_{22}^{-1}(\underline{0}) \ddot{\underline{h}}_0 + \underline{M}_L \dot{\theta}^2 \bar{\underline{H}}_{22}^{-1}(\underline{0}) (\underline{\phi}_e \underline{\phi}_e^T) \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_0 + \\ & \bar{\underline{H}}_{22}^{-1}(\underline{0}) \bar{\underline{h}}_{12}(\underline{0}) u_1 \end{aligned} \quad (26)$$

which, when substituted in (21), gives

$$\begin{aligned} \ddot{\theta} = & \frac{1}{m_{11}(\underline{0})} u_0 + \mu (-\underline{h}_{12}^T(\underline{0}) \bar{\underline{H}}_{22}^{-1}(\underline{0}) \Delta_0 \ddot{\underline{h}}_0 + \\ & \underline{h}_{12}^T(\underline{0}) \bar{\underline{H}}_{22}^{-1}(\underline{0}) \underline{M}_L \dot{\theta}^2 (\underline{\phi}_e \underline{\phi}_e^T) \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_0 + \\ & \underline{h}_{11}(\underline{0}) u_1) + O(\mu^2). \end{aligned} \quad (27)$$

The controls u_0 and u_1 can be designed and so forth. This process can be continued up to any order in μ . In the following it is assumed that the first order correction term is sufficient to account for the flexibility in the reduced order system (27).

The zero order control term can be chosen as the linearizing control

$$u_0 = m_{11}(\underline{0}) v \quad (28)$$

where v is a new input to the system.

As far as the first order control term is concerned, it turns out that, if only one mode is used to approximate the deflection ($m = 1$ in (1)), it is possible to design u_1 so as to obtain $\underline{h}_1 = 0$, i.e.

$$u_1 = \frac{1}{\bar{\underline{h}}_{12}(\underline{0})} (\Delta_0 \ddot{\underline{h}}_0 - \underline{M}_L \dot{\theta}^2 \underline{\phi}_e^T \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_0) \quad (29)$$

Extending this technique to greater order terms leads to a very interesting result: the integral manifold Σ_{μ}^h can be forced to the slow manifold Σ_0 and the reduced order system behaves by design as the rigid system.

In practice, however, more than one mode may be required to approximate the deflection. In that case $\bar{\underline{h}}_{12}(\underline{0})$ is not invertible and the above technique is not applicable anymore. This is not surprising since the flexible link arm is naturally a distributed parameter system which can never be completely "stiffened" by one control actuator co-located with joint location [3]. By examining eq. (27), however, a different strategy can be adopted. The first order control term, indeed, can be chosen as

$$u_1 = \frac{1}{\bar{\underline{h}}_{11}(\underline{0})} \underline{h}_{12}^T(\underline{0}) \bar{\underline{H}}_{22}^{-1}(\underline{0}) (\Delta_0 \ddot{\underline{h}}_0 - \underline{M}_L \dot{\theta}^2 (\underline{\phi}_e \underline{\phi}_e^T) \bar{\underline{H}}_{22}(\underline{0}) \underline{h}_0) \quad (30)$$

which cancels the term in μ . With the controls u_0 in (28) and u_1 in (30) the first order reduced order system results then

$$\ddot{\theta} = v + O(\mu^2) \quad (31)$$

which represents the overall system linearized up to order μ^2 for trajectories in the neighborhood of Σ_{μ}^h .

If a joint trajectory $\theta(t)$ is to be tracked, the new input v can be set as (inverse model technique)

$$v = \ddot{\theta} + k_v (\dot{\theta} - \dot{\theta}) + k_p (\theta - \theta) \quad (32)$$

where k_p and k_v are position and velocity gains.

In case the fast dynamics is not stable, or eventually is only lightly damped, an additional fast control term must be added to the control u given by (23), adopting a composite control strategy [7,4]. In this way solutions outside the integral manifold may

way solutions outside the integral manifold may "rapidly" flow along the fast manifold (parametrized by the slow variables) to the integral manifold which becomes an attractive set. This is a separate design issue and is beyond the purpose of this paper.

CONCLUDING REMARKS

In this paper the concept of an integral manifold has been adopted with the purpose of obtaining a more accurate reduced order model for a one link flexible arm. The effects of the flexibility along the structure have been incorporated in the reduced order model up to the first order. This issue is very important since it has been shown how a feedback linearizing control can be synthesized for the reduced order model, almost in the same way as it is done for a rigid arm. One crucial point is that using the control strategy proposed in (22)-(32) requires the measurements of the joint angle, velocity, acceleration and jerk (see (25), (28) and (32)). As a matter of fact one has position encoders and tachometers; acceleration and jerk thus need to be reconstructed and this may cause stability problems. Furthermore the fast dynamics is required to be asymptotically stable otherwise an additional fast control term must be added to the control (23). An alternative strategy may be based on a combination of 'active' modal feedback control and 'passive' damping so as to increase the structural damping [9]. All those topics will constitute the subject of future research.

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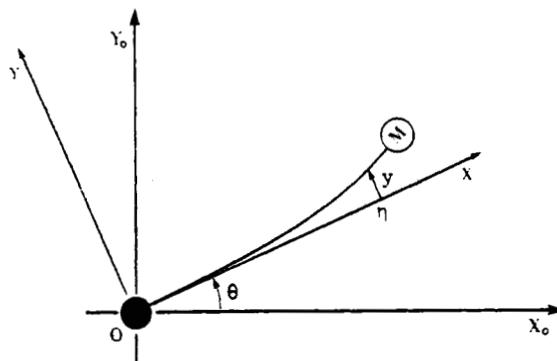


Fig. 1. The one link flexible arm