AN ALGORITHM TO COMPUTE THE REACHABLE WORKSPACE FOR 2R AND 3R PLANAR PAIR MECHANICAL ARMS

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INTRODUCTION

The determination of the workspaces of a manipulator provides a useful tool for computer aided programming of mechanical arms. Two different workspaces are commonly considered [1]: the reachable workspace is that volume of space which the end effector can reach, whereas the dexterous workspace is that subset of the reachable workspace within which the end effector can assume any given orientation.

The goal of this paper is to present a novel algorithmic approach to the reachable workspace determination. The present work is restricted to special geometries which have a 2R or a 3R planar pair, but the resulting algorithm appears to be applicable to any manipulator geometry. The supporting idea is the same as in [1], although it is logically derived from a recently established inverse kinematic solution algorithm [2-3]. Computation results are presented for a 2R and a 3R redundant planar pair.

THE INVERSE KINEMATIC SOLUTION ALGORITHM AND ITS USE IN WORKSPACE DETERMINATION

The configuration $\underline{x} \in R^m$ of the manipulator's end effector in the task space is described as a function of the joint variables $\underline{q} \in R^n$ (n DOF's) by the following nonlinear equation [4]:

$$\underline{x} = \underline{f}(\underline{q}). \tag{1}$$

The relationship between the joint velocities $\underline{q}$ and the end effector velocities $\underline{v}$ is described by the following linear equation with the Jacobian matrix $J(\underline{q}) := \partial \underline{f}/\partial \underline{q}$ as the joint-dependent coefficient matrix [4]:

$$\underline{v} = J(\underline{q})\dot{\underline{q}}. \tag{2}$$

There are two classical approaches in the literature to solving the inverse kinematics. One tries to solve equation (1), the other attempts to solve equation (2) [4].

A different approach has been recently introduced [2,3] which leads to a general solution algorithm that requires only the computation of direct kinematic functions. Consider the block diagram of the algorithm in Fig. 1. Given an $\hat{\underline{x}}$, the update law

$$\dot{\underline{q}} = KJ^T\underline{e} \tag{3}$$

assures that:

a) if $\mathrm{rank}(J^T) = m$ then $\underline{x} \to \hat{\underline{x}}$, i.e. $\dot{\underline{q}} \to \underline{0}$ and $\underline{e} \to \underline{0}$;

b) if $\mathrm{rank}(J^T) < m$, defining $\hat{\underline{x}} = \hat{\underline{x}}_n + \hat{\underline{x}}_o$ with $\hat{\underline{x}}_n \in \mathcal{N}(J^T)$ and $\hat{\underline{x}}_o \in (\mathcal{N}(J^T))^{\perp}$, then $\underline{x} \to \hat{\underline{x}}_o$, i.e. $\dot{\underline{q}} \to \underline{0}$ and $\underline{e} \to \hat{\underline{x}}_n$.

Convergence can be shown by using the Lyapunov direct method.

Notice that a solution of the kind (3) recalls the fundamental relationship between the joint torques $\underline{\tau} \in R^n$ and the task forces $\underline{\gamma} \in R^m$ applied to the end effector 4 , i.e.

$$\underline{\tau} = J^T(\underline{q})\underline{\gamma}. \tag{4}$$

The reachable workspace of a manipulator can be defined as the geometrical locus of the end effector position $\underline{p}$, when the joint variables range between the two extreme values [5]:

$$\underline{p} = \underline{p}(\underline{q}) \qquad q_{imin} \le q_i \le q_{imax}. \tag{5}$$

The volume is finite, closed, connected ($\underline{p}(\underline{q})$ is continuous) and, therefore, entirely defined by its boundary surface. This surface is made up of planar, spherical, toroidal and cylindrical surface elements. On this boundary surface a loss of mobility occurs, that is forces applied to the end effector along suitable directions are completely neutralized by the mechanical structure of the manipulator. Eq. (4) implies that a rank deficiency in the Jacobian occurs. This can be caused by any of the following situations:

a) one or more null column vectors (the corresponding joints are blocked at either of the two limits, q_{imin} or q_{imax}: $\partial \underline{p}/\partial q_i = \underline{0}$),

b) colinearity of the column vectors (the manipulator is in a singular configuration, usually two or more axes are aligned: $\partial \underline{p}/\partial q_i \times \partial \underline{p}/\partial q_j = \underline{0}$ for $i \neq j$).

All the possible combinations of two or more of the above situations lead to determine not only the so-called boundary singularities, i.e. the boundaries of the reachable workspace [4], but also the so-called internal singularities which delimit regions in the workspace characterized by different aspects of the manipulator [5].

Based upon the above premises, the above task can be accomplished by applying the inverse kinematic algorithm of Fig. 1 [3]; the sole requirement is to choose a reference trajectory $\hat{\underline{p}}$ which is unfeasible for the manipulator. In particular, only the section of the reachable workspace with the plane (r,z) of the planar pair will be derived in the present work, where r is determined through the first joint angle.

TRACING OF THE WORKSPACE FOR PLANAR PAIRS

The 2R planar pair of the manipulator of Fig. 2 is analyzed first ($a_4 = 0$, $q_4 = 0$). The trajectory $\hat{\underline{p}}(t)$ is chosen as a circle of radius $\rho > a_2 + a_3$, being a_2 and a_3 the lengths of the two links. If there are no mechanical joint limits, a complete circle will obviously force the tip to trace the circle of radius $a_2 + a_3$, plus a hole of radius $|a_2 - a_3|$. In every practical design, however, joint limits do exist. Therefore, in order to start the algorithm, assume to locate the pair at the configuration $q_2 = q_{2min}$, $q_3 = q_{3min}$ (point A of Fig. 3a). The corresponding $\hat{\underline{p}}(t=0)$ is chosen at an angle $\alpha(t=0) =$

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atan2($p_x(t=0)$,$p_y(t=0)$) with the r axis.

The logical sequence to account for the joint limits (null column vectors in J) is as follows. The stop on q_3 is released ($\partial \underline{p}/\partial q_2 = \underline{0}$); the tip of the pair $\underline{p}(t)$ draws the arc AB, as (t) increases. At point B, in fact, the limit q_{3max} is encountered. Then the stop on q_2 is released ($\partial \underline{p}/\partial q_3 = \underline{0}$); $\underline{p}(t)$ traces the arc BC, since at C the limit q_{2max} is reached. Then the stop on q_3 is released again ($\partial \underline{p}/\partial q_2 = \underline{0}$) with $\alpha(t)$ decreasing now; the corresponding arc traced is CD. The stop on q_2 is finally released again ($\partial \underline{p}/\partial q_3 = \underline{0}$), tracing the arc DA.

It is necessary now to account for the colinearity between the two columns of J. Restarting from point A, both joint stops are released; the tip of the pair will initially retrace AB, up to that point E, corresponding to $q_2 = q_{2min}$, $q_3 = 0$ ($\partial \underline{p}/\partial q_2 \times \partial \underline{p}/\partial q_3 = \underline{0}$). From there on it will trace a new arc, up to encounter the point F, corresponding to $q_2 = q_{2max}$, $q_3 = 0$. Finally, the required reachable workspace is completely determined by the area limited by the arcs AEFCGDA, where G is the intersection between arcs BC and CD. The time required to draw the workspace on an IBM PC/AT is about 10 sec.

Another facility implemented is the determination of dexterous regions inside the reachable workspace, where the pair can assume different aspects. More specifically, once a point is chosen inside the reachable workspace, its coordinates are input to the inverse kinematic algorithm of Fig. 1. Its output will be the coordinates of the joint variables which place the tip in that point. Since the solution generated from the algorithm moves the pair with continuity starting from the initial configuration, it is useful to give different initial configurations, for instance with the joints at their respective limits. In this way the different aspects are determined.

Fig. 3a shows the tracing of the workspace. It can be recognized that the area limited by AEFDA corresponds to the "elbow-down" aspect attainable by the pair, whereas the area limited by BCFEB characterizes the "elbow-up" aspect of the pair. Therefore, in the common area EBGFE the pair can assume both aspects, up and down.

The 3R planar pair of the manipulator of Fig. 2 is analyzed now. The determination of the reachable workspace is conceptually similar to the previous case. The trajectory $\hat{\underline{p}}(t)$ is chosen as a circle of radius $\rho > a_2 + a_3 + a_4$, being a_4 the length of the outer link. Assume to locate the pair at the configuration $q_i = q_{imin}$, i = 2,3,4 and let $\alpha(0)$ = atan2($p_x(0)$,$p_y(0)$). The sequence to account for the joint limits ($\partial \underline{p}/\partial q_i = \underline{0}$, i = 2,3,4) is the following (Fig. 3b):

- from q_{4min} (α increasing) to q_{4max}: the arc AB is traced,
- from q_{3min} (") to q_{3max}: " BC " ,
- from q_{2min} (") to q_{2max}: " CD " .

Then:

- from q_{4max} (α decreasing) to q_{4min}: the arc DE is traced,
- from q_{3max} (") to q_{3min}: " EF " ,
- from q_{2max} (") to q_{2min}: " FA " .

Then the sequence is modified as:

- from q_{3min} (α increasing) to q_{3max}: the arc AG is traced,
- from q_{4min} (") to q_{4max}: " GC " ,
- from q_{2min} (") to q_{2max}: " CD " .

Conversely:

- from q_{3max} (α decreasing) to q_{3min}: the arc DH is traced,
- from q_{4max} (") to q_{4min}: " HF " ,
- from q_{2max} (") to q_{2min}: " FA " .

The above sequence is seen to be sufficient to trace the internal arcs of the workspace relative to all the combinations of single null column vectors. Next, in order to get the external arcs of the reachable workspace, the pair must be taken to its maximum extension. This can be done as follows:

- from q_{4min} (α increasing) to $q_4 = 0$: the arc AK is traced,
- from q_{3min} (") to $q_3 = 0$: " KL " ,
- from q_{2min} (") to q_{2max}: " LM " .

Notice that $q_4 = 0$ and $q_3 = 0$ are automatically held by the algorithm during the last two steps, in virtue of the property discussed in the previous section. The external tracing of the workspace is finally completed by taking the following steps:

- from $q_3 = 0$ (α increasing) to q_{3max}: the arc MN is traced,
- from $q_4 = 0$ (") to q_{4max}: " ND " .

Compared to the 2R workspace, it can be recognized that the different regions characterize now families of aspects, rather than single aspects. This is obviously due to the redundancy of the pair. The total computation time is about 20 sec.

The extension to general geometries appears quite straightforward and it will constitute the subject of future investigation.

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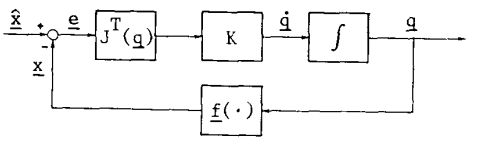

Fig. 1. The inverse kinematics scheme.

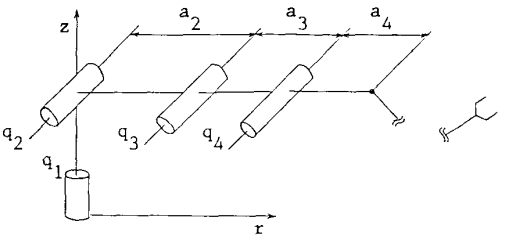

Fig. 2. A manipulator having a 3R planar pair.

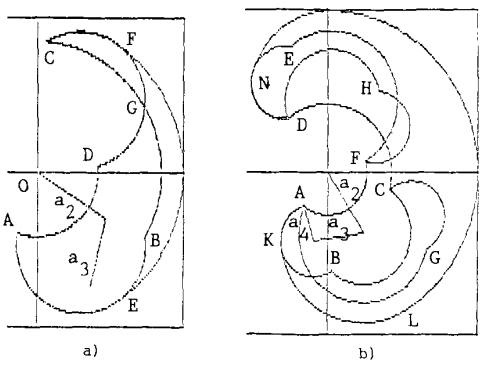

Fig. 3. The reachable workspace: a) 2R p.p., b) 3R p.p.