On the use of Cayley transform for kinematic shape reconstruction of soft continuum robots

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Abstract. A Cayley map for the special Euclidean group $SE(3)$ is introduced to relate, for a soft continuum robot, the Lie algebra of internal deformations with the Lie group of rigid–body motions. This Cayley map is used for approximated and exact kinematic shape reconstruction of soft continuum robots, under the hypothesis of constant deformations. This map could be used for deriving computationally efficient interpolation schemes for soft robots, since it does not involve transcendental functions as those introduced by the exponential parametrization of soft robot kinematics.

Keywords: differential geometry, Cayley transform, kinematics, soft robotics

1 Introduction

A soft continuum robot is a continuously deformable robot with an internal elastic structure [1]. Over the recent years, several geometrically exact models which do not involve approximation on kinematic variables have been proposed for describing their mechanics [2–9]. Despite their proven accuracy, they are still computationally inefficient for real–time simulation or control; indeed, recent model–based controllers for soft continuum robots still use approximated models [10, 11].

In this work, we propose to use the Cayley map on $SE(3)$ for kinematic parametrization and shape reconstruction of soft robots. Previous models have used the exponential map on $SE(3)$ to provide exact reconstruction of the kinematics of soft robots under the hypothesis of constant deformations [5–9]. The Cayley map is appealing from the computational point of view since it offers rational non–linearities, as opposed to the trascendental non–linearities of the exponential map.

The Cayley map for the group of rigid–body motions $SE(3)$ has been introduced in [12]. In the references [5–9] the authors have shown that rigid–body

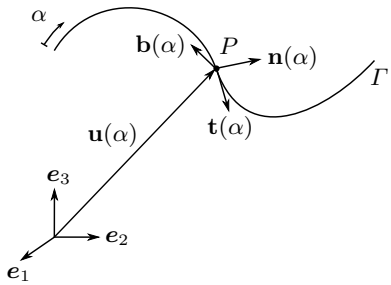

Fig. 1. Geometric description of the reference curve of a soft continuum arm.

transformations govern the three–dimensional motion of a soft arm through its internal deformations. This means that rigid–body motions still play a relevant role even for soft robots. As the Cayley map can be used for parametrization of motions [13, 14], here we will show that the Cayley map can be either used for kinematic reconstruction of low deformation robotic arms or for exact kinematic reconstruction of robots with generic internal deformations. In the literature, there is no evidence of using the Cayley map for local parametrization of soft robot kinematics.

2 Cayley transform

The Cayley transform is a function which relates a Lie group with its Lie algebra, as the exponential map. It is formally defined as

$$\begin{aligned} \mathrm{cay}(\widetilde{\boldsymbol{a}}) &= (\boldsymbol{I} + \widetilde{\boldsymbol{a}})(\boldsymbol{I} - \widetilde{\boldsymbol{a}})^{-1} = \\ &= (\boldsymbol{I} - \widetilde{\boldsymbol{a}})^{-1}(\boldsymbol{I} + \widetilde{\boldsymbol{a}}) = \boldsymbol{A} \end{aligned} \tag{1}$$

where $\boldsymbol{A}$ denotes an element belonging to a Lie group G and $\widetilde{\boldsymbol{a}} \in \mathfrak{g}$ is its corresponding Lie algebra, while $\boldsymbol{I}$ is the identity matrix.

3 Kinematics of soft robots using the Cayley transform

Let a curve Γ parametrized by the material abscissa $\alpha \in [0, L]$ be the reference curve of a continuum arm (see, e.g. Fig. 1). Let $\boldsymbol{u}(\alpha) \in \mathbb{R}^3$ be the position vector of a point P belonging to Γ. The unit tangent vector $\boldsymbol{t}(\alpha) \in \mathbb{R}^3$, the unit normal vector $\boldsymbol{n}(\alpha) \in \mathbb{R}^3$ and the unit binormal vector $\boldsymbol{b}(\alpha) \in \mathbb{R}^3$ constitute a local triad on the reference curve. Therefore, a rotation matrix $\boldsymbol{R} \in SO(3)$ can be introduced as $\boldsymbol{R}(\alpha) = [\boldsymbol{t}(\alpha)\ \ \boldsymbol{n}(\alpha)\ \ \boldsymbol{b}(\alpha)]$ for representation of its orientation. The configuration (i.e., position and orientation) along the curve can be described by using the 4×4 homogeneous matrix $\boldsymbol{H} \in SE(3)$ as

$$\boldsymbol{H}(\alpha) = \mathcal{H}(\boldsymbol{R}(\alpha), \boldsymbol{u}(\alpha)) = \begin{bmatrix} \boldsymbol{R}(\alpha) & \boldsymbol{u}(\alpha) \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix} \tag{2}$$

The homogeneous matrix in (2) evolves along the material abscissa α according to the differential kinematic relationship

$$\boldsymbol{H}'(\alpha) = \boldsymbol{H}(\alpha)\widetilde{\boldsymbol{f}}(\alpha) \tag{3}$$

where $(\cdot)'$ denotes the derivative with respect to α. The invariant vector field $\widetilde{\boldsymbol{f}}(\alpha) \in \mathfrak{se}(3)$ is called deformation twist [7, 9], and it is given by

$$\widetilde{\boldsymbol{f}}(\alpha) = \begin{bmatrix} \widetilde{\boldsymbol{f}}_\omega(\alpha) & \boldsymbol{f}_u(\alpha) \\ \boldsymbol{0}_{1\times 3} & 0 \end{bmatrix} \tag{4}$$

where $\widetilde{\boldsymbol{f}}_\omega \in \mathfrak{so}(3)$ is a 3×3 skew–symmetric matrix of angular deformations (torsion and bending deformations) and $\boldsymbol{f}_u \in \mathbb{R}^3$ is the 3×1 vector of linear deformations (axial and shear deformations). The deformation twist is a Lie algebra element belonging to $\mathfrak{se}(3)$; by using the isomorphism $\mathfrak{se}(3) \simeq \mathbb{R}^6$, we associate an axial vector $\boldsymbol{f} \in \mathbb{R}^6$ to $\widetilde{\boldsymbol{f}} \in \mathfrak{se}(3)$, and we refer to it as the six–dimensional deformation vector.

When $\boldsymbol{f}$ does not depend on α (i.e., in case of constant deformations), the differential kinematics in (3) can be integrated exactly by applying the exponential map operator on $SE(3)$ to the deformation twist $\widetilde{\boldsymbol{f}}$ [7, 9]. As a matter of fact, the exponential map introduces a local parametrization that allows to describe the behavior of elements belonging to a nonlinear space, namely the Lie group $SE(3)$, with an element belonging to a linear space, the Lie algebra $\mathfrak{se}(3)$.

According to (1), also the Cayley map can be used to relate a Lie group to its Lie algebra. Therefore, in the following we will see how to use the Cayley map for deformation–based kinematic reconstruction of soft robotic arms.

3.1 Deformation–based kinematic reconstruction using the Cayley map on $SE(3)$

Differently from the exponential map operator, the Cayley map depends on the representation of $SE(3)$, i.e. there exist different Cayley maps for the 4×4 representation and the 6×6 adjoint representation of $SE(3)$ [12]. In this work, we use the Cayley map for the 4×4 representation of $SE(3)$.

By applying $\widetilde{\boldsymbol{f}}$ in (4) to (1) for $\alpha \in [0, L]$, and by introducing a scalar value κ (which can have only two values, $\kappa = 1$ and $\kappa = 1/2$), we obtain the Cayley transform on $SE(3)$ which maps the deformations to rigid–body motions as

$$\mathrm{cay}_{SE(3)}(\alpha\widetilde{\boldsymbol{f}}) = (\boldsymbol{I}_{4\times 4} - \frac{1}{2\kappa}\alpha\widetilde{\boldsymbol{f}})^{-1}(\boldsymbol{I}_{4\times 4} + \frac{1}{2\kappa}\alpha\widetilde{\boldsymbol{f}}) = \boldsymbol{M}(\alpha) \in SE(3) \tag{5}$$

where $\boldsymbol{I}_{4\times 4}$ is the 4×4 identity matrix. Notice that the formal definition of the Cayley map in (1) is recovered by using $\kappa = 1/2$. The reason for introducing the scalar value κ will be clear in Sec. 3.2. As the exponential map, the Cayley map on $SE(3)$ returns an element belonging to $SE(3)$; however, while the exponential map applied to the deformation twist returns exactly the element $\boldsymbol{H}(\alpha)$, the

Cayley map applied to the deformation twist returns a different element, namely $\boldsymbol{M}(\alpha)$. We can express (5) in a compact form as

$$\mathrm{cay}_{SE(3)}(\widetilde{\boldsymbol{f}}) = \begin{bmatrix} \mathrm{cay}_{SO(3)}(\widetilde{\boldsymbol{f}}_\omega) & \mathrm{dcay}_{SO(3)}(\widetilde{\boldsymbol{f}}_\omega)\boldsymbol{f}_u) \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix} \tag{6}$$

which has a shape similar to the exponential map on $SE(3)$ (see [7], Eq. 10). In (6), $\mathrm{cay}_{SO(3)}(\cdot)$ is the Cayley map on $SO(3)$, the group of rotation matrices: this map can be obtained by introducing $\widetilde{\boldsymbol{f}}_\omega$ in (1); by considering again the scalar κ as done in (5), we have

$$\mathrm{cay}_{SO(3)}(\alpha\widetilde{\boldsymbol{f}}_\omega) = (\boldsymbol{I}_{3\times 3} - \frac{1}{2\kappa}\alpha\widetilde{\boldsymbol{f}}_\omega)^{-1}(\boldsymbol{I}_{3\times 3} + \frac{1}{2\kappa}\alpha\widetilde{\boldsymbol{f}}_\omega) = \boldsymbol{B}(\alpha) \in SO(3) \tag{7}$$

where $\boldsymbol{I}_{3\times 3}$ is the 3×3 identity matrix. Again, $\boldsymbol{B}(\alpha)$ in (7) is different from $\boldsymbol{R}(\alpha)$, which is obtained by applying the exponential map on $SO(3)$ to $\widetilde{\boldsymbol{f}}_\omega$. The inverse of $(\boldsymbol{I}_{3\times 3} - \alpha/2\kappa\widetilde{\boldsymbol{f}}_\omega)$ in (7) can be expressed through the Neumann series

$$(\boldsymbol{I}_{3\times 3} - \frac{1}{2\kappa}\alpha\widetilde{\boldsymbol{f}}_\omega)^{-1} = \sum_{i=0}^{\infty} \frac{1}{2\kappa}\alpha\widetilde{\boldsymbol{f}}_\omega^{i} \tag{8}$$

Let us remember that all 3×3 skew–symmetric matrices satisfy the cubic equation

$$\widetilde{\boldsymbol{f}}_\omega^{3} = -||\boldsymbol{f}_\omega||^2\widetilde{\boldsymbol{f}}_\omega \tag{9}$$

where $||\boldsymbol{f}_\omega||$ is the norm of the vector $\boldsymbol{f}_\omega \in \mathbb{R}^3$ associated to the Lie algebra $\widetilde{\boldsymbol{f}}_\omega \in \mathfrak{so}(3)$. By developing (8) with the relationship (9), and by inserting the result in (7), the Cayley map on $SO(3)$ takes the form

$$\mathrm{cay}_{SO(3)}(\alpha\widetilde{\boldsymbol{f}}_\omega) = \boldsymbol{I}_{3\times 3} + \frac{4\kappa}{4\kappa^2 + ||\boldsymbol{f}_\omega||^2}(\alpha\widetilde{\boldsymbol{f}}_\omega + \alpha\widetilde{\boldsymbol{f}}_\omega^{2}) \tag{10}$$

Being quadratic in $\widetilde{\boldsymbol{f}}_\omega$, this expression can be seen as the Cayley counterpart of the Rodrigues' formula for the exponential map on $SO(3)$ (see [7], Eq. 11); with respect to the latter, (10) does not involve trascendental functions: this is a great advantage from the computational point of view. Furthermore, $\mathrm{dcay}_{SO(3)}(\cdot)$ in (6) also involves only rational non–linearities, as it is given by

$$\mathrm{dcay}_{SO(3)}(\alpha\widetilde{\boldsymbol{f}}_\omega) = \mathrm{cay}_{SO(3)}(\alpha\widetilde{\boldsymbol{f}}_\omega) + \boldsymbol{I}_{3\times 3} \tag{11}$$

While the exponential map operator on $SE(3)$ applied to the deformation twist returns the exact shape of the robot in case of constant deformations, the Cayley map on $SE(3)$ applied to the deformation twist returns a matrix $\boldsymbol{M}$ which is different from $\boldsymbol{H}$: the point is to understand when the Cayley map, which offers computational advantages, can be used in place of the exponential map.

3.2 Geometric interpretation

Let us consider a matrix $\boldsymbol{E}$, and let $\exp(\boldsymbol{E})$ denotes its matrix exponential as

$$\exp(\boldsymbol{E}) = \sum_{i=0}^{\infty} \frac{\boldsymbol{E}^i}{i!} \tag{12}$$

If $||\boldsymbol{E}||$ is not too large, a good approximation of (12) is given by the (1,1) diagonal Padè approximants [15] equals to

$$\mathrm{appr}_{(1,1)}(\boldsymbol{E}) = (\boldsymbol{I} - \frac{1}{2}\boldsymbol{E})^{-1}(\boldsymbol{I} + \frac{1}{2}\boldsymbol{E}) \tag{13}$$

By comparing (13) with (1), we observe that the Cayley transform is, except for a scalar coefficient, the (1,1) diagonal Padè approximation of the exponential map. In order to have an exact correspondence between the Cayley map and the diagonal Padè approximation of the exponential map, often the Cayley map in (1) is defined with the coefficient 1/2. This is the reason why we have introduced the scalar factor κ in (5) and (7). In the following, we use this nomenclature for the Cayley transform:

- $\kappa = 1/2$: standard Cayley transform
- $\kappa = 1$: modified Cayley transform (coincident with the Padè approximants)

Coming back to the kinematics of a soft continuum arm, we can state that the Cayley transforms can be used for approximating the position and orientation fields of the arm when $||\widetilde{\boldsymbol{f}}||$ is not too large, i.e. in case of low, yet finite deformations of the arm. Under this hypothesis, $\boldsymbol{M}$ in (5) gives a good approximation of $\boldsymbol{H}$ in (2), and the approximated solution better reflects the exact solution of the kinematics when the deformed configuration of the arm is closer to the reference configuration (as $\mathrm{cay}_{SE(3)}(\cdot) \to \exp_{SE(3)}(\cdot)$ when $||\cdot|| \to 0$).

In summary, the Cayley map can be seen as a second–order approximation of the exponential map. However, being related to the diagonal Padè approximants, this gives a better approximation of the function than truncating its Taylor series in (12) to the second order.

As the exponential map has been used for interpolation schemes of soft robots when the exact kinematics need to be reconstructed, the Cayley map could be used instead for interpolation schemes of soft robots in case of low, yet finite deformations. Since the Cayley map is rational, meaning that it does not involve trascendental functions, it could be advantageous in numerical applications as computing trascendental functions is more time consuming.

3.3 Relationship between Cayley and exponential discretization

Does it exist a twist matrix $\widetilde{\boldsymbol{\zeta}} \in \mathfrak{se}(3)$ such that its use as argument of the Cayley map in (6) returns the exact kinematics $\boldsymbol{H}(\alpha)$ in case of constant deformations? Such twist matrix $\widetilde{\boldsymbol{\zeta}}$ should verify the equality $\mathrm{cay}_{SE(3)}(\widetilde{\boldsymbol{\zeta}}) = \exp_{SE(3)}(\widetilde{\boldsymbol{f}})$.

Bauchau and Choi in [14] found the Cayley vector of parameters which produce the same rigid body motion of the exponential parametrization. We apply this results to deformation vectors as

$$\begin{bmatrix} \boldsymbol{\zeta}_u \\ \boldsymbol{\zeta}_\omega \end{bmatrix} = \begin{bmatrix} \lambda \boldsymbol{I}_{3\times3} & \tau\lambda' \boldsymbol{I}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \lambda \boldsymbol{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_\omega \end{bmatrix} \tag{14}$$

with

$$\begin{aligned} \lambda &= 2\kappa \frac{\tan(||\boldsymbol{f}_\omega||/2)}{||\boldsymbol{f}_\omega||} \\ \tau &= \boldsymbol{u}\boldsymbol{f}_\omega = \boldsymbol{T}^T_{SO(3)}(\boldsymbol{f}_\omega)\boldsymbol{f}_u\boldsymbol{f}_\omega \end{aligned} \tag{15}$$

where $\boldsymbol{T}_{SO(3)}(\cdot)$ is the tangent operator on $SO(3)$, obtained as the transpose of the derivative of exponential map on $SO(3)$ (see [7], Eq. 13).

Therefore, the matrix $\widetilde{\boldsymbol{\zeta}}$ can be interpreted as a pseudo–deformation twist, whose application to the Cayley map (6) returns $\boldsymbol{H}(\alpha)$, i.e. the exact integration of the kinematics is achieved.

From (14) it is also clear that, for small values of $||\boldsymbol{f}||$, $||\boldsymbol{\zeta}|| \to \kappa||\boldsymbol{f}||$. When the modified Cayley transform is used ($\kappa = 1$), the pseudo–deformation twist approaches the deformation twist. This is another reason why the modified Cayley transform would be preferable over the standard Cayley transform.

Equation (14) can be regarded as a change of coordinates between physics–based coordinates and pseudo–physics coordinates. It can be useful for practical implementation of soft robot models. Indeed, the recently proposed deformation–based approaches can be used to reconstruct the shape of a soft continuum arm from the internal (and measurable) deformations. The use of the exponential mapping here is a natural way for recovering the kinematics, even if it involves the computation of multiple trascendental functions. An alternative approach would be as follows: (i) measuring the internal deformations $\boldsymbol{f}$; (ii) computing the pseudo–vector of deformations $\boldsymbol{\zeta}$ using (14); (iii) reconstructing the kinematics using (6). This approach involves an additional step, but it uses trascendental functions only in the change of coordinates.

4 Examples

In this section we consider two illustrative examples of soft continuum arms taken from [7]. For each example, we reconstruct the kinematic shape using the modified Cayley transform applied to the deformation twist and we compare the kinematic reconstruction with respect to the one obtained using the exponential mapping. The output of each example is an error e which defines the position distance between the two reconstructions as

$$e(\alpha) = \sqrt{\sum_{i=1}^{3} (u_{i,\text{cay}}(\alpha) - u_{i,\text{exp}}(\alpha))^2} \tag{16}$$

Obviously, for both the examples, computing the pseudo–deformation twist using (14) and applying it to (6) returns the exponential reconstruction.

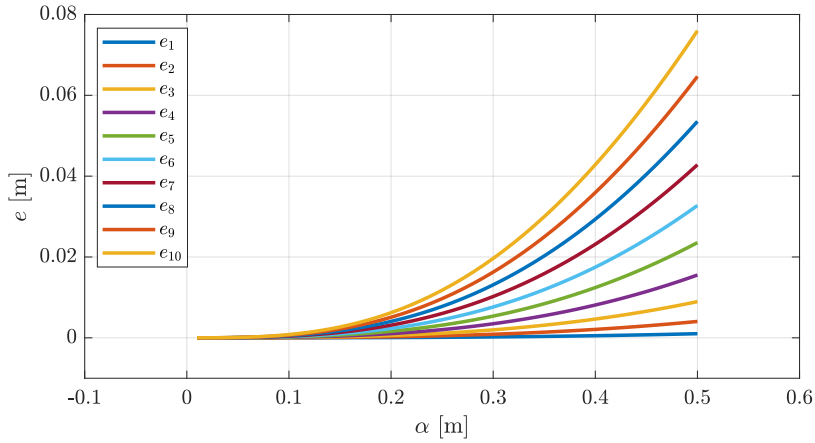

Fig. 2. Errors between the Cayley and the exponential shape reconstructions of soft continuum arms with constant curvature. 1: $K = \pi/10\,\mathrm{m}^{-1}$;...; 10: $K = \pi\,\mathrm{m}^{-1}$.

4.1 Planar bending motion

The planar bending motion is achieved by a continuum arm with constant curvature, i.e. a mechanical arm which has internal deformation equals to $\boldsymbol{f} = [1\ 0\ 0\ 0\ 0\ K]^T$, being K the degree of curvature. We consider a mechanical arm of length $L = 0.5\,\mathrm{m}$ and varying curvature $K = \pi/10 : \pi/10 : \pi\ \mathrm{m}^{-1}$. The error plot is given in Fig. 2. The maximum distance between the two models occurs, as expected, for the maximum curvature considered ($K = \pi\,\mathrm{m}^{-1}$); the maximum error occurs at the tip of the arm, and it is approximately equal to $e = 0.08\,\mathrm{m}$, which is about 16% of the total length of the robot. However, till the sixth value of curvature, the maximum error at the tip is less than 10%, if compared to the total length of the robot. We can consider this value as a barrier threshold beyond which the Cayley discretization using deformation twists would not be acceptable.

4.2 Spatial helical motion

A continuum arm subject to constant curvature and torsion produces spatial helical motions. In this case, the internal deformation vector is equal to $\boldsymbol{f} = [1\ 0\ 0\ T\ 0\ K]^T$, being T the degree of torsion and K the degree of curvature as before. We consider again a mechanical arm of length $L = 0.5\,\mathrm{m}$. The deformation parameters are: fixed torsion $T = 3\,\mathrm{m}^{-1}$ and varying curvature $K = 0 : \pi/2 : 2\pi$ m^{-1}. The error plot is given in Fig. 3. Again, the errors increase as the norms of the deformation vectors increase, and higher errors happen at the tip of the robot. Compared to before, having here spatial motions, we observe greater errors. In this example, only the first two simulations are under the barrier threshold of tip error below 10% of the total length of the robot.

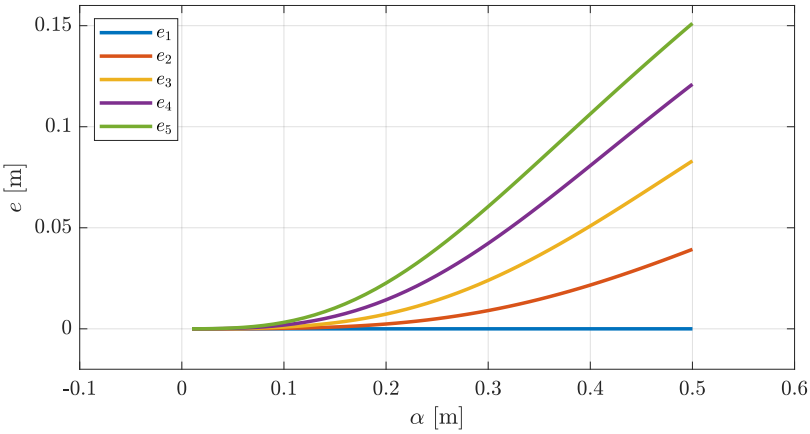

Fig. 3. Errors between the Cayley and the exponential shape reconstructions of soft continuum arms with constant curvature and torsion. 1: $T = 3\,\mathrm{m}^{-1}$ and $K = 0$;...; 5: $T = 3\,\mathrm{m}^{-1}$ and $K = 2\pi\,\mathrm{m}^{-1}$.

5 Conclusions

In this paper we have presented an initial idea of using the Cayley transform for parametrization of soft robot kinematics. We have shown that the Cayley map can be seen as a second–order approximation of the exponential map, and it can result useful when dealing with low deformations robots. Neverthless, an appropriate change of deformation coordinates allows to use the Cayley map for exact reconstruction of soft robot shapes. Thus, the Cayley map could be potentially used also as an alternative to the exponential map to recover the exact kinematics, with the advantage of reducing the use of trascendental functions.

The kinematic model of compound continuum robots will arise naturally from this modeling framework, through the product of n composite Cayley transforms as $\mathrm{cay}_{SE(3)}(\widetilde{\boldsymbol{\zeta}}) = \mathrm{cay}_{SE(3)}(\widetilde{\boldsymbol{\zeta}}_1)\cdots\cdots\mathrm{cay}_{SE(3)}(\widetilde{\boldsymbol{\zeta}}_n)$. Indeed, we know that successive rigid–body transformations can be described by successive exponentials of twists (i.e. Product of exponential formula) [16]. In the same way, since the Cayley map is a second–order approximation of the exponential map, Cayley transforms of twists can be used to describe successive rigid–body transformations, but with low, yet finite variations. There is no evidence of a "Product of Cayley" formula for rigid robotics as the joint twists usually undergone large motions, and thus the exponential parametrization is required. For soft robotics, indeed, we consider the rigid body transformations of the cross–sections: if the deformations are low, the cross sections do not undergone large motions, and thus, the kinematics can be recovered with the Cayley map. Hence, this map can be used as shape function of efficient finite element methods for soft robots. Furthermore, the Cayley map could also offer computational advantages for time integration of soft robot dynamics.

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