

An Inverse Kinematics Scheme for Flexible Manipulators

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Abstract—The inverse kinematics problem for manipulators having lightweight flexible links is considered in this work. This consists in finding the joint and deflection variables for a given tip position. The solution algorithm is based on the well-known Closed-Loop Inverse Kinematics (CLIK) scheme using the Jacobian transpose developed for rigid manipulators. The Jacobian to be used in the algorithm is obtained by correcting the equivalent rigid manipulator Jacobian with a term accounting for the steady-state deflections under arm gravity and tip load. The scheme is tested in two case studies on a planar two-link arm.

1. Introduction

The deflections induced by the lightweight nature of flexible link manipulators represent a serious problem for modelling, trajectory planning and control of such mechanical systems. The main issue to tackle is that the configuration of the system is described by an augmented number of variables, compared to the case of rigid manipulators where the joint variables suffice to describe the system configuration.

Modelling of flexible link manipulators has to be considered an assessed topic. Kinematics is derived by using suitable frame transformation matrices describing both rigid and flexible displacements [1]. Dynamics can be derived by resorting to different approaches that basically allow obtaining finite-dimensional models that approximate an inherent infinite-dimensional system, e.g. [2,3,4].

Control of flexible link vibrations has been a challenging research topic in the latest decade. The tracking problem can be successfully solved when the controlled variables are the joints, e.g. [5,6,7]. The problem becomes considerably tougher when it is desired to track a non-collocated tip trajectory with collocated joint actuators: if inversion-based controller are adopted then instability may arise along the motion [8,9].

On the other hand the regulation problem can be solved at both joint and tip level. Any desired joint configuration can be asymptotically reached under a joint PD + constant gravity compensation control [10]. If

the desired configuration is specified for the tip, an inverse kinematics problem arises: namely, find the joint and deflection variables that place the tip at the desired posture. When gravitational effects are present due to arm weight and eventually to a tip load, it is expected that the joint variables differ from the configuration of the equivalent rigid arm and should compensate for the steady-state deflections due to gravity.

The inverse kinematics problem can be formulated in differential terms by deriving a suitable Jacobian that relates the joint and deflection rates to the tip rate. Previous approaches [11,12] are based on the Jacobian inverse and utilize the Newton-Raphson method to find numerical solutions in an iterative fashion.

The present work proposes an inverse kinematics solution algorithm based on the well-known Closed-Loop Inverse Kinematics (CLIK) scheme previously developed for rigid manipulators [13,14]. Differently from above, the transpose of the Jacobian is required that naturally allows handling of singularities and redundancies [15]. Also the closed-loop fashion of the method ensures convergence of the solution, as can be proved via a simple Lyapunov argument. Further, if a slowly-varying trajectory is imposed on the tip, instead of a constant set point, the resulting joint and deflection trajectories can be used as the reference inputs to some control scheme for quasi-static motion of the arm tip.

A planar two-link flexible arm under gravity is considered to develop two case studies. The numerical results both with a constant and a time-varying tip position confirm the good performance of the scheme anticipated in theory.

2. Modelling

Without loss of generality we restrict our attention to planar n -link flexible arms with revolute joints subject only to bending deformations in the plane of motion, i.e. torsional effects are neglected. A sketch of a two-link arm is shown in Figure 1 with coordinate frame assignment. The rigid motion is described by the joint angles θ_i , while $w_i(x_i)$ denotes the transversal deflection of link i at x_i , $0 \leq x_i \leq \ell_i$, being ℓ_i the link length.

Let $p_i^d(x_i) = [x_i \quad w_i(x_i)]^T$ be the position of a point along the deflected link i with respect to frame (X_i, Y_i)

and p_i be the position of the same point in the base frame. Also let $r_{i+1}^i = p_i^i(\ell_i)$ be the position of the origin of frame (X_{i+1}, Y_{i+1}) with respect to frame (X_i, Y_i) , and r_i its position in the base frame.

The joint (rigid) rotation matrix R_i and the rotation matrix E_i of the (flexible) link at the end point are, respectively,

$$R_i = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix}, \quad E_i = \begin{bmatrix} 1 & -w'_{ie} \\ w'_{ie} & 1 \end{bmatrix}, \quad (1)$$

where $w'_{ie} = (\partial w_i / \partial x_i)|_{x_i=\ell_i}$, and the small deflection approximation $\arctan w'_{ie} \approx w'_{ie}$ has been made. Hence the above absolute position vectors can be expressed as

$$p_i = r_i + W_i p_i^i, \quad r_{i+1} = r_i + W_i r_{i+1}^i, \quad (2)$$

where W_i is the global transformation matrix from the base frame to to (X_i, Y_i) given by the recursive equation

$$W_i = W_{i-1} E_{i-1} R_i = \widehat{W}_{i-1} R_i, \quad \widehat{W}_0 = I; \quad (3)$$

On the basis of the above relations, the kinematics of any point along the arm is completely specified as a function of joint and link deflection.

A finite-dimensional model (of order m_i) of link flexibility can be obtained by the assumed modes technique. Exploiting separability in time and space of solutions to the Euler-Bernoulli equation for flexible beams

$$(EI)_i \frac{\partial^4 w_i(x_i, t)}{\partial x_i^4} + \rho_i \frac{\partial^2 w_i(x_i, t)}{\partial t^2} = 0, \quad i = 1, \dots, n. \quad (4)$$

where ρ_i is the uniform density and $(EI)_i$ is the constant flexural rigidity of link i , the link deflection can be expressed as

$$w_i(x_i, t) = \sum_{j=1}^{m_i} \phi_{ij}(x_i) \delta_{ij}(t), \quad (5)$$

where $\delta_{ij}(t)$ are the time-varying variables associated with the assumed spatial mode shapes $\phi_{ij}(x_i)$ of link i . The mode shapes have to satisfy proper boundary conditions at the base (clamped) and at the end of each link (mass).

In view of (5), a direct kinematics equation can be derived expressing the position of the arm tip point as a function of the joint variable vector $\theta = [\theta_1 \dots \theta_n]^T$ and the deflection variable vector $\delta = [\delta_{11} \dots \delta_{1m_1} \dots \delta_{n1} \dots \delta_{nm_n}]^T$, i.e.

$$p = k(\theta, \delta). \quad (6)$$

For later use in the inverse kinematics scheme, also the differential kinematics is needed. The absolute linear velocity of an arm point is

$$\dot{p}_i = \dot{r}_i + \dot{W}_i p_i^i + W_i \dot{p}_i^i, \quad (7)$$

with $\dot{r}_{i+1}^i = \dot{p}_i^i(\ell_i)$. Since the links are assumed unextensible ($\dot{x}_i = 0$), then $\dot{p}_i^i(x_i) = [0 \quad \dot{w}_i(x_i)]^T$. The computation of (7) takes advantage of the recursions

$$\dot{W}_i = \widehat{W}_{i-1} \dot{R}_i + \widehat{W}_{i-1} \dot{R}_i, \quad \widehat{W}_i = \dot{W}_i E_i + W_i \dot{E}_i. \quad (8)$$

Also, note that

$$\dot{R}_i = S R_i \dot{\theta}_i, \quad \dot{E}_i = S \dot{w}'_{ie}, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (9)$$

In view of (5), it is not difficult to show that the tip velocity can be expressed as

$$\dot{p} = J_\theta(\theta, \delta) \dot{\theta} + J_\delta(\theta, \delta) \dot{\delta}. \quad (10)$$

The above kinematics description is at the basis of the dynamic modelling of the flexible arm using the Lagrange approach that requires computation of kinetic and potential energy [2]. In a static situation the deflections are seen to satisfy the equation [10]

$$g_\delta(\theta) + K \delta = 0 \quad (11)$$

where g_δ is the gravity vector in the flexible dynamic equations that is only a function of θ and K is the link stiffness matrix

$$K = \text{diag}(k_{11}, \dots, k_{1m_1}, \dots, k_{n1}, \dots, k_{nm_n}) \quad (12)$$

with

$$k_{ij} = \int_0^{\ell_i} (EI)_i \phi_{ij}^2(x_i) dx_i. \quad (13)$$

From (11) the deflection variables can be computed as

$$\delta = -K^{-1} g_\delta(\theta). \quad (14)$$

For later use in the inverse kinematics scheme, differentiating (13) with respect to time gives

$$\dot{\delta} = -K^{-1} J_g(\theta) \dot{\theta}. \quad (15)$$

where $J_g = dg/d\theta$. Plugging (15) into (10) yields

$$\dot{p} = J_p(\theta, \delta) \dot{\theta} \quad (16)$$

where

$$J_p = J_\theta - J_\delta K^{-1} J_g \quad (17)$$

is the overall Jacobian matrix relating joint velocity to tip velocity. Notice that the Jacobian in (17) is obtained by modifying the rigid-body Jacobian J_θ with a term that accounts for the deflections induced by gravity. The differential kinematics (16) is the basic model that is used below to derive an inverse kinematics solution scheme.

3. Inverse kinematics scheme

The inverse kinematics problem for a flexible manipulator can be formulated as follows: Given a desired constant tip position, find the corresponding joint variables and deflection variables that place the arm tip under gravity at the given position.

The attractive feature of the differential kinematics equation (16) is its formal analogy with the differential kinematics equation for a rigid manipulator. Therefore any Jacobian-based inverse kinematics scheme can be adopted in principle. In this respect, one of the most effective schemes remains the CLIK scheme [13,14] that reformulates the inverse kinematics problem in terms of the convergence of a suitable closed-loop dynamic system.

Let p_d denote the desired constant tip position. A tip error can be defined as

$$e_p = p_d - p \quad (18)$$

where p can be computed as in (6). Differentiating (18) with respect to time gives

$$\dot{e}_p = -J_p(\theta, \delta)\dot{\theta} \quad (19)$$

where (16),(17) have been used. Consider the Lyapunov function

$$V = \frac{1}{2} e_p^T K_p e_p > 0 \quad (20)$$

with K_p symmetric and positive definite. The time derivative of (20) along the trajectories of the system (19) is

$$\dot{V} = -e_p^T K_p J_p(\theta, \delta)\dot{\theta}. \quad (21)$$

The Jacobian transpose joint velocity law

$$\dot{\theta} = J_p^T(\theta, \delta) K_p e_p \quad (22)$$

gives

$$\dot{V} = -e_p^T K_p J_p(\theta, \delta) J_p^T(\theta, \delta) K_p e_p \leq 0 \quad (23)$$

Hence, as long as the vector $K_p e_p$ is outside the null space of J_p^T , the tip position error e_p asymptotically tends to zero, i.e. p tends to the desired position p_d . In practical implementation of the algorithm, a suitable choice of the matrix K_p can be made to avoid that the algorithm gets stuck with $e_p \neq 0$ and $\dot{\theta} = 0$.

It should be remarked that no inverse of the Jacobian is required by the above algorithm and thus the scheme works well also in the neighbourhood of singularities of the matrix J_p . Also, if the arm possesses redundant degrees of freedom ($n > 2$), the Jacobian transpose scheme is the same and no pseudo-inverse of the Jacobian is required as instead in resolved rate schemes [15].

Finally, if the desired tip position is time-varying, a similar Lyapunov argument can be worked out to show

that the tracking error can be made arbitrarily small by augmenting the feedback gains in the matrix K_p whereas at steady state asymptotic convergence is still obtained. In practical implementation of the algorithm, bounds exist on the largest values of the gains in K_p depending on the sampling time at which the algorithm is discretized [16].

4. Case studies

In order to test the proposed inverse kinematics scheme, a planar two-link flexible arm (Fig. 1) under gravity was considered: $\theta = [\theta_1 \ \theta_2]^T$.

The following parameters were set up for the links and a payload which is assumed to be placed at the arm tip.

$$\rho_1 = \rho_2 = 1.0 \text{ kg/m (link uniform density)}$$

$$\ell_1 = \ell_2 = 0.5 \text{ m (link length)}$$

$$d_1 = d_2 = 0.25 \text{ m (link center of mass)}$$

$$m_1 = m_2 = 0.5 \text{ m (link mass)}$$

$$m_{h1} = m_{h2} = 1 \text{ kg (hub mass)}$$

$$m_p = 0.1 \text{ kg (payload mass)}$$

$$(EI)_1 = (EI)_2 = 10 \text{ N m}^2 \text{ (flexural link rigidity).}$$

An expansion with two clamped-mass assumed modes was taken for each link: $\delta = [\delta_{11} \ \delta_{12} \ \delta_{21} \ \delta_{22}]^T$. The resulting natural frequencies of vibration are:

$$f_{11} = 1.40 \text{ Hz} \quad f_{12} = 5.10 \text{ Hz}$$

$$f_{21} = 5.21 \text{ Hz} \quad f_{22} = 32.46 \text{ Hz.}$$

The stiffness coefficients of the diagonal matrix K in (13) are:

$$k_{11} = 38.79 \text{ N} \quad k_{12} = 513.37 \text{ N}$$

$$k_{21} = 536.09 \text{ N} \quad k_{22} = 20792.09 \text{ N.}$$

The link end-point deflections and their spatial derivatives can be expressed as

$$\begin{aligned} w_{1e} &= \phi_{11,e} \delta_{11} + \phi_{12,e} \delta_{12} \\ w_{2e} &= \phi_{21,e} \delta_{21} + \phi_{22,e} \delta_{22} \\ w'_{1e} &= \phi'_{11,e} \delta_{11} + \phi'_{12,e} \delta_{12} \\ w'_{2e} &= \phi'_{21,e} \delta_{21} + \phi'_{22,e} \delta_{22} \end{aligned} \quad (24)$$

where the constants are:

$$\phi_{11,e} = 0.39 \quad \phi_{12,e} = 0.36$$

$$\phi'_{11,e} = 1.34 \quad \phi'_{12,e} = -1.38$$

$$\phi_{21,e} = 1.49 \quad \phi_{22,e} = -0.75$$

$$\phi'_{21,e} = 4.30 \quad \phi'_{22,e} = -15.49$$

The tip position is expressed by

$$p = R_1(\theta_1)(r_2^1(\delta_{11}, \delta_{12}) + E_1(\delta_{11}, \delta_{12})R_2(\theta_2)r_3^2(\delta_{21}, \delta_{22})) \quad (25)$$

where the position vectors and the rotation matrices can be computed as illustrated in Section 2.

The Jacobians as in (10) resulting from (25) are:

$$J_\theta = \left[\frac{dR_1}{d\theta_1}(r_2^1 + E_1 R_2 r_3^2) \quad R_1 \left(E_1 \frac{dR_2}{d\theta_2} r_3^2 \right) \right] \quad (26)$$

$$J_\delta = \left[R_1 \left(\frac{dr_2^1}{d\delta_{11}} + \frac{dE_1}{d\delta_{11}} R_2 r_3^2 \right) \quad R_1 \left(\frac{dr_2^1}{d\delta_{12}} + \frac{dE_1}{d\delta_{12}} R_2 r_3^2 \right) \right. \\ \left. R_1 E_1 R_2 \frac{dr_3^2}{d\delta_{21}} \quad R_1 E_1 R_2 \frac{dr_3^2}{d\delta_{22}} \right] \quad (27)$$

where the required derivatives are easy to compute.

The following coefficients are also needed for the gravity term

$$v_{ij} = \int_0^L \rho_i \phi_{ij}(x_i) dx_i \quad i, j = 1, 2. \quad (28)$$

With the above data, they take on the values:

$$v_{11} = 0.069 \quad v_{12} = 0.12 \\ v_{21} = 0.28 \quad v_{22} = 0.30.$$

The resulting gravity term is (standard abbreviations are used for sine and cosine):

$$g_\delta = [g_1 \quad g_2 \quad g_3 \quad g_4]^T \quad (29)$$

with

$$g_1 = g_{11}c_1 + g_{12}c_{12} \\ g_2 = g_{21}c_1 + g_{22}c_{12} \\ g_3 = g_{31}c_{12} \\ g_4 = g_{41}c_{12}, \quad (30)$$

where the constant coefficients are:

$$g_{11} = g_0((m_2 + m_{h2} + m_p)\phi_{11,e} + v_{11}) \\ g_{12} = g_0(m_2 d_2 + m_p \ell_2)\phi'_{11,e} \\ g_{21} = g_0((m_2 + m_{h2} + m_p)\phi_{12,e} + v_{12}) \\ g_{22} = g_0(m_2 d_2 + m_p \ell_2)\phi'_{12,e} \\ g_{31} = g_0(m_p \phi_{21,e} + v_{21}) \\ g_{41} = g_0(m_p \phi_{22,e} + v_{22}), \quad (31)$$

being g_0 the gravity acceleration. It is worth noticing that g_δ is only a function of θ , as anticipated.

The Jacobian as in (15) resulting from (29) is

$$J_g = \begin{bmatrix} -g_{11}s_1 - g_{12}s_{12} & -g_{12}s_{12} \\ -g_{21}s_1 - g_{22}s_{12} & -g_{22}s_{12} \\ -g_{31}s_{12} & -g_{31}s_{12} \\ -g_{41}s_{12} & -g_{41}s_{12} \end{bmatrix} \quad (32)$$

With the expressions in (26),(27),(32) the overall Jacobian as in (17) can be computed.

In the first case study the arm was placed in the vertical equilibrium configuration

$$p = [0 \quad -1]^T \text{ [m]}.$$

The desired tip position was chosen as constant:

$$p_d = [1/\sqrt{2} \quad -1/\sqrt{2}]^T \text{ [m]}.$$

The feedback gain matrix in (22) was chosen as

$$K_p = \text{diag}(50, 50)$$

and the algorithm was discretized at a sampling time of 5 ms.

The resulting time history of joint angles, norm of tip error and link deflections are reported in Figure 2. It is easy to see that the desired position is reached in about 0.1 s. The final arm configuration is characterized by

$$\theta = [-46.66 \quad 25.28]^T \text{ [deg]}$$

$$\delta = [-0.1755 \quad -0.0048 \quad -0.0073 \quad -0.0001]^T \text{ [m]}$$

confirming the intuition that, because of gravity, the arm has to bend to properly reach the desired tip position. Actually the bending is much larger on the first link as it was natural to expect (the links have the same parameters).

It is important to point out that, for this particular case, the arm is initially placed in a configuration at which the Jacobian in (17) is singular. This does not affect the performance of the algorithm since the error e_p does have a component outside the null space of J_p . On the other hand an inverse Jacobian-based algorithm could have not been applied in this case.

In the second case study a 5th-degree polynomial trajectory is imposed from the initial to the final configuration with a duration of 1 s. The feedback gain matrices and algorithm sampling time are the same as above. The resulting time history of joint angles, norm of tip error and link deflections are reported in Figure 3. A tracking error arises in this case along the trajectory but the final position is reached soon after the end of the trajectory.

It is clear that larger gains can be tolerated if the sampling time is further reduced, leading to faster convergence in the first case study and to smaller tracking errors in the second case study, respectively.

5. Conclusions

An kinematics scheme has been proposed for flexible manipulators. The solution is based on the transpose of a suitably modified arm Jacobian so as to account

for the static effects of gravity. Convergence of the algorithm is ensured by its closed-loop fashion, as it has been formally proved via a Lyapunov argument.

The scheme is very useful for the regulation control problem. In fact, for a given constant tip position, the corresponding joint and deflection displacements can be computed, as shown in the first numerical case study. These can be used as the set points for a joint PD + constant gravity compensation, thus achieving a true tip regulation with a joint space controller.

The scheme has a potential also for the quasi-static motion tracking control problem. In fact, a slowly-varying tip trajectory can be assigned for the tip and the resulting joint and deflection variables, as shown in the second numerical case study. These could be used as the reference inputs to some dynamic model-based joint space controller.

The impact of the proposed inverse kinematics scheme on the performance of various control schemes that operate on joint (and deflection) reference values is currently under investigation and will be the subject of future papers.

Acknowledgments

This work was supported by *Consiglio Nazionale delle Ricerche* under contract 93.00904.PF67.

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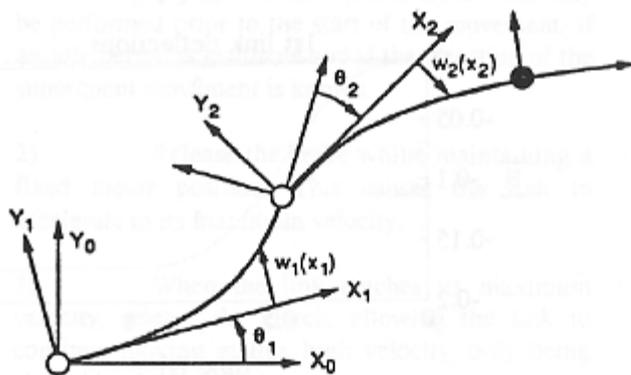


Fig. 1. A planar two-link flexible arm.

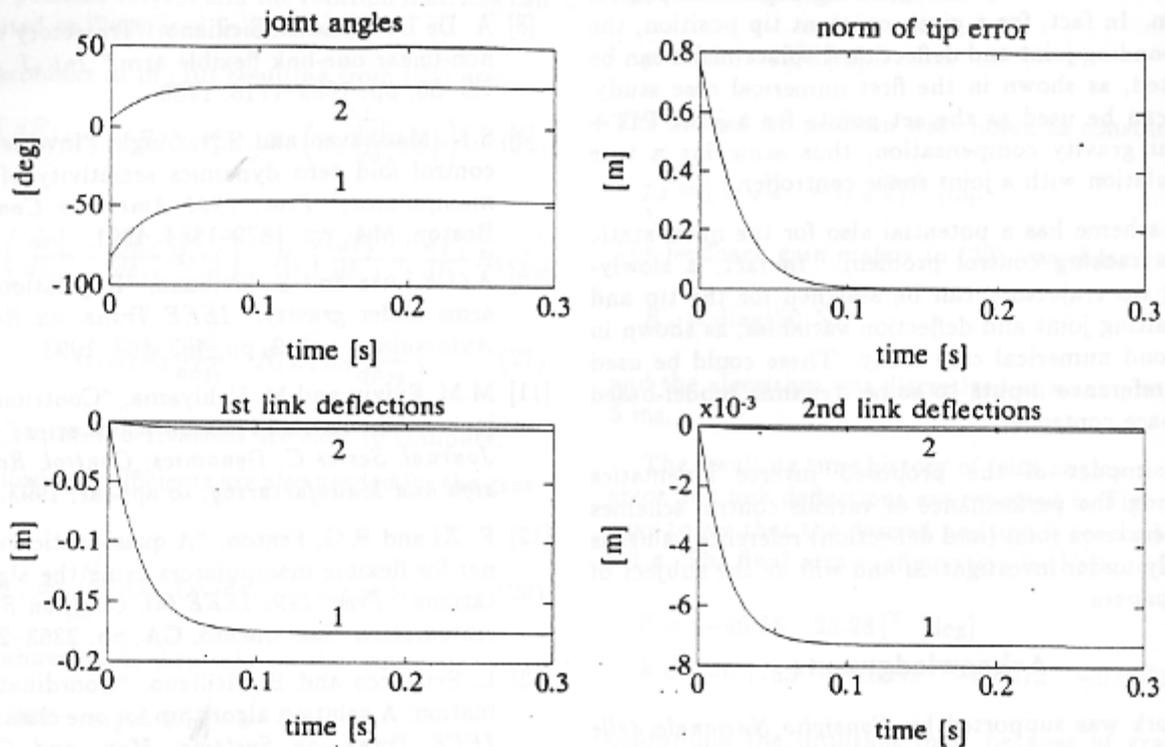


Fig. 2. Time history of joint angles, norm of tip error and link deflections for the first case study.

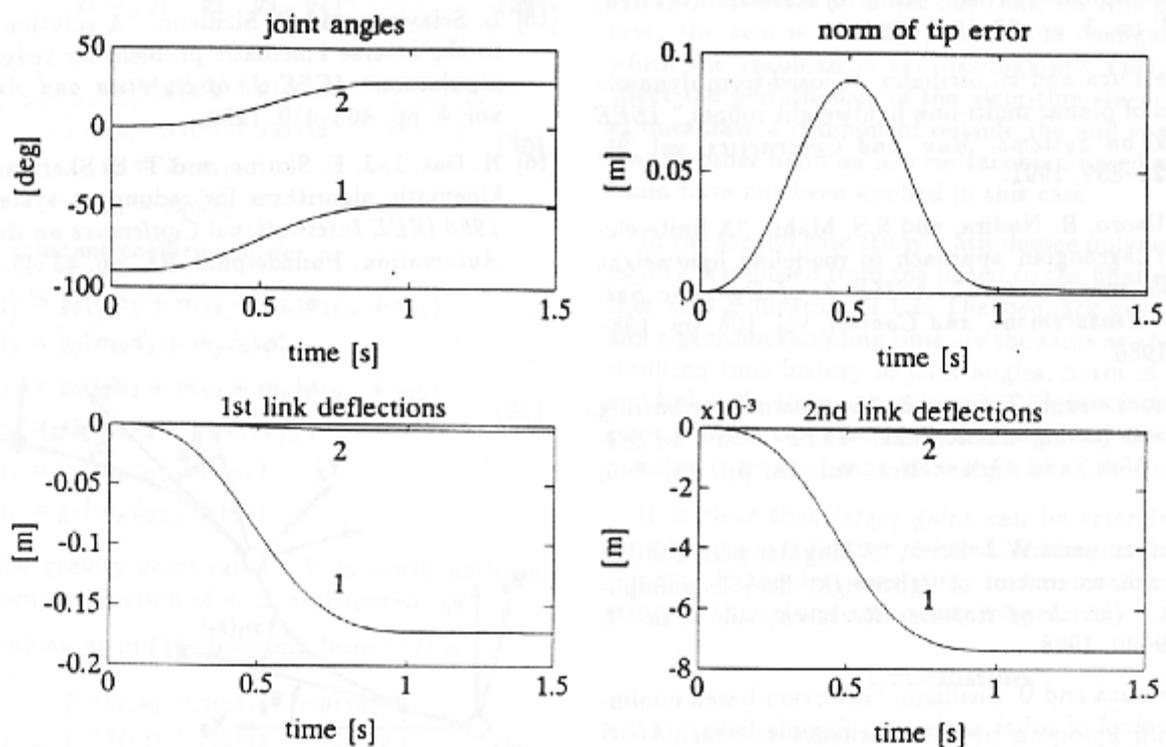


Fig. 3. Time history of joint angles, norm of tip error and link deflections for the second case study.