

## ON DYNAMIC MODELLING OF GEAR-DRIVEN RIGID ROBOT MANIPULATORS

Lorenzo SCIAVICCO, Bruno SICILIANO and Luigi VILLANI

*Dipartimento di Informatica e Sistemistica, Università degli Studi di Napoli Federico II  
Via Claudio 21, 80125 Napoli, Italy, E-mail: siciliano@na.infn.it*

**Abstract.** Dynamic modelling of gear-driven rigid robot manipulators is discussed in this work. The dynamic effects of the motion of the motors driving the joints through gears are analyzed. A complete model is derived using the Lagrange formulation in which the contributions of rotor inertias and rotor-link interactions are evidenced. The resulting equations of motion are shown to be linear in terms of a suitable set of dynamic parameters for the augmented links (links with motors). These are utilized for model derivation using the recursive Newton-Euler formulation. The example of an elbow manipulator is developed.

**Key Words.** Modelling; industrial robots; Lagrange formulation; Newton-Euler formulation; model parametrization.

### 1. INTRODUCTION

Most industrial robot manipulators are driven by motors through gears with high (from tens to a few hundred) reduction ratios. The use of gears permits an optimization of manipulator static and dynamic performance since the motors can be located on the links preceding the actuated joints along the kinematic chain. Further, typical robot applications require motions with large torques and relatively small velocities, and thus the use of gears allows joint actuation by motors of reduced size.

The inertial effects of fast spinning rotors of the motors may have a relevant influence on the dynamic behaviour of such manipulators. The effective rotor inertias are indeed multiplied by the square of gear ratios and coupling torques and forces arise from the interaction with the link motion.

The usual way to take into account the rotor inertias is to add a diagonal matrix to the inertia matrix of the dynamic model of the manipulator without actuators derived by using a Lagrange formulation (Armstrong, Khatib and Burdick, 1986; Dombre and Khalil, 1988). This corresponds to neglecting the coupling effects between rotor and link motion. Nevertheless it has been shown that additional contributions have to be introduced in the off-diagonal elements of the inertia matrix and in the Coriolis and centrifugal terms to properly account for the above effects (Chedmail, Gautier and Khalil, 1986; Otter and Türk, 1988; Chen, 1989; Murphy and Wen, 1993).

Programs allowing automatic generation of dynamic models by means of symbolic manipulation languages have been developed, e.g. (Cesareo, Nicolò and Nicosia, 1984), which include both motor inertias and joint elasticities. The resulting models may be in a scarcely compact form and thus of limited use for control design (Chiacchio, Sciacvico and Siciliano, 1990).

Recursive computation algorithms of dynamic models of geared robot manipulators have been developed both for rigid joints (Murphy, Wen and Saridis, 1990; Jain and Rodriguez, 1990) and elastic joints (Murphy and Wen, 1991), which adopt an unusual formalism based on the spatial operator algebra. A Newton-Euler formulation is adopted and the equations of motion are obtained by deriving force and moment balances separately for links and motors.

This paper is aimed at pointing out the modifications that have to be introduced both in the Lagrange and in Newton-Euler derivations of the dynamic model of a manipulator without actuators, in order to include the complete effects of the rotor inertias and rotor-link interactions. Standard notations are used for vector algebra and the Denavit-Hartenberg convention is adopted for the assignment of coordinate frames on the links. Both the cases of revolute and prismatic joints driven by revolute motors are considered and the rotor axes are supposed to be arbitrarily oriented with respect to those of the actuated joints.

The adopted Lagrange formulation naturally allows linear parametrization of the complete model

in terms of a suitable set of dynamic parameters (Nicolò and Katende, 1983). For each joint, the eleven parameters are the rotor inertia and the mass and inertial parameters of the augmented link which is obtained by considering the link and the motor mounted on it. Hence the dimension of the dynamic parameter vector is the same as in the usual approximate approaches that neglect rotor-link interactions.

Further it is shown how the augmented link concept can be conveniently used in the Newton-Euler recursive derivation of the equations of motion.

The explicit dynamic model of a gear-driven elbow manipulator is derived to illustrate the complete effects of rotor inertias and rotor-link interactions and to give an example of linear parametrization of the model.

## 2. PRELIMINARIES

The manipulator is seen as a serial chain of  $n + 1$  rigid links connected by revolute or prismatic joints. The base of the robot is numbered as link 0 and the terminal link as link  $n$ . The joint  $i$  ( $i = 1, \dots, n$ ) connects link  $i - 1$  to link  $i$  and motor  $i$  driving joint  $i$  is located on link  $i - 1$ . This is a current practice in industrial manipulators, resulting by a tradeoff between the need to locate the motors as close as possible to the manipulator base so as to reduce the structural mass of the robot, and the need of using transmission elements simple and efficient, with low backlash and wear. However this assumption is made only for clarity of presentation; the case of different motor location can be analyzed in a formally analogous way. Also mass and inertial contributions of gears can be suitably included in the corresponding terms of the motors.

The standard Denavit-Hartenberg convention is adopted for the coordinate frames on the links, i.e. a frame of rotation matrix  $R_i$  is fixed with respect to link  $i$  and its unit vector  $z_i$  lies along joint  $i + 1$  (see also Fig. 1). In addition, for each motor a frame of rotation matrix  $R_m$  is introduced which is fixed with respect to rotor  $i$  and with unit vector  $z_m$ , lying along the rotation axis.

The following notations are used for link and motor parameters:

- $p_i$  position of the origin of frame  $i$
- $r_{i-1,i}$  vector from the origin of frame  $i - 1$  to the origin of frame  $i$
- $m_l$  mass of link  $i$
- $p_l$  center of mass of link  $i$
- $I_l$  inertia tensor of link  $i$  relative to its center of mass
- $m_m$  mass of rotor  $i$
- $p_m$  center of mass of rotor  $i$

- $I_{m_i}$  inertia tensor of rotor  $i$  relative to its center of mass
- $m_i$  global mass of augmented link  $i$  (link  $i$  and rotor  $i + 1$ )
- $p_C$  center of mass of augmented link  $i$
- $r_{i-1,C_i}$  vector from the origin of frame  $i - 1$  to the center of mass of augmented link  $i$
- $r_{i,C_i}$  vector from the origin of frame  $i$  to the center of mass of augmented link  $i$
- $\bar{I}_i$  inertia tensor of augmented link  $i$  relative to its center of mass
- $\tilde{I}_i$  inertia tensor of augmented link  $i$  relative to the origin of frame  $i - 1$

where all quantities are referred to the base frame. In the following the presence of a superscript will denote a frame different from the base frame.

Let  $q_i$  denote joint  $i$  variable and  $\vartheta_m$  the angular displacement of motor  $i$ . In the assumption of rigid transmission the relation between velocities is

$$k_{r_i} \dot{q}_i = \dot{\vartheta}_{m_i}, \quad (1)$$

where  $k_{r_i}$  is the gear ratio; notice that in the case of a prismatic joint  $k_{r_i}$  is a dimensional quantity since the motor is usually taken to be revolute. Hence the angular velocity  $\omega_m$  of rotor  $i$  is

$$\omega_m = \omega_{i-1} + k_{r_i} \dot{q}_i z_{m_i}, \quad (2)$$

where  $\omega_{i-1}$  is the angular velocity of link  $i - 1$  on which the motor is located.

## 3. LAGRANGE FORMULATION

According to Lagrange formulation for dynamic modelling of mechanical systems, define the function

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \quad (3)$$

where  $\mathcal{T}$  is the global kinetic energy and  $\mathcal{U}$  is the global potential energy. The equations of motion can be obtained from the following  $n$  Lagrange equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad i = 1, \dots, n \quad (4)$$

where  $\tau_i$  is the generalized non-conservative force acting on the joint variable  $q_i$ , i.e. a force for a prismatic joint and a torque for a revolute joint. Non-conservative forces include friction, driving forces provided by motors through gears and forces arising from possible contact at the end effector.

The global kinetic energy can be obtained by adding the individual contributions of links and motors, where the contribution of each link is relative to all the parts fixed on it including the motor stator.

The kinetic energy of link  $i$  is given by

$$\mathcal{T}_i = \frac{1}{2} m_i \dot{p}_i^T \dot{p}_i + \frac{1}{2} \omega_i^T R_i I_i R_i^T \omega_i \quad (5)$$

where the inertia tensor of link  $i$  has been referred to frame  $i$  so that it is independent of the joint configuration. In order to express (5) in terms of the vector of joint coordinates  $q = [q_1 \dots q_n]^T$ , as required by (4), the following Jacobian matrices are introduced for linear and angular velocities of link  $i$ :

$$\begin{aligned} \dot{p}_i &= J_P^{(i)} \dot{q} \\ \omega_i &= J_O^{(i)} \dot{q}, \end{aligned} \quad (6)$$

with

$$\begin{aligned} J_P^{(i)} &= [J_{P1}^{(i)} \dots J_{Pi}^{(i)} \ 0 \dots 0] \\ J_O^{(i)} &= [J_{O1}^{(i)} \dots J_{Oi}^{(i)} \ 0 \dots 0] \end{aligned} \quad (7)$$

whose columns can be evaluated as

$$\begin{aligned} J_{Pj}^{(i)} &= \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_i - p_{j-1}) & \text{revolute joint} \end{cases} \\ J_{Oj}^{(i)} &= \begin{cases} 0 & \text{prismatic joint} \\ z_{j-1} & \text{revolute joint.} \end{cases} \end{aligned} \quad (8)$$

The kinetic energy contribution of motor  $i$  coincides with that of its rotor, given by

$$\mathcal{T}_m = \frac{1}{2} m_m \dot{p}_m^T \dot{p}_m + \frac{1}{2} \omega_m^T R_m I_m R_m^T \omega_m \quad (9)$$

The linear and angular velocities of motor  $i$  are:

$$\begin{aligned} \dot{p}_m &= J_P^{(m,i)} \dot{q} \\ \omega_m &= J_O^{(m,i)} \dot{q}, \end{aligned} \quad (10)$$

with

$$\begin{aligned} J_P^{(m,i)} &= [J_{P1}^{(m,i)} \dots J_{P,i-1}^{(m,i)} \ 0 \ 0 \dots 0] \\ J_O^{(m,i)} &= [J_{O1}^{(m,i)} \dots J_{O,i-1}^{(m,i)} \ J_{Oi}^{(m,i)} \ 0 \dots 0] \end{aligned} \quad (11)$$

whose columns can be evaluated as

$$\begin{aligned} J_{Pj}^{(m,i)} &= \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_m - p_{j-1}) & \text{revolute joint} \end{cases} \\ J_{Oj}^{(m,i)} &= \begin{cases} J_{Oj}^{(i)} & j = 1, \dots, i-1 \\ k_{ri} z_m & j = i. \end{cases} \end{aligned} \quad (12)$$

To obtain (12) the expression of  $\omega_m$ , in (2) has been considered.

In view of (5)-(12), the global kinetic energy can be expressed in the quadratic form

$$\begin{aligned} \mathcal{T} &= \sum_{i=1}^n (\mathcal{T}_i + \mathcal{T}_{m,i}) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_{ij}(q) \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{q}^T B(q) \dot{q}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} B(q) &= \sum_{i=1}^n (m_i J_P^{(i)T} J_P^{(i)} + J_O^{(i)T} R_i I_i R_i^T J_O^{(i)} \\ &\quad + m_{m,i} J_P^{(m,i)T} J_P^{(m,i)} + J_O^{(m,i)T} R_{m,i} I_{m,i} R_{m,i}^T J_O^{(m,i)}) \end{aligned} \quad (14)$$

is the  $(n \times n)$  inertia matrix of the manipulator which is symmetric and positive definite.

The global potential energy is the sum of the individual contributions of links and motors which, in the assumption of rigid links and joints, include only gravitational effects. Denoting by  $g_0$  the gravity acceleration vector,  $\mathcal{U}$  has the simple form

$$\begin{aligned} \mathcal{U} &= \sum_{i=1}^n (\mathcal{U}_i + \mathcal{U}_{m,i}) \\ &= - \sum_{i=1}^n (m_i g_0^T p_i + m_{m,i} g_0^T p_{m,i}). \end{aligned} \quad (15)$$

Using the expressions of kinetic energy (13) and potential energy (15), from (3),(4),(6),(7),(11),(12) the equations of motion become

$$\sum_{j=1}^n b_{ij}(q) \ddot{q}_j + \sum_{j=1}^n h_{ij}(q, \dot{q}) \dot{q}_j + g_i(q) = \tau_i \quad i = 1, \dots, n \quad (16)$$

where

$$\sum_{j=1}^n h_{ij} \dot{q}_j = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \left( \frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i} \right) \dot{q}_k \dot{q}_j \quad (17)$$

are the Coriolis and centrifugal terms, and

$$g_i = - \sum_{j=1}^n (m_i g_0^T J_{Pi}^{(i)}(q) + m_{m,i} g_0^T J_{Pi}^{(m,i)}(q)) \quad (18)$$

are the gravitational terms.

Equations (16) can be rewritten in the well-known compact form

$$B(q) \ddot{q} + H(q, \dot{q}) \dot{q} + g(q) = \tau. \quad (19)$$

#### 4. EFFECTS OF ROTOR INERTIAS

In the derivation of the dynamic model (19), link and rotor contributions have been considered separately. This distinction is needed for gear-driven manipulators only and not for direct-drive manipulators in which the rotors are integral parts of the driven links. For the latter the procedure leading to (19) still remains valid with unitary gear ratios and can be actually simplified by considering only the contributions of each link including that of the

annexed rotor. In that case rotor  $i$  is naturally associated to link  $i$ .

In order to have a clear understanding of the influence of motors on the dynamic behaviour of gear-driven manipulators, it is convenient to associate the kinetic and potential energy contributions of each rotor with that of the link on which the motor is mounted, i.e. to consider an *augmented* link  $i$  composed by link  $i$  and rotor  $i + 1$ .

In the reasonable hypothesis that rotors have a symmetric mass distribution around their rotation axes, the inertia tensor relative to the center of mass of rotor  $i$ , expressed in frame  $R_{m_i}$ , has the form

$$I_{m_i}^i = \begin{bmatrix} I_{m,xx} & 0 & 0 \\ 0 & I_{m,yy} & 0 \\ 0 & 0 & I_{m,zz} \end{bmatrix} \quad (20)$$

and thus the inertia tensor of rotor  $i + 1$  expressed in a frame fixed with respect to the supporting link  $i$  is constant, i.e. it does not vary with the angular position of the rotor. The inertia tensor of the augmented link relative to its center of mass is also constant in frame  $R_i$  and can be obtained from straightforward application of Huygens theorem as

$$\bar{I}_i^i = I_i^i + m_i S^T(r_{C_i, \ell}^i) S(r_{C_i, \ell}^i) + I_{m_{i+1}}^i + m_{m_{i+1}} S^T(r_{C_i, m_{i+1}}^i) S(r_{C_i, m_{i+1}}^i) \quad (21)$$

where  $r_{C_i, \ell}^i = p_{\ell}^i - p_{C_i}^i$ ,  $r_{C_i, m_{i+1}}^i = p_{m_{i+1}}^i - p_{C_i}^i$ , and  $S(\cdot)$  is the skew-symmetric matrix operator performing the vector product  $S(a)b = a \times b$ .

By using (2),(5),(9),(20),(21) and the following equations

$$\begin{aligned} \dot{p}_{\ell}^i &= \dot{p}_{C_i}^i + \omega_i^i \times r_{C_i, \ell}^i, \\ \dot{p}_{m_{i+1}}^i &= \dot{p}_{C_i}^i + \omega_i^i \times r_{C_i, m_{i+1}}^i, \end{aligned} \quad (22)$$

the kinetic energy of the augmented link takes on the form

$$\begin{aligned} T_i &= T_{\ell} + T_{m_{i+1}} \\ &= \frac{1}{2} m_i \dot{p}_{C_i}^{iT} \dot{p}_{C_i}^i + \frac{1}{2} \omega_i^{iT} \bar{I}_i^i \omega_i^i \\ &\quad + \frac{1}{2} k_{r,i+1}^2 \dot{q}_{i+1}^2 I_{m_{i+1}} + k_{r,i+1} \dot{q}_{i+1} I_{m_{i+1}} z_{m_{i+1}}^{iT} \omega_i^i \end{aligned} \quad (23)$$

where  $I_{m_{i+1}} = I_{m_{i+1},zz}$ . The potential energy can be easily evaluated as

$$U_i = U_{\ell} + U_{m_{i+1}} = -m_i g_0^i p_{C_i}^i. \quad (24)$$

Equation (23) reveals that the kinetic energy of the augmented link is the sum of four contributions. The first two express the kinetic energy of the augmented link when the rotor is motionless. The third one is the kinetic energy of the

augmented link when the link is motionless. The remaining term is the rotor kinetic energy deriving from the interaction of its rotation with that of the link. Equation (24) shows that the potential energy does not obviously depend on the angular position of the rotors since their centers of mass are on the rotation axes.

From the above considerations and in view of the form of Lagrange equations (4), it is possible to quantify the error made in accounting for rotor inertias effects by simply adding a term  $k_{r_i}^2 I_{m_i}$  to each diagonal element of the manipulator inertia matrix. In particular, from the last term on the right-hand side of (23) it is clear that this approximation is exact only when  $z_{m_{i+1}}^{iT} \omega_i^i = 0$ , that is for motors mounted on links with angular velocity null or orthogonal to the rotor axis for any configuration of the manipulator.

## 5. LINEARITY IN THE PARAMETERS

On the basis of the previous considerations, the dynamic model (19) can be suitably expressed in linear form with respect to the dynamic parameters of the augmented links.

The kinetic energy (23) can be rewritten in the form

$$\begin{aligned} T_i &= \frac{1}{2} m_i \dot{p}_{i-1}^{iT} \dot{p}_{i-1}^i + \dot{p}_{i-1}^{iT} S(\omega_i^i) m_i r_{i-1,C}^i \\ &\quad + \omega_i^{iT} \bar{I}_i^i \omega_i^i + k_{r,i+1} \dot{q}_{i+1} I_{m_{i+1}} z_{m_{i+1}}^{iT} \omega_i^i \\ &\quad + \frac{1}{2} k_{r,i+1}^2 \dot{q}_{i+1}^2 I_{m_{i+1}} \end{aligned} \quad (25)$$

where the equation

$$\dot{p}_{C_i}^i = \dot{p}_{i-1}^i + \omega_i^i \times r_{i-1,C}^i, \quad (26)$$

and the expression of the inertia tensor of the augmented link relative to the origin of frame  $i - 1$

$$\bar{I}_i^i = \bar{I}_i^i + m_i S^T(r_{i-1,C}^i) S(r_{i-1,C}^i) \quad (27)$$

have been exploited.

Equation (25) is linear with respect to the inertia moment of the rotor and to the mass, the six components of the inertia tensor (27) and the three components of the first moment  $m_i r_{i-1,C}^i = [m_i \ell_{C,x} \quad m_i \ell_{C,y} \quad m_i \ell_{C,z}]^T$  of the augmented link.

Similarly the potential energy (15), by means of the equation

$$p_{C_i}^i = p_{i-1}^i + r_{i-1,C}^i, \quad (28)$$

can be rewritten as

$$U_i = -g_0^i (m_i p_{i-1}^i + m_i r_{i-1,C}^i) \quad (29)$$

which is linear with respect to the mass and the three components of the first moment of the augmented link.

By adding all the kinetic and potential energy contributions, the Lagrangian function (3) can be expressed in the form

$$\mathcal{L} = \sum_{i=1}^n (\beta_{T_i}^T - \beta_{U_i}^T) \pi_i \quad (30)$$

which is linear in terms of the  $(11 \times 1)$  vector of augmented link dynamic parameters and rotor inertia

$$\pi_i = [m_i \quad m_i l_{C,x} \quad m_i l_{C,y} \quad m_i l_{C,z} \quad \dot{I}_{ixx} \quad \dot{I}_{ixy} \quad \dot{I}_{ixz} \quad \dot{I}_{iyy} \quad \dot{I}_{iyz} \quad \dot{I}_{izz} \quad I_{m,i}]^T \quad (31)$$

Notice that in  $\pi_i$  the inertia of rotor  $i$  has been associated to the dynamic parameters of augmented link  $i$  so that the vectors  $\beta_{U_i}$  and  $\beta_{T_i}$  in (31) depend only on joint coordinates  $q_1 \dots q_i$  and their derivatives, i.e.  $\beta_{U_i} = \beta_{U_i}(q_1, q_2, \dots, q_i)$  and  $\beta_{T_i} = \beta_{T_i}(q_1, q_2, \dots, q_i, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_i)$ .

In view of (30), Equations (4) become

$$\sum_{j=1}^n y_{ij}^T \pi_j = \tau_i \quad i = 1, \dots, n \quad (32)$$

where

$$y_{ij} = \frac{d}{dt} \frac{\partial \beta_{T_j}}{\partial \dot{q}_i} - \frac{\partial \beta_{T_j}}{\partial q_i} + \frac{\partial \beta_{U_{ij}}}{\partial q_i} \quad (33)$$

By virtue of the particular structure of  $\beta_{T_i}$  and  $\beta_{U_i}$ , Equations (32) have the triangular form

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix} = \begin{bmatrix} y_{11}^T & y_{12}^T & \dots & y_{1n}^T \\ 0^T & y_{22}^T & \dots & y_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0^T & 0^T & \dots & y_{nn}^T \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix} \quad (34)$$

It should be observed that the obtained parameterization is in general non-minimal. In fact each joint axis has only one degree of mobility and thus it is possible that some dynamic parameters do not influence the generalized force at the joint whereas some others may appear in the dynamic model only in linear combinations (Gautier and Khalil, 1990).

## 6. NEWTON-EULER FORMULATION

With the Newton-Euler formulation for a direct-drive manipulator, the equations of motion are obtained in a two-step recursive procedure: a forward recursion for the propagation of velocities and accelerations from the base to the end effector, and a backward recursion for the propagation of forces and moments from the end effector to the base. Thanks to the introduction of the augmented link concept, the same procedure can be

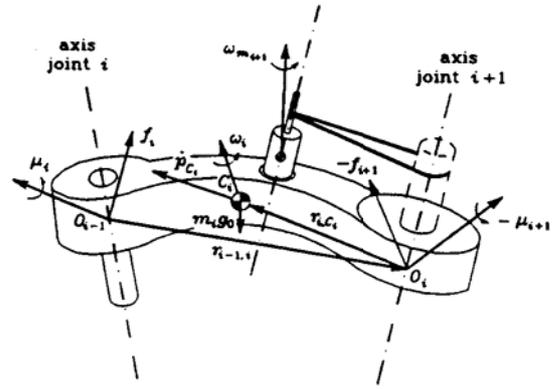


Fig. 1. Characterization of augmented link  $i$

applied also to gear-driven manipulators as shown below.

With reference to Fig. 1, the balance of forces for the augmented link can be expressed by the following Newton equation

$$f_i = f_{i+1} + m_i(\ddot{p}_{C_i} - g_0), \quad (35)$$

where  $f_i$  and  $-f_{i+1}$  are respectively the sum of the forces exerted on augmented link  $i$  by the preceding link and the subsequent link of the chain.

The balance of moments for the augmented link can be expressed by the following Euler equation

$$\mu_i = -f_i \times (\tau_{i-1,i} + \tau_{i,C_i}) + \mu_{i+1} + f_{i+1} \times \tau_{i,C_i} + \frac{dk_i}{dt} \quad (36)$$

where  $\mu_i$  and  $-\mu_{i+1}$  are respectively the sum of the moments of forces exerted on augmented link  $i$  by the preceding link and the subsequent link of the chain, evaluated with respect to the origins of frames  $i-1$  and  $i$  respectively; also  $k_i$  is the angular momentum of augmented link  $i$  with respect to its center of mass. Accounting for (20),(21) yields

$$k_i = \bar{I}_i \omega_i + k_{r,i+1} \dot{q}_{i+1} I_{m,i+1} z_{m,i+1}, \quad (37)$$

leading to

$$\begin{aligned} \mu_i = & -f_i \times (\tau_{i-1,i} + \tau_{i,C_i}) + \mu_{i+1} + f_{i+1} \times \tau_{i,C_i} \\ & + \bar{I}_i \dot{\omega}_i + \omega_i \times (\bar{I}_i \omega_i) + k_{r,i+1} \dot{q}_{i+1} I_{m,i+1} z_{m,i+1} \\ & + k_{r,i+1} \dot{q}_{i+1} I_{m,i+1} \omega_i \times z_{m,i+1}. \end{aligned} \quad (38)$$

The generalized force at joint  $i$  can be computed as the sum of the inertial torque of rotor  $i$  reported to the joint side and the component of force  $f_i$  (for a prismatic joint) or moment  $\mu_i$  (for a revolute joint) along the axis of joint  $i$ . Therefore it is

$$\tau_i = \begin{cases} f_i^T z_{i-1} + k_{r,i} I_{m,i} \dot{\omega}_m^T z_{m,i}, & \text{prismatic joint} \\ \mu_i^T z_{i-1} + k_{r,i} I_{m,i} \dot{\omega}_m^T z_{m,i}, & \text{revolute joint.} \end{cases} \quad (39)$$

Equations (35),(38),(39) require the evaluation of velocities and accelerations of links and rotors.

The angular and linear velocities of link  $i$  are respectively:

$$\omega_i = \begin{cases} \omega_{i-1} & \text{prismatic joint} \\ \omega_{i-1} + \dot{\vartheta}_i z_{i-1} & \text{revolute joint} \end{cases} \quad (40)$$

and

$$\dot{p}_i = \begin{cases} \dot{p}_{i-1} + \dot{d}_i z_{i-1} + \omega_i \times r_{i-1,i} & \text{prismatic joint} \\ \dot{p}_{i-1} + \omega_i \times r_{i-1,i} & \text{revolute joint} \end{cases} \quad (41)$$

where the joint coordinate  $q_i$  has been denoted by  $d_i$  for a prismatic joint and by  $\theta_i$  for a revolute joint.

By differentiating (40) and (41), the following expressions for the accelerations of link  $i$  are obtained

$$\dot{\omega}_i = \begin{cases} \dot{\omega}_{i-1} & \text{prismatic joint} \\ \dot{\omega}_{i-1} + \ddot{\vartheta}_i z_{i-1} + \dot{\vartheta}_i \omega_{i-1} \times z_{i-1} & \text{revolute joint} \end{cases} \quad (42)$$

and

$$\ddot{p}_i = \begin{cases} \ddot{p}_{i-1} + \ddot{d}_i z_{i-1} + 2\dot{d}_i \dot{\omega}_i \times z_{i-1} & \text{prismatic joint} \\ \ddot{p}_{i-1} + \dot{\omega}_i \times r_{i-1,i} + \omega_i \times (\omega_i \times r_{i-1,i}) & \text{joint} \\ \ddot{p}_{i-1} + \dot{\omega}_i \times r_{i-1,i} & \text{revolute joint} \\ \dot{\omega}_i \times (\omega_i \times r_{i-1,i}) & \text{joint.} \end{cases} \quad (43)$$

Finally, the acceleration of the center of mass of augmented link  $i$  and the angular acceleration of rotor  $i$  are respectively:

$$\ddot{p}_{C_i} = \ddot{p}_i + \dot{\omega}_i \times r_{i,C_i} + \omega_i \times (\omega_i \times r_{i,C_i}) \quad (44)$$

$$\dot{\omega}_{m_i} = \dot{\omega}_{i-1} + k_{r_i} \dot{q}_i z_{m_i} + k_{r_i} \dot{q}_i \omega_{i-1} \times z_{m_i} \quad (45)$$

The equations of motion are obtained through a forward recursion based on (40),(42),(43),(44),(45) starting from initial conditions  $\omega_0, \dot{p}_0 - g_0, \dot{\omega}_0$ , and a backward recursion based on (35),(38),(39) starting from terminal conditions  $f_{n+1}, \mu_{n+1}$ ; notice that augmented link  $n$  coincides with link  $n$ . For computational efficiency, all the vectors in the above equations are to be referred to the current link frame  $i$  in the actual implementation.

## 7. EXAMPLE

The explicit dynamic model for the three-degree-of-freedom elbow manipulator in Fig. 2 is presented in the following. The Denavit-Hartenberg parameters are listed in the table below:

Link	$a_i$	$\alpha_i$	$d_i$	$\vartheta_i$
1	0	$\pi/2$	0	$\vartheta_1$
2	$a_2$	0	0	$\vartheta_2$
3	$a_3$	0	0	$\vartheta_3$

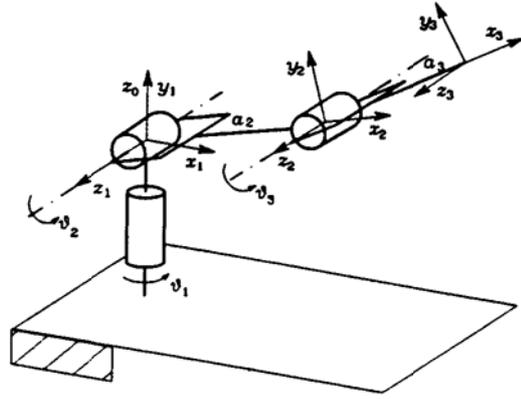


Fig. 2. Three-degree-of-freedom elbow manipulator

Let  $q = [\vartheta_1 \ \vartheta_2 \ \vartheta_3]^T$  be the vector of joint variables. The motor of joint  $i$  is located on link  $i-1$  and  $z_{m_1} = z_0, z_{m_2} = y_1, z_{m_3} = x_2$ ; note that the axes of rotors 2 and 3 are not aligned with the respective joint axes. It is assumed that  $\hat{I}_i$  is a diagonal matrix and  $m_i r_{i-1,C_i}^i = [m_i \ell_{C_i,x} \ 0 \ 0]^T, i = 2, 3$ .

It can be found that the dynamic model can be expressed in linear form with respect to the following linear combinations of augmented link dynamic parameters and rotor inertias:

$$\pi_1 = \dot{I}_{1yy} + \dot{I}_{2yy} + \dot{I}_{3yy} + a_2^2 m_3 + k_{r_1}^2 I_{m_1}$$

$$\pi_2 = m_2 \ell_{C_2,x} + a_2 m_3$$

$$\pi_3 = \dot{I}_{2xx} - \dot{I}_{2yy} - a_2^2 m_3$$

$$\pi_4 = \dot{I}_{2zz} + k_{r_2}^2 I_{m_2} + a_2^2 m_3$$

$$\pi_5 = \dot{I}_{3xx} - \dot{I}_{3yy}$$

$$\pi_6 = m_3 \ell_{C_3,x}$$

$$\pi_7 = \dot{I}_{3zz}$$

$$\pi_8 = I_{m_2}$$

$$\pi_9 = I_{m_3}$$

with obvious meaning of the quantities.

The elements of the inertia matrix  $B$  in (14) are:

$$b_{11} = \pi_1 + s_2^2 \pi_3 + s_{23}^2 \pi_5 + 2a_2 c_2 c_{23} \pi_6$$

$$b_{12} = k_{r_2} \pi_8$$

$$b_{13} = k_{r_3} s_2 \pi_9$$

$$b_{21} = b_{12}$$

$$b_{22} = \pi_4 + 2a_2 c_3 \pi_6 + \pi_7$$

$$b_{23} = a_2 c_3 \pi_6 + \pi_7$$

$$b_{31} = b_{13}$$

$$b_{32} = b_{23}$$

$$b_{33} = \pi_7 + k_{r_3}^2 \pi_9$$

where the standard abbreviations  $s_{i...j}$  and  $c_{i...j}$  have been used for  $\sin(\vartheta_i + \dots + \vartheta_j)$  and  $\cos(\vartheta_i + \dots + \vartheta_j)$  respectively.

The coefficients of the matrix  $H$  in (17) are:

$$\begin{aligned} h_{11} &= s_2 c_2 \dot{q}_2 \pi_3 + s_{23} c_{23} (\dot{q}_2 + \dot{q}_3) \pi_5 \\ &\quad - a_2 (s_2 c_{23} \dot{q}_2 + c_2 s_{23} (\dot{q}_2 + \dot{q}_3)) \pi_6 \\ h_{12} &= s_2 c_2 \dot{q}_1 \pi_3 + s_{23} c_{23} \dot{q}_1 \pi_5 \\ &\quad - a_2 (c_2 s_{23} + s_2 c_{23}) \dot{q}_1 \pi_6 + 0.5 k_{r3} c_2 \dot{q}_3 \pi_9 \\ h_{13} &= s_{23} c_{23} \dot{q}_1 \pi_5 - a_2 c_2 s_{23} \dot{q}_1 \pi_6 + 0.5 k_{r3} c_2 \dot{q}_2 \pi_9 \\ h_{21} &= -h_{12} \\ h_{22} &= -a_2 s_3 \dot{q}_3 \pi_6 \\ h_{23} &= h_{22} - h_{32} \\ h_{31} &= -s_{23} c_{23} \dot{q}_1 \pi_5 + a_2 c_2 s_{23} \dot{q}_1 \pi_6 + 0.5 k_{r3} c_2 \dot{q}_2 \pi_9 \\ h_{32} &= a_2 s_3 \dot{q}_2 \pi_6 + 0.5 k_{r3} c_2 \dot{q}_1 \pi_9 \\ h_{33} &= 0. \end{aligned}$$

The elements of the gravity vector  $g$  in (18) are:

$$\begin{aligned} g_1 &= 0 \\ g_2 &= g_3 - g c_2 \pi_2 \\ g_3 &= -g c_{23} \pi_6. \end{aligned}$$

As can be recognized from above, the elements  $b_{12}$  and  $b_{13}$  describe the effects of the accelerations of rotors 2 and 3 on joint 1. Notice that, in view of the location of motors,  $b_{12}$  is a constant whereas  $b_{13}$  depends on joint variable 2. The latter generates additional contributions in the coefficients of the Coriolis terms  $h_{ij}$ ,  $i \neq j$ . Such effects are neglected in the approximate models which are commonly used.

## 8. CONCLUSIONS

The effects of rotor inertias and rotor-link interactions on the dynamic model of gear-driven rigid manipulators have been analyzed in this work. The concept of an augmented link incorporating the link itself and the supported motor has led to evidencing how the kinetic and potential energies are modified in the Lagrange formulation due to the presence of motors, allowing also a simple linear parametrization of the model in terms of the rotor inertias and the dynamic parameters of the augmented links. The Newton-Euler formulation takes advantage of the above derivation as the balances of forces and moments are expressed directly in terms of the augmented links. An example for an elbow manipulator has been worked out to capture the effect of motor-link interaction which is typically neglected in most models.

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