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## Chapter 1

# Flexible-link Manipulators: Modeling, Nonlinear Control and Observer

The interest in flexible robot manipulators has become greater in the latest years. In order to adequately exploit the advantages of this class of manipulators, accurate models and effective control schemes are necessary. This work collects a number of recent results on modeling, nonlinear control and observer for flexible-link manipulators. The equations of motion are derived on the basis of a combined Lagrange-assumed modes approach. The resulting model shows several similarities with that of a rigid manipulator, thus allowing important properties to be derived which are used to design controllers and observers. A nonlinear control scheme based on robust control techniques is proposed in order to improve the damping of the system. Since typically link coordinate rates cannot be measured, a nonlinear observer is presented which provides estimates of both joint and link coordinate rates while keeping stability of the system.

### 1.1 Introduction

Lightweight manipulators offer many challenges in comparison to rigid and bulky robot manipulators. Energy consumption is smaller, so that the payload-to-arm weight ratio can be increased as well as faster movements can be achieved. Due to their characteristics, this class of manipulators are specially suitable for a number of nonconventional robotic applications, including space missions. On the other hand, the study of link flexibility is enforced also for some kind of heavy manipulators such as large scale systems. In either case, it is no longer possible to assume that link deformation can be neglected. All these factors make the study of flexible robot manipulators quite interesting. The present work aims at presenting some of the latest results on modeling, nonlinear control and observer in

this field.

The importance of having an accurate model that can adequately describe the dynamics of the manipulator is obvious. A common way of modeling a flexible robot manipulator consists in using a combined Lagrange-assumed modes approach, which allows deriving a dynamic model in closed form [Book (1984); De Luca and Siciliano (1991); Yuan *et al.* (1993); Canudas de Wit *et al.* (1996); Arteaga (1998)]. Just like in the case of the dynamic model of a rigid manipulator, which possesses many helpful properties [Ortega and Spong (1989); Nicosia and Tomei (1990); Canudas de Wit *et al.* (1990)], it is possible to compute a set of properties for the dynamic model of a flexible manipulator [Arteaga (1998)], whose knowledge facilitates the design of controllers and observers for this kind of system [De Luca and Siciliano (1993a); Lammerts *et al.* (1995); Arteaga (1996a); Arteaga (1996b); Arteaga (1996c)]. Perhaps the most well-known property of (rigid and flexible) manipulators is that referring to their passive structure. With the exception of those controllers based on inverse dynamics [Canudas de Wit *et al.* (1996); De Luca and Siciliano (1993a)], this property is usually employed to prove stability of several control schemes. However, there are many other properties which have been employed to design specific control laws [Ortega and Spong (1989); Nicosia and Tomei (1990); Canudas de Wit *et al.* (1990); De Luca and Siciliano (1993a); De Luca and Siciliano (1993b); De Luca and Panzieri (1994)].

Control of flexible robot manipulators shows the difficulty that there is not an independent control input for each degree of freedom. As in the case of rigid manipulators, there are mainly two goals to be achieved: point-to-point and tracking control. For the first case, some results are given in [De Luca and Siciliano (1993b); De Luca and Panzieri (1994)], where the regulation problem under gravity is studied. In [De Luca and Siciliano (1993b)] the case of no modal damping of the links is treated. By making some assumptions on the inertia matrix, it is possible to guarantee convergence of the link coordinates to certain constant values. In [De Luca and Panzieri (1994)] a solution is proposed for the case that the gravity vector is not perfectly known.

Because an arbitrary trajectory can only be assigned for the joint coordinates, the desired trajectory for the link coordinates must be computed in such a way that the control goal can be accomplished. In [De Luca and Siciliano (1993a); Lammerts *et al.* (1995)] this problem is addressed, and in particular in [Lammerts *et al.* (1995)] not only flexible links

but also flexible joints are considered, but there is no guarantee that the computed desired trajectory remains bounded; when the model parameters are not well known, an adaptive algorithm can be used [Slotine and Li (1987)]. On the other hand, in [De Luca and Siciliano (1993a)] inverse control techniques are used [Canudas de Wit *et al.* (1996)], and it is shown that the computed desired trajectory remains bounded. In none of these works the problem of no damping is treated. In this work, the tracking control of flexible robot manipulators is studied [Arteaga (1996c); Arteaga and Siciliano (2000)]. A control law is proposed which is based on the passivity-based control approach with filtered reference velocity [Ortega and Spong (1989)]. It is proven that the desired trajectory for the link coordinates remains bounded. The no damping case is also treated and robust control techniques are used to increase the damping of the system [Dawson *et al.* (1991)].

A problem which deserves special attention regards the possible lack of measurement of link deflection rates, which typically requires the use of an observer. In addition, even though joint positions can be measured very accurately, tachometers (used to measure joint velocities) may not deliver reliable signals. That is why nonlinear observers are recommended to estimate joint speeds. In [Arteaga (1996a); Arteaga (1996b)] an observer for flexible robot manipulators is proposed. Although it is possible to measure link coordinates by using a strain gauge for each coordinate [Arteaga (1995)], the observer requires only a sensor for every flexible link. However, since it is designed independently of any control scheme, the stability of this observer together with the controller proposed in [Arteaga (1996c); Arteaga and Siciliano (2000)] and presented in this work can no longer be guaranteed. To overcome this difficulty, a new observer based on that given in [Nicosia and Tomei (1990)] is proposed [Arteaga (2000)]. In order to ensure enhancing of the damping of the system, some essential modifications are necessary.

The work is organized as follows: Section 1.2 briefly describes the kinematics of flexible robot manipulators and their dynamic modelling. Some of the most important properties of the model are listed. In Section 1.3, control of flexible manipulators is studied. By using robust control techniques, the damping of the system is increased. Since it is not always possible to measure link coordinate rates, a nonlinear observer is proposed in Section 1.4 in order to estimate them. Some simulation results are presented in Section 1.5, while Section 1.6 gives some concluding remarks.

## 1.2 Modeling

A common way of modeling flexible robot manipulators is using the so-called combined Lagrange–assumed modes approach [Book (1984); De Luca and Siciliano (1991); Yuan *et al.* (1993); Canudas de Wit *et al.* (1996); Arteaga (1998)]. In this case, it is necessary to describe the kinetic and potential energy of the system adequately. In order to compute them, it is advantageous to know the kinematics of the manipulator, which can be achieved by setting coordinate frames along the joint axes. In this section, the kinematics of flexible robot manipulators is briefly studied. By using Lagrange equations of motion, the dynamic model of this class of manipulators is derived in Section 1.2.2 and in Section 1.2.3 some of its most important properties are presented.

### 1.2.1 Kinematics

It is well known that the kinematics of a rigid robot manipulator can be described by employing the Denavit–Hartenberg representation [Denavit and Hartenberg (1955)]. The main idea is to use  $4 \times 4$  transformation matrices which can be determined uniquely as a function of only 4 parameters. However, this procedure cannot be used directly to describe the kinematics of a flexible robot manipulator due to link deformation. In order to overcome this drawback, the procedure has been modified in [Book (1984); Book (1979)] by including some transformation matrices which take link elasticity into account. A description of the Denavit–Hartenberg representation for rigid manipulators is assumed to be known. Fig. 1.1 depicts a portion of the serial chain for a flexible robot manipulator. The case of revolute joints is considered.

Consider two coordinate frames  $i$  and  $j$ . Their mutual position and orientation can be expressed in terms of the homogeneous transformation matrix

$${}^j\mathbf{T}_i = \begin{bmatrix} {}^j\mathbf{R}_i & {}^j\mathbf{d}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \quad (1.1)$$

where  ${}^j\mathbf{R}_i$  is the  $3 \times 3$  rotation matrix describing the orientation of the axes of frame  $i$  and  ${}^j\mathbf{d}_i$  is the  $3 \times 1$  vector describing the origin of frame  $i$ , both with respect to frame  $j$ ; also, in (1.1)  $\mathbf{0}$  denotes a  $3 \times 1$  vector of null elements.

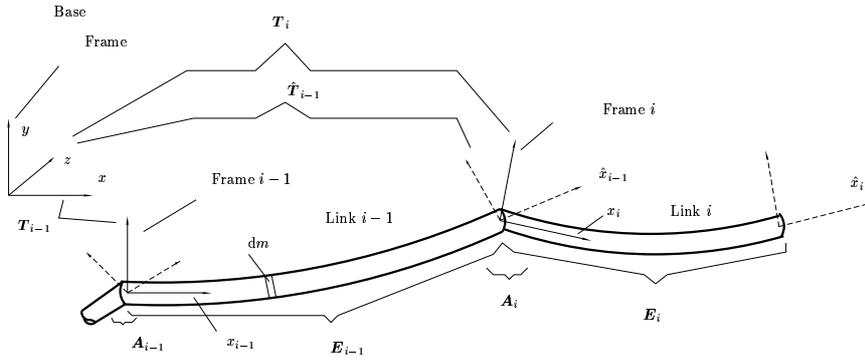


Fig. 1.1 Flexible manipulator serial chain.

The position of a point on link  $i$  with respect to frame  $i$  is given by

$${}^i \mathbf{p}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}. \quad (1.2)$$

However, it is not possible to use a homogeneous transformation with a vector of the form (1.2), so that it is necessary to rewrite it as

$${}^i \mathbf{r}_i \triangleq \begin{bmatrix} {}^i \mathbf{p}_i \\ 1 \end{bmatrix}. \quad (1.3)$$

To express the position of this point in frame  $j$ , a homogeneous transformation is used, i.e.

$${}^j \mathbf{r}_i = {}^j \mathbf{T}_i {}^i \mathbf{r}_i. \quad (1.4)$$

In the case of the base frame one has

$${}^0 \mathbf{r}_i \triangleq \mathbf{r}_i = {}^0 \mathbf{T}_i {}^i \mathbf{r}_i \triangleq \mathbf{T}_i {}^i \mathbf{r}_i, \quad (1.5)$$

where the superscript 0 has been conveniently dropped.

In general, the homogeneous transformation of frame  $i$  with respect to the base frame can be characterized through the following composition of consecutive transformations:

$${}^0 \mathbf{T}_i \triangleq \mathbf{T}_i = \mathbf{A}_1 \mathbf{E}_1 \mathbf{A}_2 \mathbf{E}_2 \dots \mathbf{A}_{i-1} \mathbf{E}_{i-1} \mathbf{A}_i \triangleq \hat{\mathbf{T}}_{i-1} \mathbf{A}_i \quad (1.6)$$

$$\hat{\mathbf{T}}_{i-1} \triangleq \mathbf{T}_{i-1} \mathbf{E}_{i-1} \quad (1.7)$$

$$\mathbf{T}_1 = \mathbf{A}_1, \quad (1.8)$$

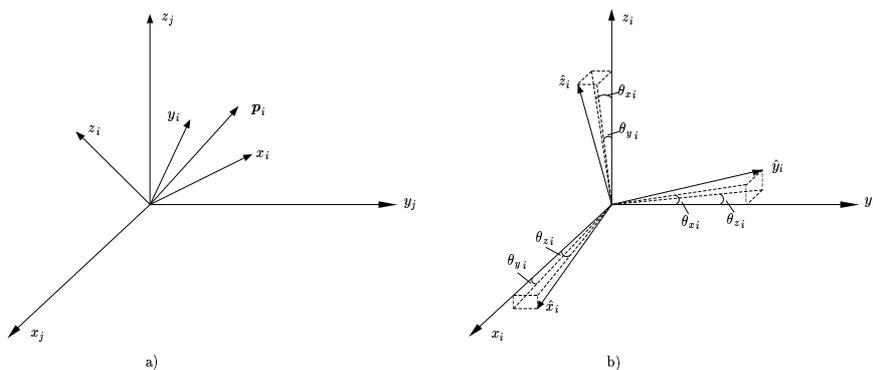


Fig. 1.2 a) Rotation of a coordinate frame; b) Rotation of a coordinate frame due to deformation of the flexible link.

where  $\mathbf{A}_i$  is the standard homogeneous transformation matrix for joint  $i$  due to rigid motion and  $\mathbf{E}_i$  is the homogeneous transformation matrix due to link  $i$  length and deflection. Notice that, even though the superscript is not explicitly indicated, each transformation matrix is referred to the frame determined by the preceding transformation.

The transformation matrix  $\mathbf{A}_i$  can be computed just like in the case of rigid robot manipulators [Sciavicco and Siciliano (2000)]. On the other hand, the transformation matrix  $\mathbf{E}_i$  deserves special attention. Firstly, consider the general form of a rotation matrix  ${}^j\mathbf{R}_i$  between two coordinate frames of common origin (see Fig. 1.2 a)) [Sciavicco and Siciliano (2000)]:

$${}^j\mathbf{R}_i = \begin{bmatrix} \mathbf{x}_i^T \mathbf{x}_j & \mathbf{y}_i^T \mathbf{x}_j & \mathbf{z}_i^T \mathbf{x}_j \\ \mathbf{x}_i^T \mathbf{y}_j & \mathbf{y}_i^T \mathbf{y}_j & \mathbf{z}_i^T \mathbf{y}_j \\ \mathbf{x}_i^T \mathbf{z}_j & \mathbf{y}_i^T \mathbf{z}_j & \mathbf{z}_i^T \mathbf{z}_j \end{bmatrix} = \begin{bmatrix} \cos(\theta_{\mathbf{x}_i \mathbf{x}_j}) & \cos(\theta_{\mathbf{y}_i \mathbf{x}_j}) & \cos(\theta_{\mathbf{z}_i \mathbf{x}_j}) \\ \cos(\theta_{\mathbf{x}_i \mathbf{y}_j}) & \cos(\theta_{\mathbf{y}_i \mathbf{y}_j}) & \cos(\theta_{\mathbf{z}_i \mathbf{y}_j}) \\ \cos(\theta_{\mathbf{x}_i \mathbf{z}_j}) & \cos(\theta_{\mathbf{y}_i \mathbf{z}_j}) & \cos(\theta_{\mathbf{z}_i \mathbf{z}_j}) \end{bmatrix}, \quad (1.9)$$

where  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  denote the unit vectors of the respective axes. Then, the relationship between  ${}^j\mathbf{p}_i$  and  ${}^i\mathbf{p}_i$  is given by

$${}^j\mathbf{p}_i = {}^j\mathbf{R}_i {}^i\mathbf{p}_i. \quad (1.10)$$

From (1.9), it can easily be understood that the knowledge of the angles  $\theta_{\mathbf{x}_i \mathbf{x}_j} \cdots \theta_{\mathbf{z}_i \mathbf{z}_j}$  is enough to compute  ${}^j\mathbf{R}_i$ . With this background, the matrices  $\mathbf{E}_i$  can be determined as follows. Consider Fig. 1.1 again and assume that the  $x$ -axis of frame  $i$  is along the link. Assuming small link deformation [Book (1979); Meirovitch (1967); Meirovitch (1975)],  $\mathbf{E}_i$  can

be expressed as [Book (1979)]

$$\mathbf{E}_i = \begin{bmatrix} 1 & \cos(\pi/2 + \theta_{z_i}) & \cos(\pi/2 - \theta_{y_i}) & l_i + \delta_{x_i} \\ \cos(\pi/2 - \theta_{z_i}) & 1 & \cos(\pi/2 + \theta_{x_i}) & \delta_{y_i} \\ \cos(\pi/2 + \theta_{y_i}) & \cos(\pi/2 - \theta_{x_i}) & 1 & \delta_{z_i} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (1.11)$$

where  $\theta_{x_i}$ ,  $\theta_{y_i}$ ,  $\theta_{z_i}$  are the angles of rotation, and  $\delta_{x_i}$ ,  $\delta_{y_i}$ ,  $\delta_{z_i}$  represent link  $i$  deformation along  $x$ ,  $y$ ,  $z$ , respectively, being  $l_i$  the length of the link without deformation. The angles of rotation  $\theta_{x_i}$ ,  $\theta_{y_i}$ ,  $\theta_{z_i}$  are depicted in Fig. 1.2 b). By taking into account the fact  $\cos(\pi/2 + \alpha) = -\sin(\alpha)$  and assuming small angles, so that  $\sin(\alpha) \approx \alpha$  is valid, the matrix  $\mathbf{E}_i$  can be approximated as

$$\mathbf{E}_i = \begin{bmatrix} 1 & -\theta_{z_i} & \theta_{y_i} & l_i + \delta_{x_i} \\ \theta_{z_i} & 1 & -\theta_{x_i} & \delta_{y_i} \\ -\theta_{y_i} & \theta_{x_i} & 1 & \delta_{z_i} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1.12)$$

By using the homogeneous transformation matrices  $\mathbf{A}_i$  and  $\mathbf{E}_i$ , the position of any point along the robot manipulator can uniquely be determined from Eqs. (1.5), (1.6) and (1.12).

### 1.2.2 Dynamics

In order to obtain a set of differential equations of motion to adequately describe the dynamics of a flexible-link manipulator, the Lagrange's approach can be used. A system with  $n$  generalized coordinates  $q_i$  must satisfy  $n$  differential equations of the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} = u_i \quad i = 1, \dots, n, \quad (1.13)$$

where  $\mathcal{L}$  is the so called Lagrangian which is given by [Wellstead (1979)]

$$\mathcal{L} = \mathcal{T} - \mathcal{U}; \quad (1.14)$$

$\mathcal{T}$  represents the kinetic energy of the system and  $\mathcal{U}$  the potential energy. Also, in (1.13)  $\mathcal{D}$  is the Rayleigh's dissipation function which allows dissipative effects to be included, and  $u_i$  is the generalized force acting on  $q_i$ .

To compute the kinetic energy of the system, the manipulator kinematics can be described systematically as explained in the previous section.

The kinetic energy of link  $i$  link can be expressed as

$$\mathcal{T}_i = \int_{\text{link}_i} d\mathcal{T}_i = \frac{1}{2} \int_{\text{link}_i} \text{Tr} \left( \frac{d\mathbf{r}_i}{dt} \frac{d\mathbf{r}_i^T}{dt} \right) dm, \quad (1.15)$$

which implies that the kinetic energy for the whole system is

$$\mathcal{T} = \sum_{i=1}^n \int_{\text{link}_i} d\mathcal{T}_i = \frac{1}{2} \sum_{i=1}^n \int_{\text{link}_i} \text{Tr} \left( \frac{d\mathbf{r}_i}{dt} \frac{d\mathbf{r}_i^T}{dt} \right) dm; \quad (1.16)$$

$\text{Tr}(\cdot)$  represents the trace operator of a square matrix. By accounting for (1.5), the kinetic energy (1.16) can be written in the form

$$\mathcal{T} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}}, \quad (1.17)$$

where

$$\begin{aligned} \mathbf{q}(t) &\triangleq [\theta_1(t) \cdots \theta_n(t) \quad \delta_{11}(t) \cdots \delta_{1m_1}(t) \cdots \delta_{n1}(t) \cdots \delta_{nm_n}(t)]^T \\ &= [q_{10}(t) \cdots q_{n0}(t) \quad q_{11}(t) \cdots q_{1m_1}(t) \cdots q_{n1}(t) \cdots q_{nm_n}(t)]^T \end{aligned} \quad (1.18)$$

is the vector of generalized coordinates which is formed by the rigid coordinates  $\theta_1 \dots \theta_n$  ( $q_{10} \dots q_{n0}$ ) and the flexible coordinates  $\delta_{11} \dots \delta_{nm_n}$  ( $q_{11} \dots q_{nm_n}$ ), and

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} \mathbf{H}_{\theta\theta}(\mathbf{q}) & \mathbf{H}_{\theta\delta}(\mathbf{q}) \\ \mathbf{H}_{\theta\delta}^T(\mathbf{q}) & \mathbf{H}_{\delta\delta}(\mathbf{q}) \end{bmatrix} \quad (1.19)$$

is the inertia matrix. In particular,  $\mathbf{H}_{\theta\theta}(\mathbf{q})$  is associated to the rigid coordinates,  $\mathbf{H}_{\theta\delta}(\mathbf{q})$  takes into account the relationship between the flexible and rigid coordinates, and  $\mathbf{H}_{\delta\delta}(\mathbf{q})$  is associated to the flexible coordinates.

In order to find an analytical form for the inertia matrix, it is necessary to describe the deflection and torsion of each link as a function of the link coordinates. These can be expressed according to the so-called assumed modes method, i.e. [De Luca and Siciliano (1991); Yuan *et al.* (1993);

Meirovitch (1967); Meirovitch (1975)]

$$\delta_{x_i} = \sum_{j=1}^{m_i} \phi_{x_{ij}} \delta_{ij} \quad \theta_{x_i} = \sum_{j=1}^{m_i} \theta_{x_{ij}} \delta_{ij} \quad (1.20)$$

$$\delta_{y_i} = \sum_{j=1}^{m_i} \phi_{y_{ij}} \delta_{ij} \quad \theta_{y_i} = \sum_{j=1}^{m_i} \theta_{y_{ij}} \delta_{ij} \quad (1.21)$$

$$\delta_{z_i} = \sum_{j=1}^{m_i} \phi_{z_{ij}} \delta_{ij} \quad \theta_{z_i} = \sum_{j=1}^{m_i} \theta_{z_{ij}} \delta_{ij}, \quad (1.22)$$

where  $\phi_{x_{ij}}, \phi_{y_{ij}}, \phi_{z_{ij}}$  ( $\theta_{x_{ij}}, \theta_{y_{ij}}, \theta_{z_{ij}}$ ) are the spatial mode shapes used to model the deflection (torsion) of link  $i$ , being  $m_i$  the number of link coordinates.

From (1.16), the elements of  $\mathbf{H}_{\theta\theta}(\mathbf{q})$  can be computed as

$$h_{\alpha 0 h 0} = \sum_{i=\max\{\alpha, h\}}^n \text{Tr} \left( \left( \hat{\mathbf{T}}_{\alpha-1} \mathbf{U}_{\alpha}{}^{\alpha} \tilde{\mathbf{T}}_i \right) \mathbf{F}_i \left( \hat{\mathbf{T}}_{h-1} \mathbf{U}_h{}^h \tilde{\mathbf{T}}_i \right)^T \right) \quad (1.23)$$

with

$${}^h \mathbf{T}_i \triangleq \mathbf{A}_{h+1} \mathbf{E}_{h+1} \mathbf{A}_{h+2} \mathbf{E}_{h+2} \cdots \mathbf{A}_{i-1} \mathbf{E}_{i-1} \mathbf{A}_i \quad (1.24)$$

$${}^h \tilde{\mathbf{T}}_i \triangleq \mathbf{E}_h{}^h \mathbf{T}_i \quad (1.25)$$

$$\mathbf{U}_h \triangleq \frac{\partial \mathbf{A}_h}{\partial q_{h0}} \quad (1.26)$$

$$\mathbf{F}_i \triangleq \mathbf{C}_i + \sum_{j=1}^{m_i} \delta_{ij} \left( (\mathbf{C}_{ij} + \mathbf{C}_{ij}^T) + \sum_{k=1}^{m_i} \delta_{ik} \mathbf{C}_{ikj} \right) = \mathbf{F}_i^T \quad (1.27)$$

$$\mathbf{C}_i \triangleq \int_{\text{link}_i} [x_i \ y_i \ z_i \ 1]^T [x_i \ y_i \ z_i \ 1] dm \quad (1.28)$$

$$\mathbf{C}_{ij} \triangleq \int_{\text{link}_i} [x_i \ y_i \ z_i \ 1]^T [\phi_{x_{ij}} \ \phi_{y_{ij}} \ \phi_{z_{ij}} \ 0] dm \quad (1.29)$$

$$\mathbf{C}_{ikj} \triangleq \int_{\text{link}_i} [\phi_{x_{ik}} \ \phi_{y_{ik}} \ \phi_{z_{ik}} \ 0]^T [\phi_{x_{ij}} \ \phi_{y_{ij}} \ \phi_{z_{ij}} \ 0] dm = \mathbf{C}_{ijk}^T, \quad (1.30)$$

the elements of  $\mathbf{H}_{\theta\delta}(\mathbf{q})$  can be computed as

$$h_{h0\alpha\beta} = \gamma_{h\alpha} + \sum_{i=\max\{h, \alpha+1\}}^n \text{Tr} \left( \left( \hat{\mathbf{T}}_{h-1} \mathbf{U}_h{}^h \tilde{\mathbf{T}}_i \right) \mathbf{F}_i \left( \mathbf{T}_{\alpha} \mathbf{N}_{\alpha\beta}{}^{\alpha} \mathbf{T}_i \right)^T \right) \quad (1.31)$$

with

$$\gamma_{h\alpha} = \begin{cases} 0 & \text{if } h > \alpha \\ \text{Tr} \left( \left( \hat{\mathbf{T}}_{h-1} \mathbf{U}_h {}^h \tilde{\mathbf{T}}_\alpha \right) \mathbf{D}_{\alpha\beta} \mathbf{T}_\alpha^T \right) & \text{if } h \leq \alpha \end{cases}$$

$$\mathbf{N}_{\alpha\beta} \triangleq \begin{bmatrix} 0 & -\theta_{z\alpha\beta} & \theta_{y\alpha\beta} & \phi_{x\alpha\beta} \\ \theta_{z\alpha\beta} & 0 & -\theta_{x\alpha\beta} & \phi_{y\alpha\beta} \\ -\theta_{y\alpha\beta} & \theta_{x\alpha\beta} & 0 & \phi_{z\alpha\beta} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.32)$$

$$\mathbf{D}_{\alpha\beta} \triangleq \mathbf{C}_{\alpha\beta} + \sum_{k=1}^{m_\alpha} \delta_{\alpha k} \mathbf{C}_{\alpha k\beta}, \quad (1.33)$$

and the elements of  $\mathbf{H}_{\delta\delta}(\mathbf{q})$  can be computed as

$$h_{hk\alpha\beta} = \eta_{h\alpha} + \sum_{i=\max\{h,\alpha\}+1}^n \text{Tr} \left( \left( \mathbf{T}_h \mathbf{N}_{hk} {}^h \mathbf{T}_i \right) \mathbf{F}_i \left( \mathbf{T}_\alpha \mathbf{N}_{\alpha\beta} {}^\alpha \mathbf{T}_i \right)^T \right) \quad (1.34)$$

with

$$\eta_{h\alpha} = \begin{cases} \text{Tr} \left( \mathbf{T}_h \mathbf{C}_{hk\beta} \mathbf{T}_h^T \right) & \text{if } h = \alpha \\ \text{Tr} \left( \left( \mathbf{T}_h \mathbf{N}_{hk} {}^h \mathbf{T}_\alpha \right) \mathbf{D}_{\alpha\beta} \mathbf{T}_\alpha^T \right) & \text{if } h < \alpha \\ \text{Tr} \left( \left( \mathbf{T}_\alpha \mathbf{N}_{\alpha\beta} {}^\alpha \mathbf{T}_h \right) \mathbf{D}_{hk} \mathbf{T}_h^T \right) & \text{if } h > \alpha. \end{cases}$$

Notice that  $\mathbf{H}_{\theta\theta}(\mathbf{q})$  and  $\mathbf{H}_{\delta\delta}(\mathbf{q})$  are symmetric, so that it is only necessary to compute the terms for which  $h \geq \alpha$ .

The next step is to compute the potential energy of the system. In a flexible-link manipulator there are two sources of potential energy: link gravity and link elasticity.

The differential element of gravity potential energy of link  $i$  is given by

$$dU_{gi} = -\mathbf{g}_0^T \mathbf{T}_i {}^i \mathbf{r}_i dm \quad (1.35)$$

where

$$\mathbf{g}_0 = [g_x \ g_y \ g_z \ 0]^T \quad (1.36)$$

is the gravity vector expressed in the base frame. The total gravitational energy is

$$\mathcal{U}_g = -\mathbf{g}_0^T \sum_{i=1}^n \mathbf{T}_i \mathbf{h}_i \quad (1.37)$$

with

$$\mathbf{h}_i \triangleq M_i \mathbf{l}_i + \sum_{k=1}^{m_i} \delta_{ik} \mathbf{s}_{ik} \quad (1.38)$$

$$\mathbf{l}_i = [l_{xi} \ l_{yi} \ l_{zi} \ 1]^T \quad (1.39)$$

$$\mathbf{s}_{ik} = \int_{\text{link}_i} [\phi_{xik} \ \phi_{yik} \ \phi_{z ik} \ 0]^T dm, \quad (1.40)$$

where  $\mathbf{l}_i$  is the vector from joint  $i$  to the center of gravity when link  $i$  is undeformed and  $M_i$  is the total mass of the link.

The strain potential energy associated to the deformation of link  $i$  is given by [Yuan *et al.* (1993)]

$$\mathcal{U}_{ei} = \frac{1}{2} \int_{\text{link}_i} \left( EI_y \left( \frac{\partial^2 \delta_{y_i}}{\partial x_i^2} \right)^2 + EI_z \left( \frac{\partial^2 \delta_{z_i}}{\partial x_i^2} \right)^2 + E_G J_x \left( \frac{\partial \theta_{x_i}}{\partial x_i} \right)^2 \right) dx_i, \quad (1.41)$$

where  $E$  is Young's modulus of elasticity,  $I_y$  ( $I_z$ ) is the area moment of inertia of the link about an axis parallel to  $y$  ( $z$ ) through the center of mass of the cross section,  $E_G$  is the shear modulus, and  $J_x$  is the polar area moment of inertia of the link about the center of mass. The integration in (1.41) is carried out along  $x$ -axis. Notice that in (1.41) the compression in the  $x$  direction has been assumed to be negligible. In view of (1.20)–(1.22), Eq. (1.41) can be rewritten as

$$\mathcal{U}_{ei} = \frac{1}{2} \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} (k_{y_{ijk}} + k_{z_{ijk}} + k_{x_{ijk}}), \quad (1.42)$$

where  $k_{x_{ijk}}$ ,  $k_{y_{ijk}}$ ,  $k_{z_{ijk}}$  are the stiffness coefficients given by

$$k_{x_{ijk}} = \int_{\text{link}_i} E_G J_x \frac{d\theta_{x_{ij}}}{dx_i} \frac{d\theta_{x_{ik}}}{dx_i} dx_i \quad (1.43)$$

$$k_{y_{ijk}} = \int_{\text{link}_i} EI_y \frac{d^2 \phi_{y_{ij}}}{dx_i^2} \frac{d^2 \phi_{y_{ik}}}{dx_i^2} dx_i \quad (1.44)$$

$$k_{z_{ijk}} = \int_{\text{link}_i} EI_z \frac{d^2 \phi_{z_{ij}}}{dx_i^2} \frac{d^2 \phi_{z_{ik}}}{dx_i^2} dx_i. \quad (1.45)$$

The total elastic energy is

$$U_e = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \delta_{ij} \delta_{ik} k_{ijk} \quad (1.46)$$

with

$$k_{ijk} \triangleq k_{y_{ijk}} + k_{z_{ijk}} + k_{x_{ijk}} = k_{ikj} \quad (1.47)$$

or in matrix form:

$$U_e = \frac{1}{2} \boldsymbol{\delta}^T \mathbf{K} \boldsymbol{\delta} = \frac{1}{2} \mathbf{q}^T \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{bmatrix} \mathbf{q} \triangleq \frac{1}{2} \mathbf{q}^T \mathbf{K}_e \mathbf{q}, \quad (1.48)$$

where

$$\boldsymbol{\delta} \triangleq [\delta_{11} \cdots \delta_{1m_1} \cdots \delta_{n1} \cdots \delta_{nm_n}]^T \quad (1.49)$$

is the vector of flexible coordinates, and

$$\mathbf{K} = \begin{bmatrix} k_{111} & \cdots & k_{11m_1} & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ k_{1m_11} & \cdots & k_{1m_1m_1} & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & k_{n11} & \cdots & k_{n1m_n} \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \cdots & & \cdots & k_{nm_n1} & \cdots & k_{nm_nm_n} \end{bmatrix} \quad (1.50)$$

is called the stiffness matrix.

By taking (1.14), (1.17), (1.37) and (1.48) into account, the Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}} - \mathbf{q}^T \mathbf{K}_e \mathbf{q}) + \mathbf{g}_0^T \sum_{i=1}^n \mathbf{T}_i \mathbf{h}_i. \quad (1.51)$$

In order to model link modal damping and joint viscous friction, the Rayleigh's dissipation function can be employed with the use of a matrix  $\mathbf{D}$  so that [Meirovitch (1967)]:

$$\mathcal{D} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{D} \dot{\mathbf{q}}. \quad (1.52)$$

Expressing (1.13) in matrix form yields

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T - \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T + \left( \frac{\partial \mathcal{D}}{\partial \dot{\mathbf{q}}} \right)^T = \mathbf{u} \quad (1.53)$$

with

$$\mathbf{u} \triangleq \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{0} \end{bmatrix}, \quad (1.54)$$

where  $\boldsymbol{\tau}$  is the  $n \times 1$  vector of the joint torques, and  $\mathbf{0}$  denotes an  $m \times 1$  vector of null elements ( $m = m_1 + \dots + m_n$ ) accounting for the fact that no generalized force acts on the flexible coordinates  $\boldsymbol{\delta}$  as long as clamped boundary conditions at the joint side are assumed.

Then, substituting (1.52) into (1.53) and accounting for (1.51) leads to

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_e \mathbf{q} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}, \quad (1.55)$$

where

$$\mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \quad (1.56)$$

is the vector of Coriolis and centrifugal forces with

$$c_{rs\alpha\beta} = \sum_{i=1}^n \sum_{j=0}^{m_i} c_{ij\alpha\beta rs} \dot{q}_{ij} \quad (1.57)$$

$$c_{ij\alpha\beta rs} \triangleq \frac{1}{2} \left( \frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} + \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} - \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}} \right) \quad (1.58)$$

$$c_{ij\alpha\beta rs} = c_{\alpha\beta ijrs} \quad (1.59)$$

$$g_{rs} = \begin{cases} -\mathbf{g}_0^T \sum_{i=r}^n \frac{\partial \mathbf{T}_i}{\partial q_{rs}} \mathbf{h}_i & \text{if } s = 0 \\ -\mathbf{g}_0^T \left( \sum_{i=r+1}^n \frac{\partial \mathbf{T}_i}{\partial q_{rs}} \mathbf{h}_i + \mathbf{T}_r \mathbf{s}_{rs} \right) & \text{if } s \neq 0. \end{cases} \quad (1.60)$$

### 1.2.3 Model Properties

In this section some properties of model (1.55) are presented. Many of them are rather physical properties while other arise from the procedure used to derive the dynamic model of the manipulator. Several of these properties are similar to those of rigid manipulators. As a matter of fact,

the properties presented in the following apply to rigid manipulators just by letting the link deformation be zero.

Hereafter, the Euclidean norm for vectors is used, i.e.

$$\|\mathbf{q}\| \triangleq \sqrt{\sum_{i=1}^n q_i^2}. \quad (1.61)$$

The norm of a matrix  $\mathbf{A}$  is the corresponding induced norm

$$\|\mathbf{A}\| \triangleq \sqrt{\lambda_{\max}(\mathbf{A}^T \mathbf{A})}, \quad (1.62)$$

where  $\lambda_{\max}(\cdot)$  ( $\lambda_{\min}(\cdot)$ ) denotes the largest (smallest) eigenvalue of a matrix. Since all norms in  $\mathfrak{R}^n$  are equivalent [Desoer and Vidyasagar (1975)], the results presented in this section are valid for any norm in  $\mathfrak{R}^n$ .

A well-known property of the dynamic model of a robot manipulator is the following one.

**Property 1.1** The inertia matrix  $\mathbf{H}(\mathbf{q})$  is symmetric positive definite.

*Proof:* It can be seen directly from (1.19), (1.23), (1.31) and (1.34) that  $\mathbf{H}(\mathbf{q})$  is symmetric. Since the kinetic energy of any mechanical system can be zero if and only if the system is in a steady state, and otherwise it is always greater than zero, it follows from (1.17) that  $\mathbf{H}(\mathbf{q})$  is positive definite. △

The next property is very important. It is related to the passive structure of robot manipulators and it is frequently used in the proof of many control schemes. It gives a relationship between the inertia matrix  $\mathbf{H}(\mathbf{q})$  and the matrix  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  employed to compute the vector of Coriolis and centrifugal torques.

**Property 1.2** The matrix  $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  is skew symmetric.

*Proof:* Every element of  $\dot{\mathbf{H}}(\mathbf{q})$  satisfies

$$\dot{h}_{rs\alpha\beta} = \sum_{i=1}^n \sum_{j=0}^{m_i} \frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} \dot{q}_{ij}. \quad (1.63)$$

By taking (1.57) and (1.58) into account, the elements of  $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$  can be

computed as

$$\begin{aligned}
 n_{rs\alpha\beta} &\triangleq \dot{h}_{rs\alpha\beta} - 2c_{rs\alpha\beta} \\
 &= \sum_{i=1}^n \sum_{j=0}^{m_i} \left( \frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} - \left( \frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} + \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} - \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}} \right) \right) \dot{q}_{ij} \\
 &= \sum_{i=1}^n \sum_{j=0}^{m_i} \left( \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}} - \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} \right) \dot{q}_{ij}.
 \end{aligned} \tag{1.64}$$

Since  $h_{rs\alpha\beta} = h_{\alpha\beta rs}$ , the property holds true.  $\triangle$

Note that Property 1.2 has been proven using the definition of  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  which is in terms of the Christoffel symbols. Since there are many possible definitions for  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ , it is worth pointing out that

$$\dot{\mathbf{q}}^T (\dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{q}} = 0 \tag{1.65}$$

is always true no matter what definition of  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  is used [Ortega and Spong (1989)]. To show this, rewrite (1.53) and (1.55) as

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T - \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T = \boldsymbol{\psi} = \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{K}_e \mathbf{q} + \mathbf{g}(\mathbf{q}) \tag{1.66}$$

with

$$\boldsymbol{\psi} \triangleq \mathbf{u} - \mathbf{D} \dot{\mathbf{q}}. \tag{1.67}$$

The Hamiltonian of the system is given by [Ortega and Spong (1989); Greenwood (1977)]

$$\mathcal{H} = \boldsymbol{\pi}^T \dot{\mathbf{q}} - \mathcal{L}, \tag{1.68}$$

where the generalized momentum  $\boldsymbol{\pi}$  is defined as

$$\boldsymbol{\pi} = \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T. \tag{1.69}$$

On the other hand, by using (1.51), (1.68) and (1.69), the Hamiltonian can be expressed as the sum of the kinetic and the potential energy of the system, i.e.

$$\mathcal{H} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \mathbf{q}^T \mathbf{K}_e \mathbf{q} + \mathcal{U}_g = \mathcal{T} + \mathcal{U}, \tag{1.70}$$

while the Hamilton's equations are given by

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad (1.71)$$

$$\dot{\pi}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \psi_i \quad i = 1, \dots, n+m. \quad (1.72)$$

By employing (1.71) and (1.72), the derivative of  $\mathcal{H}$  can be computed as

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{n+m} \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n+m} \frac{\partial \mathcal{H}}{\partial \pi_i} \dot{\pi}_i = \dot{\mathbf{q}}^T \boldsymbol{\psi}. \quad (1.73)$$

Eqs. (1.66) and (1.70) can be used as well to obtain  $d\mathcal{H}/dt$ , i.e.

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{H}}(\mathbf{q}) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{K}_e \mathbf{q} + \dot{\mathbf{q}}^T \left( \frac{\partial \mathcal{U}_g}{\partial \mathbf{q}} \right)^T \\ &= \dot{\mathbf{q}}^T \boldsymbol{\psi} + \frac{1}{2} \dot{\mathbf{q}}^T \left( \dot{\mathbf{H}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \right) \dot{\mathbf{q}}. \end{aligned} \quad (1.74)$$

By comparing (1.73) and (1.74), one can conclude that (1.65) is valid for any possible choice of  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ .

The following property holds for the vector  $\mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}})$  of Coriolis and centrifugal torques.

**Property 1.3** The vector  $\mathbf{h}_c(\mathbf{q}, \dot{\mathbf{q}})$  of Coriolis and centrifugal torques satisfies the equalities:

$$\mathbf{h}_c(\mathbf{q}, \mathbf{x}, \mathbf{y}) = \mathbf{C}(\mathbf{q}, \mathbf{x})\mathbf{y} = \mathbf{C}(\mathbf{q}, \mathbf{y})\mathbf{x} = \mathbf{h}_c(\mathbf{q}, \mathbf{y}, \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathfrak{R}^{n+m}. \quad (1.75)$$

*Proof:* The element  $rs$  of vector  $\mathbf{h}_c(\mathbf{q}, \mathbf{x}, \mathbf{y})$  can be expressed as (see (1.56) and (1.57))

$$\begin{aligned} h_{c_{rs}}(\mathbf{q}, \mathbf{x}, \mathbf{y}) &= \sum_{\alpha=1}^n \sum_{\beta=0}^{m_\alpha} \left( \sum_{i=1}^n \sum_{j=0}^{m_i} c_{ij\alpha\beta rs} x_{ij} \right) y_{\alpha\beta} \\ &= \sum_{i=1}^n \sum_{j=0}^{m_i} \left( \sum_{\alpha=1}^n \sum_{\beta=0}^{m_\alpha} c_{ij\alpha\beta rs} y_{\alpha\beta} \right) x_{ij} \\ &= \sum_{i=1}^n \sum_{j=0}^{m_i} \left( \sum_{\alpha=1}^n \sum_{\beta=0}^{m_\alpha} c_{\alpha\beta ijrs} y_{\alpha\beta} \right) x_{ij} = h_{c_{rs}}(\mathbf{q}, \mathbf{y}, \mathbf{x}). \end{aligned} \quad (1.76)$$

△

Due to the orthogonality of the modes shapes of a flexible-link manipulator, the following property can be obtained.

Property 1.4 The stiffness matrix  $\mathbf{K}$  is diagonal and positive definite and satisfies

$$\lambda_{\min}(\mathbf{K})\|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{K} \mathbf{x} \leq \lambda_{\max}(\mathbf{K})\|\mathbf{x}\|^2 \quad \forall \mathbf{x} \in \mathfrak{R}^m. \quad (1.77)$$

*Proof:* To prove the positive definiteness of the stiffness matrix, the definition of its elements can be used (see (1.43)–(1.45) and (1.47)). Since the mode shapes are orthogonal [Meirovitch (1967)], the result of the integrals must be zero if  $j \neq k$  and positive otherwise. Eq. (1.77) follows from the fact that  $\mathbf{K}$  is positive definite.  $\triangle$

Regarding the matrix  $\mathbf{D}$  of link modal damping and joint viscous friction, the following property can be established.

Property 1.5 The matrix  $\mathbf{D}$  is diagonal positive semidefinite and satisfies

$$\lambda_{\min}(\mathbf{D})\|\mathbf{x}\|^2 \leq \mathbf{x}^T \mathbf{D} \mathbf{x} \leq \lambda_{\max}(\mathbf{D})\|\mathbf{x}\|^2 \quad \forall \mathbf{x} \in \mathfrak{R}^{n+m}. \quad (1.78)$$

*Proof:*  $\mathbf{D}$  is positive semidefinite because it is defined on the basis of the Rayleigh's dissipation function (see (1.52)). Assuming it to be diagonal is actually a special but very important and common case of the definition of the Rayleigh's dissipation function [Meirovitch (1967)].  $\triangle$

Finding bounds on the norms of the matrices of model (1.55) plays an important role in the control of robot manipulators because such bounds are helpful for design of many control schemes. Norm bounds are especially advantageous when Lyapunov theory is used. As a matter of fact, for any mechanical system, the vectors  $\mathbf{q}$  and  $\dot{\mathbf{q}}$  are bounded. Taking only link deformation into account and in view of the small deformation assumption, the potential energy due to elasticity cannot be infinite, i.e.  $U_e < \infty$  [De Luca and Siciliano (1993b); De Luca and Panzieri (1994)], so that it is possible to find a bound for the vector of link coordinates  $\boldsymbol{\delta}$ .

Property 1.6 The norm of  $\boldsymbol{\delta}$  is bounded by

$$\|\boldsymbol{\delta}\| \leq \sqrt{\frac{2U_{e,\max}}{\lambda_{\min}(\mathbf{K})}} \triangleq \bar{\delta}, \quad (1.79)$$

where  $U_{e,\max}$  is the maximum of the link strain potential energy.

*Proof:* In view of the assumption of small link deformation, there must be a maximum link strain potential energy. Directly from (1.48) it is

$$\boldsymbol{\delta}^T \mathbf{K} \boldsymbol{\delta} \leq 2U_{e,\max} < \infty, \quad (1.80)$$

from which Property 1.6 follows by taking Property 1.4 into account.  $\triangle$

Notice that Property 1.6 means that  $\boldsymbol{\delta}$  belongs to a set  $\Delta$  whose elements are bounded. For simplicity, every vector  $\mathbf{q} \in \mathfrak{R}^n \times \Delta \subset \mathfrak{R}^{n+m}$  will be assumed to belong to a set  $\mathcal{Q}^{n+m}$ .

The next four properties are related to the inertia matrix and can easily be derived from Property 1.1.

Property 1.7 The inertia matrix  $\mathbf{H}(\mathbf{q})$  satisfies

$$\lambda_{\min}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \leq \mathbf{y}^T \mathbf{H}(\mathbf{q}) \mathbf{y} \leq \lambda_{\max}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \quad \forall \mathbf{y} \in \mathfrak{R}^{n+m}. \quad (1.81)$$

*Proof:* Since  $\mathbf{H}(\mathbf{q})$  is positive definite, each vector  $\mathbf{y}$  in  $\mathfrak{R}^{n+m}$  can be expressed in terms of an orthonormal basis  $(\mathbf{y}_1, \dots, \mathbf{y}_{n+m})$  as

$$\mathbf{y} = \sum_{i=1}^{n+m} c_i \mathbf{y}_i, \quad (1.82)$$

implying that

$$\mathbf{y}^T \mathbf{H}(\mathbf{q}) \mathbf{y} = c_1^2 \lambda_1(\mathbf{H}(\mathbf{q})) + \dots + c_{n+m}^2 \lambda_{n+m}(\mathbf{H}(\mathbf{q})) \quad (1.83)$$

$$\mathbf{y}^T \mathbf{y} = \|\mathbf{y}\|^2 = c_1^2 + \dots + c_{n+m}^2, \quad (1.84)$$

from which (1.81) follows.  $\triangle$

Property 1.8 The matrix  $\mathbf{H}^{-1}(\mathbf{q})$  exists and satisfies

$$\lambda_{\max}^{-1}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \leq \mathbf{y}^T \mathbf{H}^{-1}(\mathbf{q}) \mathbf{y} \leq \lambda_{\min}^{-1}(\mathbf{H}(\mathbf{q}))\|\mathbf{y}\|^2 \quad \forall \mathbf{y} \in \mathfrak{R}^{n+m}. \quad (1.85)$$

*Proof:* This property follows directly from Property 1.7.  $\triangle$

Property 1.9 The inertia matrix satisfies

$$\lambda_h \leq \|\mathbf{H}(\mathbf{q})\| \leq \lambda_H < \infty. \quad (1.86)$$

*Proof:* Since the vector of generalized coordinates is bounded, i.e.  $\|\mathbf{q}\| < \infty$ , it is easy to see from (1.81) that

$$\lambda_h = \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}(\mathbf{H}(\mathbf{q})) \quad (1.87)$$

$$\lambda_H = \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}(\mathbf{H}(\mathbf{q})). \quad (1.88)$$

△

Property 1.10 The inverse of the inertia matrix satisfies

$$\sigma_h \leq \|\mathbf{H}^{-1}(\mathbf{q})\| \leq \sigma_H < \infty. \quad (1.89)$$

*Proof:* The proof is the same as in Property 1.9 with

$$\sigma_h = \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}^{-1}(\mathbf{H}(\mathbf{q})) \quad (1.90)$$

$$\sigma_H = \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}^{-1}(\mathbf{H}(\mathbf{q})). \quad (1.91)$$

△

It is easy to recognize that Properties 1.7 to 1.10 are closely related. Of course, by taking only Property 1.7 into account, it is not difficult to develop the other three properties. These properties are very important because many Lyapunov functions employed to prove the stability of a control approach make use of the inertia matrix and its boundedness properties.

Since  $\mathbf{q}$  is bounded, it is possible to find the following bound for the matrix  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ .

Property 1.11 The matrix  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  satisfies

$$\|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\| \leq k_c \|\dot{\mathbf{q}}\|. \quad (1.92)$$

*Proof:* From (1.57), it can be seen that matrix  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  can be written as

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \mathbf{C}_{ij}(\mathbf{q}) \dot{q}_{ij}, \quad (1.93)$$

so that each element of matrix  $\mathbf{C}_{ij}(\mathbf{q})$  is given by

$$\frac{\partial h_{rs\alpha\beta}}{\partial q_{ij}} + \frac{\partial h_{rsij}}{\partial q_{\alpha\beta}} - \frac{\partial h_{ij\alpha\beta}}{\partial q_{rs}}.$$

Computing the norm of  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  leads to

$$\begin{aligned} \|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\| &= \frac{1}{2} \left\| \sum_{i=1}^n \sum_{j=0}^{m_i} \mathbf{C}_{ij}(\mathbf{q}) \dot{q}_{ij} \right\| & (1.94) \\ &\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q}) \dot{q}_{ij}\| \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q})\| |\dot{q}_{ij}| \\ &\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q})\| \|\dot{\mathbf{q}}\|. \end{aligned}$$

With

$$k_c \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \sum_{i=1}^n \sum_{j=0}^{m_i} \|\mathbf{C}_{ij}(\mathbf{q})\|, \quad (1.95)$$

Property 1.11 follows.  $\triangle$

It is worth noticing that Property 1.11 applies to every vector  $\mathbf{y} \in \mathfrak{R}^{n+m}$ .

Because the vector of gravitational torques  $\mathbf{g}(\mathbf{q})$  is only a function of  $\mathbf{q}$ , one can find a bound related to it as well.

**Property 1.12** The vector of gravitational torques  $\mathbf{g}(\mathbf{q})$  is bounded by a constant  $\sigma_g > 0$ , i.e.

$$\|\mathbf{g}(\mathbf{q})\| \leq \sigma_g. \quad (1.96)$$

*Proof:* Since

$$\|\mathbf{g}(\mathbf{q})\| = \sqrt{\sum_{r=1}^n \sum_{s=0}^{m_r} g_{rs}^2}, \quad (1.97)$$

it should be proven that each term  $g_{rs}$  is bounded. By recalling that  $\mathbf{q}$  is bounded and taking (1.60) into account, it can easily be seen that each  $g_{rs}$  is bounded, so that (1.96) follows.  $\triangle$

### 1.3 Nonlinear Control

Perhaps the most relevant issue in controlling flexible robot manipulators resides in the fact that there are fewer inputs than degrees of freedom, thus making impossible to choose a desired trajectory for each generalized coordinate. An obvious solution to this problem consists in renouncing to choose an arbitrary trajectory for the link coordinates and select only one for the joint coordinates. By doing so, a new difficulty might arise: since the desired trajectory for the link coordinates is computed to achieve some control goal for the joint coordinates, it is necessary to guarantee, at least, that it remains bounded. In Section 1.3.1, some stability theorems related to the boundedness of the state of a system are presented. Although they are mainly used to design robust controllers, it will be shown in Sections 1.3.2 and 1.3.3 that they can be employed to design controllers for flexible robot manipulators as well. Section 1.3.2 presents a controller based on the well-known approach with filtered reference velocity at the basis of passivity. The damping problem is specifically treated in Section 1.3.3, where a solution to increase damping is proposed.

#### 1.3.1 State Boundedness

Consider a dynamical continuous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1.98)$$

where  $\mathbf{f}(\cdot) : \mathfrak{R}^n \times \mathfrak{R} \rightarrow \mathfrak{R}^n$  is known and  $\mathbf{x}$  is an  $n \times 1$  vector.

**Definition 1.1** [Leitmann (1981)] Given a solution  $\mathbf{x}(\cdot) : [t_0, t_1] \rightarrow \mathfrak{R}^n$  of (1.98), we say that it is *uniformly bounded* (UB) if there is a positive constant  $d(\mathbf{x}_0) < \infty$ , possibly dependent on  $\mathbf{x}_0$  but not on  $t_0$ , such that

$$\|\mathbf{x}(t)\| \leq d(\mathbf{x}_0) \quad \forall t \in [t_0, t_1]. \quad (1.99)$$

△

**Definition 1.2** [Leitmann (1981)] Given a solution  $\mathbf{x}(\cdot) : [t_0, \infty) \rightarrow \mathfrak{R}^n$  of (1.98), we say that it is *uniformly ultimately bounded* (UUB) with respect to the set  $S$  if there is a non-negative constant  $T(\mathbf{x}_0, S) < \infty$ , possibly dependent on  $\mathbf{x}_0$  and  $S$  but not on  $t_0$ , such that

$$\mathbf{x}(t) \in S \quad \forall t \geq t_0 + T(\mathbf{x}_0, S). \quad (1.100)$$

△

Notice that if the constants  $d(\mathbf{x}_0)$  in Definition 1.1 and  $T(\mathbf{x}_0, S)$  in Definition 1.2 are independent of  $\mathbf{x}_0$ , then we say that the solution is *globally uniformly bounded* (GUB) or *globally uniformly ultimately bounded* (GUUB). The next two theorems are related to the stability of a system of form (1.98).

**Theorem 1.1** [Dawson et al. (1991)] *Let a continuous system be described by (1.98) and let  $V(\mathbf{x}, t)$  be an associated Lyapunov function with the following properties:*

$$\lambda_1 \|\mathbf{x}(t)\|^2 \leq V(\mathbf{x}, t) \leq \lambda_2 \|\mathbf{x}(t)\|^2 \quad (1.101)$$

$$\dot{V}(\mathbf{x}, t) \leq -\lambda_3 \|\mathbf{x}(t)\|^2 + \epsilon e^{-\beta t} \quad (1.102)$$

$\forall (\mathbf{x}, t) \in \mathfrak{R}^n \times \mathfrak{R}$ , where  $\lambda_1, \lambda_2, \lambda_3$  and  $\epsilon$  are positive scalar constants. If  $\beta = 0$  in (1.102), then the state  $\mathbf{x}$  is GUUB in the sense that

$$\|\mathbf{x}(t)\| \leq \left( \frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1 \lambda} (1 - e^{-\lambda t}) \right)^{1/2}, \quad (1.103)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are defined in (1.101) and (1.102) and  $\lambda = \lambda_3/\lambda_2$ . If  $\beta > 0$  in (1.102), then the state  $\mathbf{x}$  is globally exponentially stable (GES) in the sense that

$$\|\mathbf{x}(t)\| \leq \begin{cases} \left( \frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1} t e^{-\lambda t} \right)^{1/2} & \text{if } \beta = \lambda \\ \left( \frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1 (\lambda - \beta)} (e^{-\beta t} - e^{-\lambda t}) \right)^{1/2} & \text{if } \beta \neq \lambda \end{cases} \quad (1.104)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are defined in (1.101) and (1.102) and  $\lambda = \lambda_3/\lambda_2$ .  $\triangle$

**Theorem 1.2** [Dawson et al. (1991)] *Let a continuous system be described by (1.98) and let  $V(\mathbf{x}, t)$  be an associated Lyapunov function with the following properties:*

$$\lambda_1 \|\mathbf{x}(t)\|^2 \leq V(\mathbf{x}, t) \leq \lambda_2 \|\mathbf{x}(t)\|^2 \quad (1.105)$$

$$\dot{V}(\mathbf{x}, t) \leq -\lambda_3 \|\mathbf{x}(t)\|^2 + \|\mathbf{x}(t)\| \sigma e^{-\gamma t} \quad (1.106)$$

$\forall (\mathbf{x}, t) \in \mathfrak{R}^n \times \mathfrak{R}$ , where  $\lambda_1, \lambda_2, \lambda_3$  and  $\sigma$  are positive scalar constants. If  $\gamma = 0$  in (1.106), then the state  $\mathbf{x}$  is GUUB in the sense that

$$\|\mathbf{x}(t)\| \leq \frac{1}{\sqrt{\lambda_1}} \left( \sqrt{\lambda_2} \|\mathbf{x}_0\| e^{-\lambda t/2} + \frac{\zeta}{\lambda} (1 - e^{-\lambda t/2}) \right), \quad (1.107)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are defined in (1.105) and (1.106),  $\zeta = \sigma/\sqrt{\lambda_1}$  and  $\lambda = \lambda_3/\lambda_2$ . If  $\gamma > 0$ , then the state  $\mathbf{x}$  is GES in the sense that

$$\|\mathbf{x}(t)\| \leq \begin{cases} \frac{1}{\sqrt{\lambda_1}} \left( \sqrt{\lambda_2} \|\mathbf{x}_0\| e^{-\lambda t/2} + \frac{\zeta}{2} t e^{-\lambda t/2} \right) & \text{if } \lambda = 2\gamma \\ \frac{1}{\sqrt{\lambda_1}} \left( \sqrt{\lambda_2} \|\mathbf{x}_0\| e^{-\lambda t/2} + \frac{\zeta}{\lambda - 2\gamma} \left( e^{-\gamma t} - e^{-\lambda t/2} \right) \right) & \text{if } \lambda \neq 2\gamma \end{cases} \quad (1.108)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are defined in (1.105) and (1.106),  $\zeta = \sigma/\sqrt{\lambda_1}$  and  $\lambda = \lambda_3/\lambda_2$ .  $\triangle$

### 1.3.2 Tracking Control

In this section, the tracking control problem of flexible manipulators is studied. Consider model (1.55) again:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}_e\mathbf{q} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \quad (1.109)$$

with  $\mathbf{u}$  as in (1.54).

Given a bounded continuous desired trajectory  $\mathbf{q}_d$ , with bounded velocity  $\dot{\mathbf{q}}_d$  and acceleration  $\ddot{\mathbf{q}}_d$ , the tracking errors  $\tilde{\mathbf{q}}$  and  $\dot{\tilde{\mathbf{q}}}$  can be defined as

$$\tilde{\mathbf{q}} \triangleq \mathbf{q} - \mathbf{q}_d \quad (1.110)$$

$$\dot{\tilde{\mathbf{q}}} \triangleq \dot{\mathbf{q}} - \dot{\mathbf{q}}_d. \quad (1.111)$$

Before the controller can be introduced, the following definitions are

necessary:

$$\dot{\mathbf{q}}_r \triangleq \dot{\mathbf{q}}_d - \mathbf{\Lambda}\tilde{\mathbf{q}} \triangleq \begin{bmatrix} \dot{\boldsymbol{\theta}}_r \\ \dot{\boldsymbol{\delta}}_r \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_d - \mathbf{\Lambda}_\theta \tilde{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\delta}}_d - \mathbf{\Lambda}_\delta \tilde{\boldsymbol{\delta}} \end{bmatrix} \quad (1.112)$$

$$\mathbf{s} \triangleq \dot{\mathbf{q}} - \dot{\mathbf{q}}_r = \dot{\tilde{\mathbf{q}}} + \mathbf{\Lambda}\tilde{\mathbf{q}} \triangleq \begin{bmatrix} \mathbf{s}_\theta \\ \mathbf{s}_\delta \end{bmatrix} \quad (1.113)$$

$$\mathbf{\Lambda} \triangleq \begin{bmatrix} \mathbf{\Lambda}_\theta & \mathbf{O} \\ \mathbf{O} & \mathbf{\Lambda}_\delta \end{bmatrix} \quad (1.114)$$

$$\mathbf{K}_p \triangleq \begin{bmatrix} \mathbf{K}_{p\theta} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{p\delta} \end{bmatrix} \quad (1.115)$$

$$\mathbf{D} \triangleq \begin{bmatrix} \mathbf{D}_\theta & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_\delta \end{bmatrix} \quad (1.116)$$

$$\mathbf{K}_{pD} \triangleq \mathbf{K}_p + \mathbf{D} = \begin{bmatrix} \mathbf{K}_{p\theta} + \mathbf{D}_\theta & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{p\delta} + \mathbf{D}_\delta \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{K}_{pD\theta} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{pD\delta} \end{bmatrix} \quad (1.117)$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \begin{bmatrix} \mathbf{C}_{\theta\theta}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{C}_{\theta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} \quad (1.118)$$

$$\mathbf{g}(\mathbf{q}) \triangleq \begin{bmatrix} \mathbf{g}_\theta \\ \mathbf{g}_\delta \end{bmatrix}, \quad (1.119)$$

where  $\mathbf{\Lambda}$  and  $\mathbf{K}_p$  are diagonal and positive definite.

The proposed controller is given by

$$\boldsymbol{\tau} = \mathbf{H}_{\theta\theta}\ddot{\boldsymbol{\theta}}_r + \mathbf{H}_{\theta\delta}\ddot{\boldsymbol{\delta}}_r + \mathbf{C}_{\theta\theta}\dot{\boldsymbol{\theta}}_r + \mathbf{C}_{\theta\delta}\dot{\boldsymbol{\delta}}_r + \mathbf{D}_\theta\dot{\boldsymbol{\theta}}_r + \mathbf{g}_\theta - \mathbf{K}_{p\theta}\mathbf{s}_\theta, \quad (1.120)$$

while the desired trajectory  $\boldsymbol{\delta}_d$  is computed from

$$\ddot{\boldsymbol{\delta}}_d = \mathbf{\Lambda}_\delta \dot{\tilde{\boldsymbol{\delta}}} - \mathbf{H}_{\delta\delta}^{-1}(\mathbf{C}_{\delta\delta}\dot{\boldsymbol{\delta}}_r + \mathbf{D}_\delta\dot{\boldsymbol{\delta}}_r - \mathbf{K}_{p\delta}\mathbf{s}_\delta + \mathbf{K}\boldsymbol{\delta}_d + \mathbf{H}_{\theta\delta}^T\ddot{\boldsymbol{\theta}}_r + \mathbf{C}_{\delta\theta}\dot{\boldsymbol{\theta}}_r + \mathbf{g}_\delta) \quad (1.121)$$

with initial condition

$$\boldsymbol{\delta}_d(0) = \dot{\boldsymbol{\delta}}_d(0) = \mathbf{0}. \quad (1.122)$$

It is not difficult to realize that Eqs. (1.120) and (1.121) are equivalent to

$$\mathbf{u} = \mathbf{H}(\mathbf{q})\ddot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_r + \mathbf{K}_e\mathbf{q}_d + \mathbf{D}\dot{\mathbf{q}}_r + \mathbf{g}(\mathbf{q}) - \mathbf{K}_p\mathbf{s}.$$

Notice that in contrast with the control given in [Lammerts *et al.* (1995)], the term  $\mathbf{K}_e$  is not multiplied by  $\mathbf{q}_r$  but by  $\mathbf{q}_d$ . It will be proven that  $\boldsymbol{\delta}_d$  and  $\dot{\boldsymbol{\delta}}_d$  computed from (1.121)–(1.122) are bounded.

By substituting (1.123) in (1.109), the error dynamics can be expressed as

$$\mathbf{H}(\mathbf{q})\dot{\mathbf{s}} = -(\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{s} + \mathbf{K}_e\tilde{\mathbf{q}} + \mathbf{K}_{pD}\mathbf{s}). \quad (1.123)$$

In order to simplify the stability discussion, the state  $\mathbf{x}$  is introduced:

$$\mathbf{x} \triangleq \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix}. \quad (1.124)$$

By taking advantage of the fact that for any mechanical system the velocity vector  $\dot{\mathbf{q}}$  is bounded, the following theorem establishes the boundedness of  $\delta_d$  and  $\dot{\delta}_d$  and the stability of  $\mathbf{x}$ .

**Theorem 1.3** *Given a bounded continuous desired trajectory  $\theta_d$  with bounded velocity and acceleration, if  $\|\dot{\mathbf{q}}\| \leq v_m$ , where  $v_m$  is a positive scalar constant, then the desired trajectory  $\delta_d$  and  $\dot{\delta}_d$  given by (1.121) and (1.122) remains bounded. In addition, by using the input vector (1.120), the equilibrium point  $\mathbf{x} = \mathbf{0}$  of (1.123) is globally asymptotically stable.*

*Proof:*

a) In order to prove the asymptotic stability of  $\mathbf{x}$ , consider the Lyapunov function  $V(\mathbf{x}, t) = V(\mathbf{x})$ :

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{s}^T\mathbf{H}(\mathbf{q})\mathbf{s} + \tilde{\mathbf{q}}^T\left(\Lambda\mathbf{K}_{pD} + \frac{1}{2}\mathbf{K}_e\right)\tilde{\mathbf{q}}.$$

Its derivative along (1.123) is

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \frac{1}{2}\mathbf{s}^T\dot{\mathbf{H}}(\mathbf{q})\mathbf{s} + \dot{\tilde{\mathbf{q}}}^T\Lambda\mathbf{K}_{pD}\tilde{\mathbf{q}} + \tilde{\mathbf{q}}^T\Lambda\mathbf{K}_{pD}\dot{\tilde{\mathbf{q}}} + \dot{\tilde{\mathbf{q}}}^T\mathbf{K}_e\tilde{\mathbf{q}} \quad (1.125) \\ &\quad - \mathbf{s}^T(\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{s} + \mathbf{K}_e\tilde{\mathbf{q}} + \mathbf{K}_{pD}\mathbf{s}) \\ &= -\mathbf{s}^T\mathbf{K}_{pD}\mathbf{s} + \dot{\tilde{\mathbf{q}}}^T\Lambda\mathbf{K}_{pD}\tilde{\mathbf{q}} + \tilde{\mathbf{q}}^T\Lambda\mathbf{K}_{pD}\dot{\tilde{\mathbf{q}}} - \mathbf{s}^T\mathbf{K}_e\tilde{\mathbf{q}} + \dot{\tilde{\mathbf{q}}}^T\mathbf{K}_e\tilde{\mathbf{q}} \\ &= -(\dot{\tilde{\mathbf{q}}}^T + \tilde{\mathbf{q}}^T\Lambda)\mathbf{K}_{pD}(\dot{\tilde{\mathbf{q}}} + \Lambda\tilde{\mathbf{q}}) + \dot{\tilde{\mathbf{q}}}^T\Lambda\mathbf{K}_{pD}\tilde{\mathbf{q}} + \tilde{\mathbf{q}}^T\Lambda\mathbf{K}_{pD}\dot{\tilde{\mathbf{q}}} \\ &\quad - (\dot{\tilde{\mathbf{q}}}^T + \tilde{\mathbf{q}}^T\Lambda)\mathbf{K}_e\tilde{\mathbf{q}} + \dot{\tilde{\mathbf{q}}}^T\mathbf{K}_e\tilde{\mathbf{q}} \\ &= -\dot{\tilde{\mathbf{q}}}^T\mathbf{K}_{pD}\dot{\tilde{\mathbf{q}}} - \tilde{\mathbf{q}}^T(\Lambda\mathbf{K}_{pD}\Lambda + \Lambda\mathbf{K}_e)\tilde{\mathbf{q}} \\ &= -\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ &\leq 0 \end{aligned}$$

with

$$\mathbf{Q} \triangleq \begin{bmatrix} \Lambda\mathbf{K}_{pD}\Lambda + \Lambda\mathbf{K}_e & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{pD} \end{bmatrix}.$$

To compute (1.125), Property 1.2 has been used. Because  $\mathbf{Q} > \mathbf{O}$ ,  $\dot{V}(\mathbf{x})$  is 0 if and only if  $\mathbf{x} = \mathbf{0}$ , which implies that the equilibrium point is asymptotically stable ([Vidyasagar (1978)], p. 154).

b) To prove the boundedness of  $\delta_d$  and  $\dot{\delta}_d$ , consider the following notation:

$$\mathbf{s}_0 \triangleq \dot{\delta}_d + \Lambda_\delta \delta_d \quad (1.126)$$

$$\mathbf{x}_\delta \triangleq \begin{bmatrix} \delta_d \\ \dot{\delta}_d \end{bmatrix} \quad (1.127)$$

and simplify (1.121) to get

$$\mathbf{H}_{\delta\delta} \dot{\mathbf{s}}_0 = -(\mathbf{C}_{\delta\delta} \mathbf{s}_0 + \mathbf{K}_{pD\delta} \mathbf{s}_0 + \mathbf{K} \delta_d + \mathbf{f}_r) \quad (1.128)$$

$$\begin{aligned} \mathbf{f}_r \triangleq & \mathbf{H}_{\theta\delta}^T \ddot{\theta}_r + \mathbf{C}_{\delta\theta} \dot{\theta}_r + \mathbf{g}_\delta - \mathbf{H}_{\delta\delta} \Lambda_\delta \dot{\delta} - \mathbf{C}_{\delta\delta} \Lambda_\delta \delta \\ & - \mathbf{D}_\delta \Lambda_\delta \delta - \mathbf{K}_{p\delta} (\dot{\delta} + \Lambda_\delta \delta). \end{aligned} \quad (1.129)$$

Since  $\|\mathbf{q}\| < \infty$  and  $\|\dot{\mathbf{q}}\| \leq v_m$ , the vector  $\mathbf{f}_r$  is bounded by a positive constant  $f_{r,\max}$ , i.e.  $\|\mathbf{f}_r\| \leq f_{r,\max}$  (see Properties 1.9 to 1.12).

Consider the Lyapunov function  $V_\delta(\mathbf{x}_\delta, t) = V_\delta(\mathbf{x}_\delta)$ :

$$V_\delta(\mathbf{x}_\delta) = \frac{1}{2} \mathbf{s}_0^T \mathbf{H}_{\delta\delta} \mathbf{s}_0 + \delta_d^T \left( \Lambda_\delta \mathbf{K}_{pD\delta} + \frac{1}{2} \mathbf{K} \right) \delta_d = \frac{1}{2} \mathbf{x}_\delta^T \mathbf{M} \mathbf{x}_\delta, \quad (1.130)$$

with

$$\mathbf{M} \triangleq \begin{bmatrix} 2\Lambda_\delta \mathbf{K}_{pD\delta} + \mathbf{K} + \Lambda_\delta \mathbf{H}_{\delta\delta} \Lambda_\delta & \Lambda_\delta \mathbf{H}_{\delta\delta} \\ \mathbf{H}_{\delta\delta} \Lambda_\delta & \mathbf{H}_{\delta\delta} \end{bmatrix}.$$

Since  $\mathbf{M} > \mathbf{O}$ , there exist two constants  $\lambda_1$  and  $\lambda_2$  such that

$$\lambda_1 \|\mathbf{x}_\delta\|^2 \leq V_\delta(\mathbf{x}_\delta) \leq \lambda_2 \|\mathbf{x}_\delta\|^2 \quad (1.131)$$

$$\lambda_1 \triangleq \frac{1}{2} \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}(\mathbf{M}) \quad (1.132)$$

$$\lambda_2 \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}(\mathbf{M}). \quad (1.133)$$

The derivative of (1.130) is

$$\begin{aligned} \dot{V}_\delta(\mathbf{x}_\delta) = & \frac{1}{2} \mathbf{s}_0^T \dot{\mathbf{H}}_{\delta\delta} \mathbf{s}_0 + \dot{\delta}_d^T \Lambda_\delta \mathbf{K}_{pD\delta} \delta_d + \delta_d^T \Lambda_\delta \mathbf{K}_{pD\delta} \dot{\delta}_d \\ & + \dot{\delta}_d^T \mathbf{K} \delta_d + \mathbf{s}_0^T \mathbf{H}_{\delta\delta} \dot{\mathbf{s}}_0. \end{aligned} \quad (1.134)$$

Evaluating (1.134) along (1.128) and taking Property 1.2 into account yields

$$\begin{aligned}
\dot{V}_\delta(\mathbf{x}_\delta) &= \frac{1}{2} \mathbf{s}_0^T \dot{\mathbf{H}}_{\delta\delta} \mathbf{s}_0 + \dot{\delta}_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \delta_d + \delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \dot{\delta}_d + \dot{\delta}_d^T \mathbf{K} \delta_d \quad (1.135) \\
&\quad - \mathbf{s}_0^T (\mathbf{C}_{\delta\delta} \mathbf{s}_0 + \mathbf{K}_{pD\delta} \mathbf{s}_0 + \mathbf{K} \delta_d + \mathbf{f}_r) \\
&= -\mathbf{s}_0^T \mathbf{K}_{pD\delta} \mathbf{s}_0 + \dot{\delta}_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \delta_d + \delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \dot{\delta}_d - \mathbf{s}_0^T \mathbf{K} \delta_d \\
&\quad + \dot{\delta}_d^T \mathbf{K} \delta_d - \mathbf{s}_0^T \mathbf{f}_r \\
&= -(\dot{\delta}_d^T + \delta_d^T \mathbf{\Lambda}_\delta) \mathbf{K}_{pD\delta} (\delta_d + \mathbf{\Lambda}_\delta \delta_d) + \dot{\delta}_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \delta_d + \delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \dot{\delta}_d \\
&\quad - (\dot{\delta}_d^T + \delta_d^T \mathbf{\Lambda}_\delta) \mathbf{K} \delta_d + \dot{\delta}_d^T \mathbf{K} \delta_d - \mathbf{s}_0^T \mathbf{f}_r \\
&= -\dot{\delta}_d^T \mathbf{K}_{pD\delta} \delta_d - \delta_d^T (\mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta + \mathbf{\Lambda}_\delta \mathbf{K}) \delta_d - \mathbf{s}_0^T \mathbf{f}_r \\
&= -\mathbf{x}_\delta^T \mathbf{P} \mathbf{x}_\delta - \mathbf{s}_0^T \mathbf{f}_r \\
&\leq -\lambda_3 \|\mathbf{x}_\delta\|^2 - \mathbf{s}_0^T \mathbf{f}_r
\end{aligned}$$

with

$$\mathbf{P} \triangleq \begin{bmatrix} \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta + \mathbf{\Lambda}_\delta \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{pD\delta} \end{bmatrix}$$

and

$$\begin{aligned}
\lambda_3 &\triangleq \lambda_{\min}(\mathbf{P}) \quad (1.136) \\
&= \min\{\lambda_{\min}(\mathbf{K}_{pD\delta}), \lambda_{\min}(\mathbf{\Lambda}_\delta)^2 \lambda_{\min}(\mathbf{K}_{pD\delta}) + \lambda_{\min}(\mathbf{\Lambda}_\delta) \lambda_{\min}(\mathbf{K})\}.
\end{aligned}$$

Since

$$\begin{aligned}
\|\mathbf{s}_0\| &= \|\dot{\delta}_d + \mathbf{\Lambda}_\delta \delta_d\| \leq \|\dot{\delta}_d\| + \lambda_{\max}(\mathbf{\Lambda}_\delta) \|\delta_d\| \quad (1.137) \\
&\leq (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) \|\mathbf{x}_\delta\|,
\end{aligned}$$

Eq. (1.135) can be rewritten as

$$\begin{aligned}
\dot{V}_\delta(\mathbf{x}_\delta) &\leq -\lambda_3 \|\mathbf{x}_\delta\|^2 + (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) f_{r,\max} \|\mathbf{x}_\delta\| \quad (1.138) \\
&\triangleq -\lambda_3 \|\mathbf{x}_\delta\|^2 + \sigma \|\mathbf{x}_\delta\|.
\end{aligned}$$

By applying Theorem 1.2 with  $\gamma = 0$ , it can be proven that  $\|\mathbf{x}_\delta\|$  is bounded.  $\triangle$

Now, suppose that the desired trajectory  $\boldsymbol{\theta}_d$  is constant and assume that  $\mathbf{x} = \mathbf{0}$ . In this case (1.121) becomes

$$\mathbf{H}_{\delta\delta} \ddot{\delta}_d + \mathbf{C}_{\delta\delta} \dot{\delta}_d + \mathbf{D}_\delta \delta_d + \mathbf{K} \delta_d + \mathbf{g}_\delta = \mathbf{0}.$$

Assume that the vector  $\mathbf{g}_\delta$  is only a function of  $\boldsymbol{\theta}$  [De Luca and Siciliano (1993b); De Luca and Panzieri (1994)], i.e.  $\mathbf{g}_\delta(\mathbf{q}) = \mathbf{g}_\delta(\boldsymbol{\theta})$ . Since  $\boldsymbol{\theta}_d$  is constant and  $\boldsymbol{\theta}_d \equiv \boldsymbol{\theta}$ , the vector  $\mathbf{g}_\delta$  is constant as well, so that a new variable  $\mathbf{y}$  can be defined:

$$\mathbf{y} \triangleq \boldsymbol{\delta}_d + \mathbf{K}^{-1}\mathbf{g}_\delta, \quad (1.139)$$

and (1.139) can be rewritten as

$$\mathbf{H}_{\delta\delta}\ddot{\mathbf{y}} + \mathbf{C}_{\delta\delta}\dot{\mathbf{y}} + \mathbf{D}_\delta\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0}. \quad (1.140)$$

Consider the Lyapunov function

$$V_y(\mathbf{y}, \dot{\mathbf{y}}) = \frac{1}{2}\mathbf{y}^T \mathbf{K}\mathbf{y} + \frac{1}{2}\dot{\mathbf{y}}^T \mathbf{H}_{\delta\delta}\dot{\mathbf{y}}. \quad (1.141)$$

Its derivative along (1.140) is

$$\begin{aligned} \dot{V}_y &= \dot{\mathbf{y}}^T \mathbf{K}\mathbf{y} + \frac{1}{2}\dot{\mathbf{y}}^T \dot{\mathbf{H}}_{\delta\delta}\dot{\mathbf{y}} + \dot{\mathbf{y}}^T \mathbf{H}_{\delta\delta}\ddot{\mathbf{y}} \\ &= \dot{\mathbf{y}}^T \mathbf{K}\mathbf{y} + \frac{1}{2}\dot{\mathbf{y}}^T \dot{\mathbf{H}}_{\delta\delta}\dot{\mathbf{y}} - \dot{\mathbf{y}}^T (\mathbf{C}_{\delta\delta}\dot{\mathbf{y}} + \mathbf{D}_\delta\dot{\mathbf{y}} + \mathbf{K}\mathbf{y}) \\ &= -\dot{\mathbf{y}}^T \mathbf{D}_\delta\dot{\mathbf{y}} \leq 0. \end{aligned} \quad (1.142)$$

Property 1.2 has been used to compute (1.142). Assuming that  $\mathbf{D}_\delta > \mathbf{O}$ ,  $\dot{V}_y$  is 0 if and only if  $\dot{\mathbf{y}}$  is  $\mathbf{0}$ . By applying the invariant set theorem, it can be proven that the equilibrium point  $\dot{\mathbf{y}} = \mathbf{y} = \mathbf{0}$  of (1.140) is asymptotically stable [De Luca and Siciliano (1993a)]. When  $\mathbf{y} = \mathbf{0}$ ,  $\boldsymbol{\delta}_d$  becomes  $-\mathbf{K}^{-1}\mathbf{g}_\delta$ , which is the same result as in [De Luca and Siciliano (1993b); De Luca and Panzieri (1994)].

### 1.3.3 No Damping

In the preceding section, a nonlinear controller for the tracking problem of flexible robot manipulators was introduced. By studying the case when the desired trajectory for the joint coordinates is constant and the error vector  $\mathbf{x} = \mathbf{0}$ , Eq. (1.139) was obtained. Assuming that  $\mathbf{D}_\delta > \mathbf{O}$ , it was proven that  $\boldsymbol{\delta}_d \rightarrow -\mathbf{K}^{-1}\mathbf{g}_\delta$ . Although this is the case for any mechanical system, if the elements of  $\mathbf{D}_\delta$  are small, it may last before possible oscillations disappear. Therefore, it is desirable that the controller increases the damping of the system.

Suppose that  $\mathbf{D}_\delta = \mathbf{O}$ . Since (1.139) belongs to the controller rather than to the manipulator equations of motion, it is always possible to add,

arbitrarily, a term  $\mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d$ , with  $\mathbf{D}_\Delta$  diagonal and positive definite. If  $\mathbf{x} \rightarrow \mathbf{0}$  were still true, then the desired trajectory of the link coordinates would be damped and as a direct consequence of this, the real trajectory  $\boldsymbol{\delta}$  too. Of course, by doing so, the stability analysis of Section 1.3.2 is no longer valid. However, if the term  $\mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d$  is treated as a well-known perturbation, some robust control techniques [Dawson *et al.* (1991); Leitmann (1981); Corless and Leitmann (1981); Qu and Dawson (1991); Spong (1992); Yaz (1993)] can be used to guarantee that  $\mathbf{x} \rightarrow \mathbf{0}$ .

Consider the following equation to compute the desired trajectory  $\boldsymbol{\delta}_d$  and  $\dot{\boldsymbol{\delta}}_d$ :

$$\begin{aligned} \ddot{\boldsymbol{\delta}}_d = \Lambda_\delta \dot{\boldsymbol{\delta}} - \mathbf{H}_{\delta\delta}^{-1} (\mathbf{C}_{\delta\delta} \dot{\boldsymbol{\delta}}_r + \mathbf{D}_\delta \dot{\boldsymbol{\delta}}_r - \mathbf{K}_{p\delta} \mathbf{s}_\delta + \mathbf{K} \boldsymbol{\delta}_d \\ + \mathbf{H}_{\theta\delta}^T \ddot{\boldsymbol{\theta}}_r + \mathbf{C}_{\delta\theta} \dot{\boldsymbol{\theta}}_r + \mathbf{g}_\delta + \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f}) \end{aligned} \quad (1.143)$$

with initial condition

$$\boldsymbol{\delta}_d(0) = \dot{\boldsymbol{\delta}}_d(0) = \mathbf{0} \quad (1.144)$$

and

$$f_{ij} \triangleq -\dot{\delta}_{dij} \frac{(\dot{\delta}_{dij} s_{\delta ij}) d_{ij}}{\|\dot{\delta}_{dij} s_{\delta ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \quad i = 1, \dots, n \quad j = 1, \dots, m_i, \quad (1.145)$$

where  $s_{\delta ij}$  is an element of  $\mathbf{s}_\delta$ ,  $\dot{\delta}_{dij}$  is an element of  $\dot{\boldsymbol{\delta}}_d$ ,  $f_{ij}$  is an element of  $\mathbf{f}$  and  $d_{ij}$  is an element of  $\mathbf{D}_\Delta$ . Control (1.120) together with (1.143) can be expressed as

$$\mathbf{u} = \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_r + \mathbf{K}_e \mathbf{q}_d + \mathbf{D} \dot{\mathbf{q}}_r + \mathbf{g}(\mathbf{q}) - \mathbf{K}_p \mathbf{s} + \mathbf{f}_1, \quad (1.146)$$

with

$$\mathbf{f}_1 \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f} \end{bmatrix}. \quad (1.147)$$

By taking (1.146) into account, the error dynamics becomes

$$\mathbf{H}(\mathbf{q}) \dot{\mathbf{s}} = -(\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{s} + \mathbf{K}_e \tilde{\mathbf{q}} + \mathbf{K}_{pD} \mathbf{s}) + \mathbf{f}_1. \quad (1.148)$$

Theorem 1.4 establishes the stability of the state  $\mathbf{x}$  of (1.148).

**Theorem 1.4** *Given a bounded continuous desired trajectory  $\boldsymbol{\theta}_d$  with bounded velocity and acceleration, if  $\|\dot{\mathbf{q}}\| \leq v_m$ , where  $v_m$  is a positive scalar*

constant, then the desired trajectory  $\delta_d$  and  $\dot{\delta}_d$  given by (1.143) and (1.144) remains bounded if  $\lambda_3 \triangleq \lambda_{\min}(\mathbf{P}) - (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))\sqrt{m} \lambda_{\max}(\mathbf{D}_\Delta) > 0$ , with

$$\mathbf{P} \triangleq \begin{bmatrix} \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta + \mathbf{\Lambda}_\delta \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{pD\delta} + \mathbf{D}_\Delta \end{bmatrix}$$

and

$$\lambda_{\min}(\mathbf{P}) = \min\{\lambda_{\min}(\mathbf{K}_{pD\delta}) + \lambda_{\min}(\mathbf{D}_\Delta), \lambda_{\min}(\mathbf{\Lambda}_\delta)^2 \lambda_{\min}(\mathbf{K}_{pD\delta}) + \lambda_{\min}(\mathbf{\Lambda}_\delta) \lambda_{\min}(\mathbf{K})\}. \quad (1.149)$$

In addition, by using the input vector (1.120) the state  $\mathbf{x}$  of (1.148) is GES in the sense of Theorem 1.1.

*Proof:* The proof is similar at all to that of Theorem 1.3.

a) For the stability analysis of  $\mathbf{x}$ , the Lyapunov function (1.125) is used, which can be rewritten as

$$V(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{N} \mathbf{x} \quad (1.150)$$

with

$$\mathbf{N} \triangleq \begin{bmatrix} 2\mathbf{\Lambda} \mathbf{K}_{pD} + \mathbf{K}_e + \mathbf{\Lambda} \mathbf{H}(\mathbf{q}) \mathbf{\Lambda} & \mathbf{\Lambda} \mathbf{H}(\mathbf{q}) \\ \mathbf{H}(\mathbf{q}) \mathbf{\Lambda} & \mathbf{H}(\mathbf{q}) \end{bmatrix}.$$

Notice that  $\mathbf{N} > \mathbf{O}$ , implying that

$$\hat{\lambda}_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq \hat{\lambda}_2 \|\mathbf{x}\|^2 \quad (1.151)$$

$$\hat{\lambda}_1 \triangleq \frac{1}{2} \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}(\mathbf{N}) \quad (1.152)$$

$$\hat{\lambda}_2 \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}(\mathbf{N}). \quad (1.153)$$

The derivative  $\dot{V}(\mathbf{x})$  can be obtained from (1.125) and (1.148), using Prop-

erty 1.2, i.e.

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= -\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{s}_\delta^T (\mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f}) \quad (1.154) \\
&\leq -\lambda_{\min}(\mathbf{Q}) \|\mathbf{x}\|^2 + \sum_{i=1}^n \sum_{j=1}^{m_i} \left( \dot{\delta}_{dij} s_{\delta ij} d_{ij} - d_{ij} \frac{\|\dot{\delta}_{dij} s_{\delta ij}\|^2}{\|\dot{\delta}_{dij} s_{\delta ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\
&\leq -\lambda_{\min}(\mathbf{Q}) \|\mathbf{x}\|^2 + \sum_{i=1}^n \sum_{j=1}^{m_i} \left( \|\dot{\delta}_{dij} s_{\delta ij}\| d_{ij} - d_{ij} \frac{\|\dot{\delta}_{dij} s_{\delta ij}\|^2}{\|\dot{\delta}_{dij} s_{\delta ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\
&= -\lambda_{\min}(\mathbf{Q}) \|\mathbf{x}\|^2 + \sum_{i=1}^n \sum_{j=1}^{m_i} \left( d_{ij} \frac{\|\dot{\delta}_{dij} s_{\delta ij}\| \epsilon_{ij} e^{-\beta_{ij} t}}{\|\dot{\delta}_{dij} s_{\delta ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\
&\leq -\lambda_{\min}(\mathbf{Q}) \|\mathbf{x}\|^2 + \sum_{i=1}^n \sum_{j=1}^{m_i} (d_{ij} \epsilon_{ij} e^{-\beta_{ij} t}) \\
&= -\hat{\lambda}_3 \|\mathbf{x}\|^2 + \epsilon e^{-\beta t}
\end{aligned}$$

with

$$\hat{\lambda}_3 \triangleq \lambda_{\min}(\mathbf{Q}) \quad (1.155)$$

$$\epsilon \triangleq \sum_{i=1}^n \sum_{j=1}^{m_i} d_{ij} \epsilon_{ij} \quad (1.156)$$

$$\beta \triangleq \min\{\beta_{ij}, i = 1, \dots, n, j = 1, \dots, m_i\}. \quad (1.157)$$

The proof concludes by applying Theorem 1.1 with  $\hat{\lambda}_1$ ,  $\hat{\lambda}_2$ ,  $\hat{\lambda}_3$ .

b) To prove the boundedness of  $\boldsymbol{\delta}_d$  and  $\dot{\boldsymbol{\delta}}_d$ , the variables  $\mathbf{x}_\delta$  and  $\mathbf{s}_0$  are used again (see (1.126) and (1.127)), such that the dynamics of  $\boldsymbol{\delta}_d$  and  $\dot{\boldsymbol{\delta}}_d$  can be expressed as

$$\mathbf{H}_{\delta\delta} \dot{\mathbf{s}}_0 = -(\mathbf{C}_{\delta\delta} \mathbf{s}_0 + \mathbf{K}_{pD\delta} \mathbf{s}_0 + \mathbf{K} \boldsymbol{\delta}_d + \mathbf{f}_r + \mathbf{f} + \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d). \quad (1.158)$$

The term  $\mathbf{f}_r$  is the same as in Theorem 1.3 (see (1.129)). Note that  $\|\mathbf{f}_r\| \leq f_{r,\max}$ . The following Lyapunov function is employed:

$$V_\delta(\mathbf{x}_\delta) = \frac{1}{2} \mathbf{s}_0^T \mathbf{H}_{\delta\delta} \mathbf{s}_0 + \frac{1}{2} \boldsymbol{\delta}_d^T (2\boldsymbol{\Lambda}_\delta \mathbf{K}_{pD\delta} + \mathbf{K} + \boldsymbol{\Lambda}_\delta \mathbf{D}_\Delta) \boldsymbol{\delta}_d = \frac{1}{2} \mathbf{x}_\delta^T \mathbf{M} \mathbf{x}_\delta \quad (1.159)$$

with

$$\mathbf{M} \triangleq \begin{bmatrix} 2\boldsymbol{\Lambda}_\delta \mathbf{K}_{pD\delta} + \mathbf{K} + \boldsymbol{\Lambda}_\delta \mathbf{H}_{\delta\delta} \boldsymbol{\Lambda}_\delta + \boldsymbol{\Lambda}_\delta \mathbf{D}_\Delta & \boldsymbol{\Lambda}_\delta \mathbf{H}_{\delta\delta} \\ \mathbf{H}_{\delta\delta} \boldsymbol{\Lambda}_\delta & \mathbf{H}_{\delta\delta} \end{bmatrix}.$$

Since  $\mathbf{M} > \mathbf{O}$ , there exist two constants  $\lambda_1$  and  $\lambda_2$  such that

$$\lambda_1 \|\mathbf{x}_\delta\|^2 \leq V_\delta(\mathbf{x}_\delta) \leq \lambda_2 \|\mathbf{x}_\delta\|^2 \quad (1.160)$$

$$\lambda_1 \triangleq \frac{1}{2} \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}(\mathbf{M}) \quad (1.161)$$

$$\lambda_2 \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}(\mathbf{M}). \quad (1.162)$$

It is not difficult to obtain  $\dot{V}_\delta(\mathbf{x}_\delta)$  from (1.135) and (1.158), using Property 1.2, i.e.

$$\begin{aligned} \dot{V}_\delta(\mathbf{x}_\delta) &= -\dot{\boldsymbol{\delta}}_d^T \mathbf{K}_{pD\delta} \dot{\boldsymbol{\delta}}_d - \boldsymbol{\delta}_d^T (\boldsymbol{\Lambda}_\delta \mathbf{K}_{pD\delta} \boldsymbol{\Lambda}_\delta + \boldsymbol{\Lambda}_\delta \mathbf{K}) \boldsymbol{\delta}_d \\ &\quad + \dot{\boldsymbol{\delta}}_d^T \boldsymbol{\Lambda}_\delta \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d - \mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f} + \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d) \\ &\leq -\lambda_{\min}(\mathbf{P}) \|\mathbf{x}_\delta\|^2 - \mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f}), \end{aligned} \quad (1.163)$$

where  $\lambda_{\min}(\mathbf{P})$  is given by (1.149). By noting that  $\mathbf{f}$  can be written as

$$\mathbf{f} = -\mathbf{D}_\Delta \text{diag}\{\dot{\delta}_{d11}, \dots, \dot{\delta}_{dnm_n}\} \hat{\mathbf{f}} \quad (1.164)$$

with

$$\hat{\mathbf{f}} \triangleq [\hat{f}_{11} \dots \hat{f}_{nm_n}]^T \quad (1.165)$$

$$\hat{f}_{ij} \triangleq \frac{\dot{\delta}_{dij} s_{\delta ij}}{\|\dot{\delta}_{dij} s_{\delta ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}}, \quad (1.166)$$

and since  $\|\hat{f}_{ij}\| < 1$ ,  $\forall i = 1, \dots, n$ ,  $j = 1, \dots, m_i$ , it can be seen from (1.164) that

$$\|\mathbf{f}\| \leq \lambda_{\max}(\mathbf{D}_\Delta) \sqrt{m} \|\dot{\boldsymbol{\delta}}_d\|, \quad (1.167)$$

such that (see (1.137))

$$\|\mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f})\| \leq (1 + \lambda_{\max}(\boldsymbol{\Lambda}_\delta)) \|\mathbf{x}_\delta\| (f_{r,\max} + \lambda_{\max}(\mathbf{D}_\Delta) \sqrt{m} \|\mathbf{x}_\delta\|). \quad (1.168)$$

Eq. (1.163) then becomes

$$\begin{aligned} \dot{V}_\delta(\mathbf{x}_\delta) &\leq -\lambda_{\min}(\mathbf{P}) \|\mathbf{x}_\delta\|^2 + (1 + \lambda_{\max}(\boldsymbol{\Lambda}_\delta)) \|\mathbf{x}_\delta\| (f_{r,\max} \\ &\quad + \lambda_{\max}(\mathbf{D}_\Delta) \sqrt{m} \|\mathbf{x}_\delta\|) \\ &= -(\lambda_{\min}(\mathbf{P}) - (1 + \lambda_{\max}(\boldsymbol{\Lambda}_\delta)) \lambda_{\max}(\mathbf{D}_\Delta) \sqrt{m}) \|\mathbf{x}_\delta\|^2 \\ &\quad + (1 + \lambda_{\max}(\boldsymbol{\Lambda}_\delta)) f_{r,\max} \|\mathbf{x}_\delta\| \\ &\triangleq -\lambda_3 \|\mathbf{x}_\delta\|^2 + \sigma \|\mathbf{x}_\delta\|. \end{aligned} \quad (1.169)$$

By using  $\lambda_1$ ,  $\lambda_2$  given in (1.161), (1.162) and taking into account that  $\lambda_3 > 0$  (by assumption), Theorem 1.2 can be applied with  $\gamma = 0$  to prove the boundedness of  $\|\mathbf{x}_\delta\|$ .  $\triangle$

The control approach of Theorem 1.4 is similar to that given in [Dawson *et al.* (1991)], although here it is used to increase damping, rather than to consider possible uncertainties in the manipulator model. Notice that it is always possible to get  $\lambda_3 > 0$  just by letting the elements of  $\mathbf{K}_{p\delta}$  be large enough.

In order to show that the damping of the system becomes greater, assume that  $\mathbf{x}$  is negligible and that  $\mathbf{D}_\delta = \mathbf{O}$ . In this case, the vector  $\mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f}$  represents the damping of the system. By factorizing every element of this vector as

$$\dot{\delta}_{dij} d_{ij} \left( 1 - \frac{\dot{\delta}_{dij} s \delta_{ij}}{\|\dot{\delta}_{dij} s \delta_{ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right),$$

it can be seen that  $\boldsymbol{\delta}_d$  is damped because the second factor belongs to the open set  $(0, 2)$ . This becomes clearer if one considers again the case when  $\boldsymbol{\theta}_d$  is constant. The dynamics of  $\boldsymbol{\delta}_d$  is described by

$$\mathbf{H}_{\delta\delta} \ddot{\boldsymbol{\delta}}_d + \mathbf{C}_{\delta\delta} \dot{\boldsymbol{\delta}}_d + \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{K} \boldsymbol{\delta}_d + \mathbf{g}_\delta + \mathbf{f} = \mathbf{0}. \quad (1.170)$$

If  $\mathbf{g}_\delta(\mathbf{q}) = \mathbf{g}_\delta(\boldsymbol{\theta})$ , the vector  $\mathbf{y}$  (see (1.139)) can be employed, so that (1.170) becomes

$$\mathbf{H}_{\delta\delta} \ddot{\mathbf{y}} + \mathbf{C}_{\delta\delta} \dot{\mathbf{y}} + \mathbf{D}_\Delta \dot{\mathbf{y}} + \mathbf{K} \mathbf{y} + \mathbf{f}(\dot{\mathbf{y}}) = \mathbf{0}. \quad (1.171)$$

Using the Lyapunov function (1.141) leads to (see (1.142))

$$\begin{aligned} \dot{V}_y &= -\dot{\mathbf{y}}^T \mathbf{D}_\Delta \dot{\mathbf{y}} - \dot{\mathbf{y}}^T \mathbf{f}(\dot{\mathbf{y}}) \\ &= -\sum_{i=1}^n \sum_{j=1}^{m_i} \left( d_{ij} \dot{y}_{ij}^2 - d_{ij} \dot{y}_{ij}^2 \frac{\dot{y}_{ij} s \delta_{ij}}{\|\dot{y}_{ij} s \delta_{ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\ &= -\sum_{i=1}^n \sum_{j=1}^{m_i} d_{ij} \dot{y}_{ij}^2 \left( 1 - \frac{\dot{y}_{ij} s \delta_{ij}}{\|\dot{y}_{ij} s \delta_{ij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\ &\leq 0. \end{aligned} \quad (1.172)$$

Because  $\dot{V}_y$  is 0 if and only if  $\dot{\mathbf{y}} = \mathbf{0}$ , the asymptotic stability of the equilibrium point  $\mathbf{y}$  and  $\dot{\mathbf{y}}$  of (1.171) can be proven by using the same argument as before. Note that  $\boldsymbol{\delta}_d$  becomes  $-\mathbf{K}^{-1} \mathbf{g}_\delta$  as well and that this would not be true if  $\mathbf{D}_\Delta = \mathbf{O}$ .

## 1.4 Control with Nonlinear Observer

In the previous section, the tracking control problem of flexible robot manipulators was studied. It was shown how to get a bounded desired trajectory for the link coordinates  $(\delta_d, \dot{\delta}_d)$  and how to increase the damping of the system by resorting to a robust control technique. However, it was assumed that link and joint coordinate rates were available. Although one can use tachometers to measure joint speeds, it is not always possible to measure link coordinate rates. In this section, the controller given in Section 1.3 is slightly modified and a nonlinear observer is proposed to estimate link and joint coordinate rates.

### 1.4.1 Nonlinear Observer

In this section, the tracking control problem with a nonlinear observer is studied. Consider model (1.55) again:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}_e\mathbf{q} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \quad (1.173)$$

and the tracking errors  $\tilde{\mathbf{q}}$  and  $\dot{\tilde{\mathbf{q}}}$  given in Section 1.3.2, together with definitions (1.112) to (1.119). In addition, define:

$$\mathbf{z} \triangleq \mathbf{q} - \hat{\mathbf{q}} = \begin{bmatrix} \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \\ \boldsymbol{\delta} - \hat{\boldsymbol{\delta}} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_\theta \\ \mathbf{z}_\delta \end{bmatrix} \quad (1.174)$$

$$\begin{aligned} \ddot{\tilde{\mathbf{q}}}_r &= \begin{bmatrix} \ddot{\tilde{\boldsymbol{\theta}}}_r \\ \ddot{\tilde{\boldsymbol{\delta}}}_r \end{bmatrix} \triangleq \ddot{\tilde{\mathbf{q}}}_d - \boldsymbol{\Lambda}(\dot{\tilde{\mathbf{q}}} - \dot{\tilde{\mathbf{q}}}_d) = \ddot{\tilde{\mathbf{q}}}_r + \boldsymbol{\Lambda}\dot{\mathbf{z}} \\ &= \begin{bmatrix} \ddot{\boldsymbol{\theta}}_d - \boldsymbol{\Lambda}_\theta(\dot{\hat{\boldsymbol{\theta}}} - \dot{\boldsymbol{\theta}}_d) \\ \ddot{\boldsymbol{\delta}}_d - \boldsymbol{\Lambda}_\delta(\dot{\hat{\boldsymbol{\delta}}} - \dot{\boldsymbol{\delta}}_d) \end{bmatrix} = \begin{bmatrix} \ddot{\boldsymbol{\theta}}_r + \boldsymbol{\Lambda}_\theta\dot{\mathbf{z}}_\theta \\ \ddot{\boldsymbol{\delta}}_r + \boldsymbol{\Lambda}_\delta\dot{\mathbf{z}}_\delta \end{bmatrix} \end{aligned} \quad (1.175)$$

$$\hat{\mathbf{s}} \triangleq \dot{\tilde{\mathbf{q}}} - \dot{\tilde{\mathbf{q}}}_r = \mathbf{s} - \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{s}_\theta - \dot{\mathbf{z}}_\theta \\ \mathbf{s}_\delta - \dot{\mathbf{z}}_\delta \end{bmatrix} \triangleq \begin{bmatrix} \hat{\mathbf{s}}_\theta \\ \hat{\mathbf{s}}_\delta \end{bmatrix} = \begin{bmatrix} \dot{\hat{\boldsymbol{\theta}}} - \dot{\boldsymbol{\theta}}_r \\ \dot{\hat{\boldsymbol{\delta}}} - \dot{\boldsymbol{\delta}}_r \end{bmatrix} \quad (1.176)$$

$$\mathbf{s}_x \triangleq \dot{\hat{\boldsymbol{\delta}}} - \dot{\boldsymbol{\delta}}_d + \boldsymbol{\Lambda}_\delta(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}_d) \quad (1.177)$$

$$\mathbf{K}_q \triangleq \begin{bmatrix} \mathbf{K}_{q\theta} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{q\delta} \end{bmatrix} \quad (1.178)$$

$$\mathbf{K}_{qe} \triangleq \mathbf{K}_q + \mathbf{K}_e = \begin{bmatrix} \mathbf{K}_{q\theta} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{q\delta} + \mathbf{K} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{K}_{q\theta} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{qe\delta} \end{bmatrix}, \quad (1.179)$$

where  $\hat{(\cdot)}$  denotes the estimate of  $(\cdot)$  and  $\mathbf{z}$  and  $\dot{\mathbf{z}}$  are the observation errors. In view of Property 1.11, there exists a constant  $k_{c\delta\delta}$  such that

$$\|\mathbf{C}_{\delta\delta}(\mathbf{q}, \mathbf{y})\| \leq k_{c\delta\delta} \|\mathbf{y}\| \quad \forall \mathbf{q} \in \mathcal{Q}^{n+m}, \forall \mathbf{y} \in \mathfrak{R}^{n+m}. \quad (1.180)$$

The proposed controller is given by

$$\boldsymbol{\tau} = \mathbf{H}_{\theta\theta} \ddot{\boldsymbol{\theta}}_r + \mathbf{H}_{\theta\delta} \ddot{\boldsymbol{\delta}}_r + \mathbf{C}_{\theta\theta}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\boldsymbol{\theta}}_r + \mathbf{C}_{\theta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\boldsymbol{\delta}}_r + \mathbf{D}_\theta \dot{\boldsymbol{\theta}}_r + \mathbf{g}_\theta - \mathbf{K}_{q\theta} \tilde{\boldsymbol{\theta}} - \mathbf{K}_{p\theta} \hat{\mathbf{s}}_\theta, \quad (1.181)$$

while the desired trajectory for the link coordinates is computed from

$$\begin{aligned} \ddot{\boldsymbol{\delta}}_d = \boldsymbol{\Lambda}_\delta (\dot{\boldsymbol{\delta}} - \dot{\boldsymbol{\delta}}_d) - \mathbf{H}_{\delta\delta}^{-1} \left( \mathbf{H}_{\theta\delta}^T \ddot{\boldsymbol{\theta}}_r + \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\boldsymbol{\theta}}_r + \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\boldsymbol{\delta}}_r \right. \\ \left. + \mathbf{K}_\delta \boldsymbol{\delta}_d + \mathbf{D}_\delta \dot{\boldsymbol{\delta}}_r + \mathbf{g}_\delta - \mathbf{K}_{q\delta} \tilde{\boldsymbol{\delta}} - \mathbf{K}_{p\delta} \hat{\mathbf{s}}_\delta + \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f} \right), \end{aligned} \quad (1.182)$$

with initial condition

$$\boldsymbol{\delta}_d(0) = \dot{\boldsymbol{\delta}}_d(0) = \mathbf{0} \quad (1.183)$$

and

$$f_{ij} \triangleq -\dot{\delta}_{dij} \frac{(\dot{\delta}_{dij} s_{xij}) d_{ij}}{\|\dot{\delta}_{dij} s_{xij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \quad i = 1, \dots, n, j = 1, \dots, m_i, \quad (1.184)$$

where  $s_{xij}$  is an element of the vector  $\mathbf{s}_x$ . The inclusion of the positive definite matrix  $\mathbf{K}_q$  should be noticed. As pointed out before,  $\mathbf{f}$  helps increase the damping of the system. Eqs. (1.181) and (1.182) are equivalent to

$$\mathbf{u} = \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_r + \mathbf{K}_e \mathbf{q}_d + \mathbf{D} \dot{\mathbf{q}}_r + \mathbf{g}(\mathbf{q}) - \mathbf{K}_q \tilde{\mathbf{q}} - \mathbf{K}_p \hat{\mathbf{s}} + \mathbf{f}_1 \quad (1.185)$$

with

$$\mathbf{f}_1 \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f} \end{bmatrix}. \quad (1.186)$$

It is not difficult to show that control (1.185) can be rewritten as

$$\begin{aligned} \mathbf{u} = \mathbf{H}(\mathbf{q}) \ddot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_r + \mathbf{K}_e \mathbf{q}_d + \mathbf{D} \dot{\mathbf{q}}_r + \mathbf{g}(\mathbf{q}) - \mathbf{K}_q \tilde{\mathbf{q}} \\ - \mathbf{K}_p \mathbf{s} + \mathbf{f}_1 + \mathbf{H}(\mathbf{q}) \boldsymbol{\Lambda} \dot{\mathbf{z}} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \dot{\mathbf{q}}_r + \mathbf{K}_p \dot{\mathbf{z}}. \end{aligned} \quad (1.187)$$

By substituting (1.187) into (1.173), the error dynamics is given by

$$\begin{aligned} \mathbf{H}(\mathbf{q}) \dot{\mathbf{s}} = -(\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{s} + \mathbf{K}_{pD} \mathbf{s} + \mathbf{K}_{qe} \tilde{\mathbf{q}}) + \mathbf{f}_1 + \mathbf{H}(\mathbf{q}) \boldsymbol{\Lambda} \dot{\mathbf{z}} \\ - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{z}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s} + \mathbf{K}_p \dot{\mathbf{z}}. \end{aligned} \quad (1.188)$$

In the following, an observer based on that given in [Nicosia and Tomei (1990)] is presented, although some essential modifications are necessary in order to guarantee the stability of the whole system and the improvement of link damping [Arteaga (2000)]. Consider the following definitions:

$$\dot{\mathbf{q}}_o \triangleq \dot{\mathbf{q}} - \mathbf{\Lambda}z \quad (1.189)$$

$$\mathbf{r} \triangleq \dot{\mathbf{q}} - \dot{\mathbf{q}}_o = \dot{\mathbf{q}} - \dot{\mathbf{q}} + \mathbf{\Lambda}z = \dot{z} + \mathbf{\Lambda}z. \quad (1.190)$$

By using an estimate of  $\mathbf{q}_o$ , the proposed observer is given by

$$\dot{\hat{\mathbf{q}}} = \dot{\hat{\mathbf{q}}}_o + \mathbf{\Lambda}z + k_d z \quad (1.191)$$

$$\mathbf{H}(\mathbf{q})\ddot{\hat{\mathbf{q}}}_o + \mathbf{C}(\mathbf{q}, \dot{\hat{\mathbf{q}}})\dot{\hat{\mathbf{q}}}_o + \mathbf{K}_e \hat{\mathbf{q}} + \mathbf{D}\dot{\hat{\mathbf{q}}}_o + \mathbf{g}(\mathbf{q}) = \mathbf{u} + \mathbf{K}_v z + \mathbf{f}_1 \quad (1.192)$$

where  $\mathbf{K}_v$  is diagonal and positive definite and  $k_d > 0$ . Since one has

$$\ddot{\hat{\mathbf{q}}} = \ddot{\hat{\mathbf{q}}}_o + \mathbf{\Lambda}\dot{z} + k_d \dot{z}, \quad (1.193)$$

Eqs. (1.191) and (1.192) are equivalent to

$$\begin{aligned} \mathbf{H}(\mathbf{q})\ddot{\hat{\mathbf{q}}}_o + \mathbf{C}(\mathbf{q}, \dot{\hat{\mathbf{q}}})\dot{\hat{\mathbf{q}}}_o + \mathbf{K}_e \hat{\mathbf{q}} + \mathbf{D}\dot{\hat{\mathbf{q}}}_o + \mathbf{g}(\mathbf{q}) &= \mathbf{u} + \mathbf{K}_v z + \mathbf{f}_1 \\ &+ k_d \mathbf{H}(\mathbf{q})\dot{z} + \mathbf{C}(\mathbf{q}, \dot{z})\dot{\hat{\mathbf{q}}}_o. \end{aligned} \quad (1.194)$$

Subtracting (1.194) from (1.173) yields

$$\mathbf{H}(\mathbf{q})\dot{\mathbf{r}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{r} + \mathbf{K}_e z + \mathbf{D}\mathbf{r} = -\mathbf{K}_v z - \mathbf{f}_1 - k_d \mathbf{H}(\mathbf{q})\dot{z} - \mathbf{C}(\mathbf{q}, \dot{z})\dot{\hat{\mathbf{q}}}_o. \quad (1.195)$$

Due to Property 1.3, the following equality is valid:

$$\mathbf{C}(\mathbf{q}, \dot{z})\dot{\hat{\mathbf{q}}}_o = \mathbf{C}(\mathbf{q}, \dot{z})(\dot{\hat{\mathbf{q}}} - \mathbf{r}) = \mathbf{C}(\mathbf{q}, \dot{\hat{\mathbf{q}}})\dot{z} - \mathbf{C}(\mathbf{q}, \dot{z})\mathbf{r}, \quad (1.196)$$

so that (1.195) can be rewritten as

$$\mathbf{H}(\mathbf{q})\dot{\mathbf{r}} = -(\mathbf{C}(\mathbf{q}, \dot{\hat{\mathbf{q}}})\mathbf{r} + \mathbf{K}_{ve} z + \mathbf{D}\mathbf{r} + k_d \mathbf{H}(\mathbf{q})\dot{z}) + \mathbf{C}(\mathbf{q}, \dot{z})\mathbf{r} - \mathbf{C}(\mathbf{q}, \dot{\hat{\mathbf{q}}})\dot{z} - \mathbf{f}_1 \quad (1.197)$$

with  $\mathbf{K}_{ve} \triangleq \mathbf{K}_v + \mathbf{K}_e$ .

In order to simplify the stability discussion, the following definition is introduced:

$$\mathbf{x} \triangleq \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \\ z \\ \dot{z} \end{bmatrix}. \quad (1.198)$$

By taking advantage again of the fact that  $\dot{\mathbf{q}}$  is bounded, Theorem 1.5 establishes that the state  $\mathbf{x}$  of (1.188) and (1.197) is *exponentially stable* (ES). In order to prove it, Theorems 1.1 and 1.2 can be employed. Although they are helpful to prove whether the state of a system is GUUB or GES, they can be used as well to analyze if it is UUB or ES. To show this, suppose that there exists a ball  $B_r = \{\mathbf{x} : \|\mathbf{x}\| \leq b_r\}$  so that conditions (1.101)–(1.102) of Theorem 1.1 or (1.105)–(1.106) of Theorem 1.2 are satisfied. If it is possible to find a region of attraction  $S = \{\mathbf{x} : \|\mathbf{x}\| \leq b_s\}$  with  $0 < b_s \leq b_r$ , so that  $\|\mathbf{x}\|$  never abandons the ball  $B_r$ , then it can be proven that  $\|\mathbf{x}\|$  is UUB or ES.

**Theorem 1.5** *Given a bounded continuous desired trajectory  $\theta_d$  with bounded velocity and acceleration, if  $\|\dot{\mathbf{q}}\| \leq v_m$  where  $v_m$  is a positive scalar constant, then the state  $\mathbf{x}$  of (1.188) and (1.197) is ES in the sense of Theorem 1.1 as long as the following inequalities are satisfied:*

$$k_d > \frac{9}{4} \frac{(\lambda_M \lambda_H + k_c v_m + \lambda_P)^2}{\lambda_{pd} \lambda_h} + \frac{k_c v_m - \lambda_d}{\lambda_h} \quad (1.199)$$

$$\lambda_q > \frac{9}{4} \frac{\lambda_M (\lambda_M \lambda_H + k_c v_m + \lambda_P)}{\lambda_m (\lambda_d + k_d \lambda_h - k_c v_m)} - \lambda_m \lambda_{pd} - \lambda_{\min}(\mathbf{K}_e) \quad (1.200)$$

$$\lambda_v > \frac{9}{4} \frac{(k_d \lambda_M \lambda_H + \lambda_M k_c v_m)^2}{\lambda_m (\lambda_d + k_d \lambda_h - k_c v_m)} - \lambda_m \lambda_d - \lambda_{\min}(\mathbf{K}_e) \quad (1.201)$$

$$\varphi^2 z_m^2 > \frac{\epsilon e^{-1}}{\lambda_1 \lambda} \quad \text{if } \beta = \lambda \quad (1.202)$$

$$\varphi^2 z_m^2 > \frac{\epsilon}{\lambda_1 (\lambda - \beta)} \left( e^{-\beta \left( \frac{\ln(\lambda/\beta)}{\lambda - \beta} \right)} - e^{-\lambda \left( \frac{\ln(\lambda/\beta)}{\lambda - \beta} \right)} \right) \quad \text{if } \beta \neq \lambda \quad (1.203)$$

with

$$\begin{aligned} \lambda_m &\triangleq \lambda_{\min}(\mathbf{A}) & \lambda_M &\triangleq \lambda_{\max}(\mathbf{A}) & \lambda_{pd} &\triangleq \lambda_{\min}(\mathbf{K}_{pD}) \\ \lambda_d &\triangleq \lambda_{\min}(\mathbf{D}) & \lambda_{qe} &\triangleq \lambda_{\min}(\mathbf{K}_{qe}) & \lambda_q &\triangleq \lambda_{\min}(\mathbf{K}_q) \\ \lambda_P &\triangleq \lambda_{\max}(\mathbf{K}_p) & \lambda_{ve} &\triangleq \lambda_{\min}(\mathbf{K}_{ve}) & \lambda_v &\triangleq \lambda_{\min}(\mathbf{K}_v) \end{aligned}$$

and  $\lambda_h, \lambda_H$  are defined in Property 1.9. A region of attraction is given by

$$S = \left\{ \mathbf{x} \in \mathfrak{R}^{4(n+m)} : \|\mathbf{x}\| \leq \sqrt{\frac{\lambda_1}{\lambda_2}} \left( \varphi^2 z_m^2 - \frac{\epsilon e^{-1}}{\lambda_1 \lambda} \right)^{\frac{1}{2}} \right\} \quad (1.204)$$

if  $\beta = \lambda$ , and by

$$S = \left\{ \mathbf{x} \in \mathfrak{R}^{4(n+m)} : \right\} \quad (1.205)$$

$$\|\mathbf{x}\| \leq \sqrt{\frac{\lambda_1}{\lambda_2} \left( \varphi^2 z_m^2 - \frac{\epsilon}{\lambda_1(\lambda - \beta)} \left( e^{-\beta \left( \frac{\ln(\lambda/\beta)}{\lambda - \beta} \right)} - e^{-\lambda \left( \frac{\ln(\lambda/\beta)}{\lambda - \beta} \right)} \right) \right)^{\frac{1}{2}}}$$

if  $\beta \neq \lambda$ , where

$$\lambda_1 \triangleq \frac{1}{2} \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\min}(\mathbf{N}) \quad (1.206)$$

$$\lambda_2 \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} \lambda_{\max}(\mathbf{N}) \quad (1.207)$$

$$\lambda_3 \triangleq \min \lambda_{\min}(\mathbf{Q}) \quad 0 \leq \|\dot{\mathbf{z}}\| \leq \varphi z_m, \quad 0 < \varphi < 1 \quad (1.208)$$

$$\lambda \triangleq \frac{\lambda_3}{\lambda_2} \quad (1.209)$$

$$\epsilon \triangleq \sum_{i=1}^n \sum_{j=1}^{m_i} d_{ij} \epsilon_{ij} \quad (1.210)$$

$$\beta \triangleq \min\{\beta_{ij}, i = 1, \dots, n; j = 1, \dots, m_i\} \quad (1.211)$$

$$z_m \triangleq \min \left\{ \begin{array}{l} \frac{\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe}}{\lambda_M^2 k_c}, \frac{\lambda_{pd}}{k_c}, \frac{\lambda_m^2 \lambda_d + \lambda_m \lambda_{ve}}{\lambda_M^2 k_c}, \frac{\lambda_d + k_d \lambda_h - k_c v_m}{k_c}, \\ \frac{(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe})(\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4} \lambda_M (\lambda_M \lambda_H + k_c v_m + \lambda_P)}{k_c (\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} + \lambda_M^2 (\lambda_d + k_d \lambda_h - k_c v_m))}, \\ \frac{\lambda_{pd} (\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4} (\lambda_M \lambda_H + k_c v_m + \lambda_P)^2}{k_c (\lambda_{pd} + \lambda_d + k_d \lambda_h - k_c v_m)}, \\ \left( \frac{\lambda_m^2 \lambda_{pd}^2 + \lambda_m \lambda_{qe} \lambda_{pd}}{8 \lambda_M^2 k_c^2} + \left( \frac{2 \lambda_M^2 \lambda_{pd} + \lambda_m \lambda_{qe}}{16 \lambda_M^2 k_c} \right)^2 \right)^{\frac{1}{2}} - \frac{2 \lambda_M^2 \lambda_{pd} + \lambda_m \lambda_{qe}}{16 \lambda_M^2 k_c}, \\ \left( \frac{(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d)(\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4} (k_d \lambda_M \lambda_H + \lambda_M k_c v_m)^2}{8 \lambda_M^2 k_c^2} \right. \\ \left. + \left( \frac{\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d + \lambda_M^2 (\lambda_d + k_d \lambda_h - k_c v_m) + 9 \lambda_M^2 (k_d \lambda_H + k_c v_m)}{16 k_c \lambda_M^2} \right)^2 \right)^{\frac{1}{2}} \\ \frac{\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d + \lambda_M^2 (\lambda_d + k_d \lambda_h - k_c v_m) + 9 \lambda_M^2 (k_d \lambda_H + k_c v_m)}{16 k_c \lambda_M^2} \end{array} \right\} \quad (1.212)$$

$$Q \triangleq \begin{bmatrix}
\Lambda K_{pD} \Lambda & -\frac{1}{2} \Lambda C^T(q, \dot{z}) & O & -\frac{1}{2} \Lambda H(q) \Lambda \\
+\Lambda K_{qe} & -\frac{1}{2} \Lambda C(q, \dot{z}) & & -\frac{1}{2} \Lambda K_p \\
-\frac{1}{2} \Lambda C(q, \dot{z}) \Lambda & & & +\frac{1}{2} \Lambda C(q, \dot{q}) \\
-\frac{1}{2} \Lambda C^T(q, \dot{z}) \Lambda & & & \\
-\frac{1}{2} C(q, \dot{z}) \Lambda & K_{pD} & O & -\frac{1}{2} H(q) \Lambda \\
-\frac{1}{2} C^T(q, \dot{z}) \Lambda & -\frac{1}{2} C(q, \dot{z}) & & -\frac{1}{2} K_p \\
& -\frac{1}{2} C^T(q, \dot{z}) & & +\frac{1}{2} C(q, \dot{q}) \\
O & O & \Lambda K_{ve} & \frac{1}{2} k_d \Lambda H(q) \\
& & +\Lambda D \Lambda & -\frac{1}{2} \Lambda C^T(q, \dot{z}) \\
& & -\frac{1}{2} \Lambda C(q, \dot{z}) \Lambda & -\frac{1}{2} \Lambda C(q, \dot{z}) \\
& & -\frac{1}{2} \Lambda C^T(q, \dot{z}) \Lambda & +\frac{1}{2} \Lambda C(q, \dot{q}) \\
-\frac{1}{2} \Lambda H(q) \Lambda & -\frac{1}{2} \Lambda H(q) & \frac{1}{2} k_d H(q) \Lambda & D \\
-\frac{1}{2} K_p \Lambda & -\frac{1}{2} K_p & -\frac{1}{2} C(q, \dot{z}) \Lambda & +k_d H(q) \\
+\frac{1}{2} C^T(q, \dot{q}) \Lambda & +\frac{1}{2} C^T(q, \dot{q}) & -\frac{1}{2} C^T(q, \dot{z}) \Lambda & -\frac{1}{2} C(q, \dot{z}) \\
& & +\frac{1}{2} C^T(q, \dot{q}) \Lambda & -\frac{1}{2} C^T(q, \dot{z}) \\
& & & +\frac{1}{2} C(q, \dot{q}) \\
& & & +\frac{1}{2} C^T(q, \dot{q})
\end{bmatrix} \quad (1.213)$$

$$N \triangleq \begin{bmatrix}
2\Lambda K_{pD} + K_{qe} \Lambda H(q) & O & O \\
+\Lambda H(q) \Lambda & & \\
H(q) \Lambda & H(q) & O & O \\
O & O & 2\Lambda D + K_{ve} \Lambda H(q) \\
& & +\Lambda H(q) \Lambda \\
O & O & H(q) \Lambda & H(q)
\end{bmatrix}. \quad (1.214)$$

In addition, the desired trajectory for the link coordinates  $\delta_d$  and  $\dot{\delta}_d$  given by (1.182) and (1.183) remains bounded if

$$\hat{\lambda}_3 \triangleq \lambda_{\min}(P) - (1 + \lambda_{\max}(\Lambda_\delta)) (\lambda_{\max}(D_\Delta) \sqrt{m} + k_{c\delta\delta} \varphi z_m (1 + \lambda_{\max}(\Lambda_\delta))) > 0 \quad (1.215)$$

with

$$\mathbf{P} \triangleq \begin{bmatrix} \Lambda_\delta \mathbf{K}_{qe\delta} + \Lambda_\delta \mathbf{K}_{pD\delta} \Lambda_\delta & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_\Delta + \mathbf{K}_{pD\delta} \end{bmatrix} \quad (1.216)$$

$$\lambda_{\min}(\mathbf{P}) = \min\{\lambda_{\min}(\Lambda_\delta)\lambda_{\min}(\mathbf{K}_{qe\delta}) + \lambda_{\min}^2(\Lambda_\delta)\lambda_{\min}(\mathbf{K}_{pD\delta}), \lambda_{\min}(\mathbf{K}_{pD\delta}) + \lambda_{\min}(\mathbf{D}_\Delta)\}. \quad (1.217)$$

*Proof:*

a) Firstly, the stability of the whole system will be proven. Consider the following Lyapunov function:

$$\begin{aligned} V(\mathbf{x}, t) \triangleq V(\mathbf{x}) &= \frac{1}{2} \mathbf{s}^T \mathbf{H}(\mathbf{q}) \mathbf{s} + \frac{1}{2} \tilde{\mathbf{q}}^T (2\Lambda \mathbf{K}_{pD} + \mathbf{K}_{qe}) \tilde{\mathbf{q}} \\ &\quad + \frac{1}{2} \mathbf{r}^T \mathbf{H}(\mathbf{q}) \mathbf{r} + \frac{1}{2} \mathbf{z}^T (2\Lambda \mathbf{D} + \mathbf{K}_{ve}) \mathbf{z} \\ &= \frac{1}{2} \mathbf{x}^T \mathbf{N} \mathbf{x}, \end{aligned} \quad (1.218)$$

whose derivative is given by

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \frac{1}{2} \mathbf{s}^T \dot{\mathbf{H}}(\mathbf{q}) \mathbf{s} + \tilde{\mathbf{q}}^T (2\Lambda \mathbf{K}_{pD} + \mathbf{K}_{qe}) \dot{\tilde{\mathbf{q}}} + \frac{1}{2} \mathbf{r}^T \dot{\mathbf{H}}(\mathbf{q}) \mathbf{r} \\ &\quad + \mathbf{z}^T (2\Lambda \mathbf{D} + \mathbf{K}_{ve}) \dot{\mathbf{z}} + \mathbf{s}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{s}} + \mathbf{r}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{r}}. \end{aligned} \quad (1.219)$$

Substituting (1.188) and (1.197) into (1.219) leads to

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \frac{1}{2} \mathbf{s}^T \dot{\mathbf{H}}(\mathbf{q}) \mathbf{s} + \tilde{\mathbf{q}}^T (2\Lambda \mathbf{K}_{pD} + \mathbf{K}_{qe}) \dot{\tilde{\mathbf{q}}} + \frac{1}{2} \mathbf{r}^T \dot{\mathbf{H}}(\mathbf{q}) \mathbf{r} \\ &\quad + \mathbf{z}^T (2\Lambda \mathbf{D} + \mathbf{K}_{ve}) \dot{\mathbf{z}} \\ &\quad - \mathbf{s}^T (\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{s} + \mathbf{K}_{pD} \mathbf{s} + \mathbf{K}_{qe} \tilde{\mathbf{q}}) \\ &\quad + \mathbf{s}^T (\mathbf{f}_1 + \mathbf{H}(\mathbf{q}) \Lambda \dot{\mathbf{z}} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{z}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s} + \mathbf{K}_p \dot{\mathbf{z}}) \\ &\quad - \mathbf{r}^T (\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{r} + \mathbf{K}_{ve} \mathbf{z} + \mathbf{D} \mathbf{r} + k_d \mathbf{H}(\mathbf{q}) \dot{\mathbf{z}}) \\ &\quad + \mathbf{r}^T (\mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{r} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{z}} - \mathbf{f}_1). \end{aligned} \quad (1.220)$$

Since

$$\begin{aligned} -\mathbf{s}^T \mathbf{K}_{pD} \mathbf{s} - \mathbf{s}^T \mathbf{K}_{qe} \tilde{\mathbf{q}} &= -(\dot{\tilde{\mathbf{q}}}^T + \tilde{\mathbf{q}}^T \Lambda) \mathbf{K}_{pD} (\dot{\tilde{\mathbf{q}}} + \Lambda \tilde{\mathbf{q}}) \\ &\quad - (\dot{\tilde{\mathbf{q}}}^T + \tilde{\mathbf{q}}^T \Lambda) \mathbf{K}_{qe} \tilde{\mathbf{q}} \\ &= -\dot{\tilde{\mathbf{q}}}^T \mathbf{K}_{pD} \dot{\tilde{\mathbf{q}}} - \tilde{\mathbf{q}}^T \Lambda \mathbf{K}_{pD} \dot{\tilde{\mathbf{q}}} - \dot{\tilde{\mathbf{q}}}^T \mathbf{K}_{pD} \Lambda \tilde{\mathbf{q}} \\ &\quad - \tilde{\mathbf{q}}^T \Lambda \mathbf{K}_{pD} \Lambda \tilde{\mathbf{q}} - \dot{\tilde{\mathbf{q}}}^T \mathbf{K}_{qe} \tilde{\mathbf{q}} - \tilde{\mathbf{q}}^T \Lambda \mathbf{K}_{qe} \tilde{\mathbf{q}} \end{aligned} \quad (1.221)$$

and

$$\begin{aligned}
-\mathbf{r}^T \mathbf{D} \mathbf{r} - \mathbf{r}^T \mathbf{K}_{ve} \mathbf{z} &= -(\dot{\mathbf{z}}^T + \mathbf{z}^T \mathbf{\Lambda}) \mathbf{D} (\dot{\mathbf{z}} + \mathbf{\Lambda} \mathbf{z}) \\
&\quad - (\dot{\mathbf{z}}^T + \mathbf{z}^T \mathbf{\Lambda}) \mathbf{K}_{ve} \mathbf{z} \\
&= -\dot{\mathbf{z}}^T \mathbf{D} \dot{\mathbf{z}} - \mathbf{z}^T \mathbf{\Lambda} \mathbf{D} \dot{\mathbf{z}} - \dot{\mathbf{z}}^T \mathbf{D} \mathbf{\Lambda} \mathbf{z} \\
&\quad - \mathbf{z}^T \mathbf{\Lambda} \mathbf{D} \mathbf{\Lambda} \mathbf{z} - \dot{\mathbf{z}}^T \mathbf{K}_{ve} \mathbf{z} - \mathbf{z}^T \mathbf{\Lambda} \mathbf{K}_{ve} \mathbf{z},
\end{aligned} \tag{1.222}$$

by taking Property 1.2 into account, one gets from (1.220)–(1.222)

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= -\tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{K}_{pD} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{K}_{qe}) \tilde{\mathbf{q}} - \dot{\tilde{\mathbf{q}}}^T \mathbf{K}_{pD} \dot{\tilde{\mathbf{q}}} \\
&\quad - \mathbf{z}^T (\mathbf{\Lambda} \mathbf{D} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{K}_{ve}) \mathbf{z} - \dot{\mathbf{z}}^T (\mathbf{D} + k_d \mathbf{H}(\mathbf{q})) \dot{\mathbf{z}} \\
&\quad + \mathbf{s}^T (\mathbf{H}(\mathbf{q}) \mathbf{\Lambda} \dot{\mathbf{z}} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{z}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s} + \mathbf{K}_p \dot{\mathbf{z}}) \\
&\quad - k_d \mathbf{z}^T \mathbf{\Lambda} \mathbf{H}(\mathbf{q}) \dot{\mathbf{z}} + \mathbf{r}^T (\mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{r} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{z}}) + (\mathbf{s}^T - \mathbf{r}^T) \mathbf{f}_1 \\
&= -\tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{K}_{pD} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{K}_{qe} - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{\Lambda}) \tilde{\mathbf{q}} - \dot{\tilde{\mathbf{q}}}^T (\mathbf{K}_{pD} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}})) \dot{\tilde{\mathbf{q}}} \\
&\quad - \mathbf{z}^T (\mathbf{\Lambda} \mathbf{D} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{K}_{ve} - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{\Lambda}) \mathbf{z} \\
&\quad - \dot{\mathbf{z}}^T (\mathbf{D} + k_d \mathbf{H}(\mathbf{q}) - \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{z}} \\
&\quad + \tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{C}^T(\mathbf{q}, \dot{\mathbf{z}}) + \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}})) \dot{\tilde{\mathbf{q}}} \\
&\quad + \mathbf{z}^T (-k_d \mathbf{\Lambda} \mathbf{H}(\mathbf{q}) + \mathbf{\Lambda} \mathbf{C}^T(\mathbf{q}, \dot{\mathbf{z}}) + \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{z}} \\
&\quad + \tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{H}(\mathbf{q}) \mathbf{\Lambda} - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{\Lambda} \mathbf{K}_p) \dot{\mathbf{z}} \\
&\quad + \dot{\tilde{\mathbf{q}}}^T (\mathbf{H}(\mathbf{q}) \mathbf{\Lambda} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_p) \dot{\mathbf{z}} + \mathbf{s}_x^T (\mathbf{D}_\Delta \dot{\delta}_d + \mathbf{f}) \\
&= -\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{s}_x^T (\mathbf{D}_\Delta \dot{\delta}_d + \mathbf{f}).
\end{aligned} \tag{1.223}$$

Finally, from (1.184) it is

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= -\mathbf{x}^T \mathbf{Q} \mathbf{x} \\
&\quad + \sum_{i=1}^n \sum_{j=1}^{m_i} \left( \dot{\delta}_{dij} s_{xij} d_{ij} - d_{ij} \frac{\|\dot{\delta}_{dij} s_{xij}\|^2}{\|\dot{\delta}_{dij} s_{xij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\
&\leq -\mathbf{x}^T \mathbf{Q} \mathbf{x} + \sum_{i=1}^n \sum_{j=1}^{m_i} \left( \|\dot{\delta}_{dij} s_{xij}\| d_{ij} - d_{ij} \frac{\|\dot{\delta}_{dij} s_{xij}\|^2}{\|\dot{\delta}_{dij} s_{xij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \\
&= -\mathbf{x}^T \mathbf{Q} \mathbf{x} + \sum_{i=1}^n \sum_{j=1}^{m_i} \left( d_{ij} \frac{\|\dot{\delta}_{dij} s_{xij}\|}{\|\dot{\delta}_{dij} s_{xij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \epsilon_{ij} e^{-\beta_{ij} t} \right) \\
&\leq -\mathbf{x}^T \mathbf{Q} \mathbf{x} + \sum_{i=1}^n \sum_{j=1}^{m_i} d_{ij} \epsilon_{ij} e^{-\beta_{ij} t} \\
&\leq -\mathbf{x}^T \mathbf{Q} \mathbf{x} + \epsilon e^{-\beta t}.
\end{aligned} \tag{1.224}$$

The stability of the system can be proven by using Theorem 1.1 with  $\lambda_3$  given by (1.208) and assuming  $\mathbf{Q}$  to be positive definite. It can be shown (see Section 1.4.2), that conditions (1.199) to (1.201) together with the definition of  $\lambda_3$  guarantee that  $\mathbf{Q} > \mathbf{O}$ . Since

$$\|\mathbf{x}\|^2 \leq \varphi^2 z_m^2 \Rightarrow \|\dot{\mathbf{z}}\|^2 \leq \varphi^2 z_m^2, \quad (1.225)$$

a proper region of attraction should be selected in such a way that (1.225) is always true for  $t \geq 0$ . If  $\beta = \lambda$  and  $\mathbf{Q} > \mathbf{O}$ , the following must hold:

$$\|\mathbf{x}\|^2 \leq \frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1} t e^{-\lambda t}. \quad (1.226)$$

In view of (1.225) and (1.226), it should be satisfied that

$$\frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1} t e^{-\lambda t} \leq \varphi^2 z_m^2. \quad (1.227)$$

Since the maximum of the left-hand side of (1.227) is a function of  $\|\mathbf{x}_0\|$ , it is easier, although more conservative, to compute the maxima of both terms separately, so that one has

$$\frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 + \frac{\epsilon}{\lambda_1 \lambda} e^{-1} \leq \varphi^2 z_m^2. \quad (1.228)$$

The region of attraction (1.204) comes from (1.228). If  $\beta \neq \lambda$ , it should be true that

$$\|\mathbf{x}\|^2 \leq \frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1 (\lambda - \beta)} (e^{-\beta t} - e^{-\lambda t}). \quad (1.229)$$

From (1.225) it is

$$\frac{\lambda_2}{\lambda_1} \|\mathbf{x}_0\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1 (\lambda - \beta)} (e^{-\beta t} - e^{-\lambda t}) \leq \varphi^2 z_m^2. \quad (1.230)$$

By calculating the maxima of both terms on the right-hand side of (1.230) as before, the region of attraction (1.205) can be derived.

*b)* It must be proven that the desired trajectory given by (1.182) and (1.183) remains bounded. In view of part *a)* of the proof, a constant  $\hat{v}_m$  exists such that  $\|\dot{\hat{\mathbf{q}}}\| < \hat{v}_m$ . By using again

$$\mathbf{s}_0 = \dot{\hat{\delta}}_d + \mathbf{\Lambda}_\delta \delta_d \quad (1.231)$$

$$\mathbf{x}_\delta = \begin{bmatrix} \delta_d \\ \dot{\delta}_d \end{bmatrix}, \quad (1.232)$$

Eq. (1.182) can be written as

$$\begin{aligned} \mathbf{H}_{\delta\delta}\dot{\mathbf{s}}_0 = & -(\mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{s}_0 + \mathbf{K}_{pD\delta}\mathbf{s}_0 + \mathbf{K}_{qe\delta}\boldsymbol{\delta}_d + \mathbf{D}_\Delta\dot{\boldsymbol{\delta}}_d + \mathbf{f}_r) \\ & - \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}})\mathbf{s}_0 + \mathbf{f}), \end{aligned} \quad (1.233)$$

with

$$\begin{aligned} \mathbf{f}_r \triangleq & \mathbf{H}_{\theta\delta}^T\ddot{\boldsymbol{\theta}}_r + \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{q}})\dot{\boldsymbol{\theta}}_r + \mathbf{g}_\delta - \mathbf{H}_{\delta\delta}\boldsymbol{\Lambda}_\delta\dot{\boldsymbol{\delta}} - \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}})\boldsymbol{\Lambda}_\delta\boldsymbol{\delta} \\ & - \mathbf{D}_\delta\boldsymbol{\Lambda}_\delta\boldsymbol{\delta} - \mathbf{K}_{q\delta}\boldsymbol{\delta} - \mathbf{K}_{p\delta}\dot{\boldsymbol{\delta}} - \mathbf{K}_{p\delta}\boldsymbol{\Lambda}_\delta\boldsymbol{\delta} + \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}})\boldsymbol{\Lambda}_\delta\boldsymbol{\delta}. \end{aligned} \quad (1.234)$$

To compute (1.233) and (1.234), Property 1.3 has been used in the form:

$$\begin{aligned} \begin{bmatrix} \mathbf{C}_{\theta\theta}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{C}_{\theta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}}_r \\ \dot{\boldsymbol{\delta}}_r \end{bmatrix} = & \begin{bmatrix} \mathbf{C}_{\theta\theta}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{C}_{\theta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}}_r \\ \dot{\boldsymbol{\delta}}_r \end{bmatrix} \\ & - \begin{bmatrix} \mathbf{C}_{\theta\theta}(\mathbf{q}, \dot{\mathbf{z}}) & \mathbf{C}_{\theta\delta}(\mathbf{q}, \dot{\mathbf{z}}) \\ \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{z}}) & \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}}) \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}}_r \\ \dot{\boldsymbol{\delta}}_r \end{bmatrix}. \end{aligned} \quad (1.235)$$

Since  $\|\mathbf{q}\| < \infty$ ,  $\|\dot{\mathbf{q}}\| \leq v_m$  and  $\|\dot{\mathbf{q}}\| \leq \hat{v}_m$ ,  $\mathbf{f}_r$  is only a function of bounded variables, which implies that a positive constant  $f_{r,\max}$  must exist such that (see Properties 1.9 to 1.12)

$$\|\mathbf{f}_r\| \leq f_{r,\max}. \quad (1.236)$$

A Lyapunov function for system (1.233) is

$$\begin{aligned} V_\delta(\mathbf{x}_\delta, t) \triangleq V_\delta(\mathbf{x}_\delta) = & \frac{1}{2}\mathbf{s}_0^T \mathbf{H}_{\delta\delta}\mathbf{s}_0 + \frac{1}{2}\boldsymbol{\delta}_d^T (2\boldsymbol{\Lambda}_\delta\mathbf{K}_{pD\delta} + \mathbf{K}_{qe\delta} \\ & + \boldsymbol{\Lambda}_\delta\mathbf{D}_\Delta)\boldsymbol{\delta}_d \\ = & \frac{1}{2}\mathbf{x}_\delta^T \begin{bmatrix} \boldsymbol{\Lambda}_\delta\mathbf{H}_{\delta\delta}\boldsymbol{\Lambda}_\delta + 2\boldsymbol{\Lambda}_\delta\mathbf{K}_{pD\delta} + \mathbf{K}_{qe\delta} + \boldsymbol{\Lambda}_\delta\mathbf{D}_\Delta & \boldsymbol{\Lambda}_\delta\mathbf{H}_{\delta\delta} \\ \mathbf{H}_{\delta\delta}\boldsymbol{\Lambda}_\delta & \mathbf{H}_{\delta\delta} \end{bmatrix} \mathbf{x}_\delta \\ \triangleq & \frac{1}{2}\mathbf{x}_\delta^T \mathbf{M}\mathbf{x}_\delta. \end{aligned} \quad (1.237)$$

In view of Property 1.9, one has

$$\hat{\lambda}_1\|\mathbf{x}_\delta\|^2 \leq V_\delta(\mathbf{x}_\delta) \leq \hat{\lambda}_2\|\mathbf{x}_\delta\|^2 \quad (1.238)$$

$$\hat{\lambda}_1 \triangleq \frac{1}{2} \min_{\mathbf{q} \in \mathcal{Q}^{n+m}} (\lambda_{\min}(\mathbf{M})) \quad (1.239)$$

$$\hat{\lambda}_2 \triangleq \frac{1}{2} \max_{\mathbf{q} \in \mathcal{Q}^{n+m}} (\lambda_{\max}(\mathbf{M})). \quad (1.240)$$

The derivative of (1.237) is

$$\dot{V}_\delta(\mathbf{x}_\delta) = \frac{1}{2} \mathbf{s}_0^T \dot{\mathbf{H}}_{\delta\delta} \mathbf{s}_0 + \delta_d^T (2\mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} + \mathbf{K}_{qe\delta} + \mathbf{\Lambda}_\delta \mathbf{D}_\Delta) \dot{\delta}_d + \mathbf{s}_0^T \mathbf{H}_{\delta\delta} \dot{\mathbf{s}}_0. \quad (1.241)$$

Substituting (1.233) into (1.241) and taking Property 1.2 into account leads to

$$\begin{aligned} \dot{V}_\delta(\mathbf{x}_\delta) &= \frac{1}{2} \mathbf{s}_0^T \dot{\mathbf{H}}_{\delta\delta} \mathbf{s}_0 + \delta_d^T (2\mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} + \mathbf{K}_{qe\delta} + \mathbf{\Lambda}_\delta \mathbf{D}_\Delta) \dot{\delta}_d \quad (1.242) \\ &\quad - \mathbf{s}_0^T \left( \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}}) \mathbf{s}_0 + \mathbf{K}_{qe\delta} \delta_d + \mathbf{K}_{pD\delta} \mathbf{s}_0 + \mathbf{D}_\Delta \dot{\delta}_d \right) \\ &\quad - \mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f} - \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s}_0) \\ &= -\delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{qe\delta} \delta_d - \dot{\delta}_d^T \mathbf{D}_\Delta \dot{\delta}_d - \dot{\delta}_d^T \mathbf{K}_{pD\delta} \dot{\delta}_d \\ &\quad - \dot{\delta}_d^T \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta \delta_d - \delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \dot{\delta}_d - \delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta \delta_d \\ &\quad + \dot{\delta}_d^T \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta \delta_d + \delta_d^T \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \dot{\delta}_d - \mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f} - \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s}_0) \\ &= -\delta_d^T (\mathbf{\Lambda}_\delta \mathbf{K}_{qe\delta} + \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta) \delta_d - \dot{\delta}_d^T (\mathbf{D}_\Delta + \mathbf{K}_{pD\delta}) \dot{\delta}_d \\ &\quad - \mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f} - \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s}_0) \\ &= -\mathbf{x}_\delta^T \mathbf{P} \mathbf{x}_\delta - \mathbf{s}_0^T (\mathbf{f}_r + \mathbf{f} - \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{s}_0), \end{aligned}$$

where  $\mathbf{P}$  is given by (1.216). Notice that

$$\|\mathbf{s}_0\| = \|\dot{\delta}_d + \mathbf{\Lambda}_\delta \delta_d\| \leq \|\dot{\delta}_d\| + \lambda_{\max}(\mathbf{\Lambda}_\delta) \|\delta_d\| \leq (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) \|\mathbf{x}_\delta\|. \quad (1.243)$$

Rewrite  $\mathbf{f}$  as

$$\mathbf{f} = -\mathbf{D}_\Delta \text{diag}\{\dot{\delta}_{d11}, \dots, \dot{\delta}_{dnm_n}\} \begin{bmatrix} \frac{\dot{\delta}_{d11} s_{x11}}{\|\dot{\delta}_{d11} s_{x11}\| + \epsilon_{11} e^{-\beta_{11} t}} \\ \vdots \\ \frac{\dot{\delta}_{dnm_n} s_{xnm_n}}{\|\dot{\delta}_{dnm_n} s_{xnm_n}\| + \epsilon_{nm_n} e^{-\beta_{nm_n} t}} \end{bmatrix}, \quad (1.244)$$

to show that

$$\|\mathbf{f}\| \leq \lambda_{\max}(\mathbf{D}_\Delta) \sqrt{m} \|\mathbf{x}_d\|. \quad (1.245)$$

To find a bound for  $\mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}})$ , consider (1.180). From part *a*) of the proof, it is

$$\|\mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{z}})\| \leq k_{c\delta\delta} \varphi z_m. \quad (1.246)$$

By taking Eqs. (1.243)–(1.246) into account, (1.242) can be written as

$$\begin{aligned} \dot{V}_\delta(\mathbf{x}_\delta) &\leq -\lambda_{\min}(\mathbf{P})\|\mathbf{x}_\delta\|^2 + f_{r,\max}(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))\|\mathbf{x}_\delta\| \\ &\quad + (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))(\lambda_{\max}(\mathbf{D}_\Delta)\sqrt{m} + k_{c\delta\delta}\varphi z_m(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)))\|\mathbf{x}_\delta\|^2 \\ &\triangleq -\hat{\lambda}_3\|\mathbf{x}_\delta\|^2 + \sigma\|\mathbf{x}_\delta\|. \end{aligned} \quad (1.247)$$

Since  $\hat{\lambda}_3 > 0$  (by assumption), the boundedness of  $\|\mathbf{x}_\delta\|$  can be proven using Theorem 1.2 with  $\gamma = 0$ .  $\triangle$

In the case that the term  $\mathbf{f}$  is not to be used, i.e.  $\mathbf{D}_\Delta = \mathbf{O}$ ,  $\delta_d$  can be computed from

$$\begin{aligned} \ddot{\delta}_d &= \mathbf{\Lambda}_\delta(\dot{\delta} - \dot{\delta}_d) - \mathbf{H}_{\delta\delta}^{-1}(\mathbf{H}_{\theta\delta}^T \ddot{\theta}_r + \mathbf{C}_{\delta\theta}(\mathbf{q}, \dot{\mathbf{q}})\dot{\theta}_r + \mathbf{C}_{\delta\delta}(\mathbf{q}, \dot{\mathbf{q}})\dot{\delta}_r) \\ &\quad + \mathbf{K}\delta_d + \mathbf{D}_\delta\dot{\delta}_r + \mathbf{g}_\delta - \mathbf{K}_{q\delta}\ddot{\mathbf{q}} - \mathbf{K}_{p\delta}\dot{\mathbf{s}}_\delta \end{aligned} \quad (1.248)$$

with initial condition

$$\delta_d(0) = \dot{\delta}_d(0) = \mathbf{0}. \quad (1.249)$$

Eqs. (1.188) and (1.197) then become

$$\begin{aligned} \mathbf{H}(\mathbf{q})\dot{\mathbf{s}} &= -(\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{s} + \mathbf{K}_{pD}\mathbf{s} + \mathbf{K}_{qe}\tilde{\mathbf{q}}) + \mathbf{H}(\mathbf{q})\mathbf{\Lambda}\dot{\mathbf{z}} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{z}} \\ &\quad + \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}})\mathbf{s} + \mathbf{K}_p\dot{\mathbf{z}} \\ \mathbf{H}(\mathbf{q})\dot{\mathbf{r}} &= -(\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{r} + \mathbf{K}_{ve}\mathbf{z} + \mathbf{D}\mathbf{r} + k_d\mathbf{H}(\mathbf{q})\dot{\mathbf{z}}) + \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}})\mathbf{r} \\ &\quad - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{z}}. \end{aligned} \quad (1.250)$$

The next Corollary establishes the stability of system (1.250) and (1.251) when  $\mathbf{D}_\Delta = \mathbf{O}$ .

### Corollary 1.1

Given a bounded continuous desired trajectory  $\theta_d$  with bounded velocity and acceleration, if  $\|\dot{\mathbf{q}}\| \leq v_m$  where  $v_m$  is a positive scalar constant, then the equilibrium point  $\mathbf{x} = \mathbf{0}$  of (1.250) and (1.251) is asymptotically stable if conditions (1.199)–(1.201) are satisfied. A region of attraction is given by

$$S = \left\{ \mathbf{x} \in \mathfrak{R}^{4(n+m)} : \|\mathbf{x}\| \leq \sqrt{\frac{\lambda_1}{\lambda_2}}\varphi z_m \right\}, \quad (1.252)$$

where  $\lambda_1$ ,  $\lambda_2$ ,  $\varphi$  and  $z_m$  are the same as in Theorem 1.5. In addition, the desired trajectory  $\delta_d$  and  $\dot{\delta}_d$  given by (1.248) and (1.249) remains bounded

if

$$\hat{\lambda}_3 \triangleq \lambda_{\min}(\mathbf{P}) - k_{c\delta\delta}\varphi z_m(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))^2 > 0 \quad (1.253)$$

with

$$\mathbf{P} \triangleq \begin{bmatrix} \mathbf{\Lambda}_\delta \mathbf{K}_{qe\delta} + \mathbf{\Lambda}_\delta \mathbf{K}_{pD\delta} \mathbf{\Lambda}_\delta & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{pD\delta} \end{bmatrix} \quad (1.254)$$

$$\lambda_{\min}(\mathbf{P}) = \min\{\lambda_{\min}(\mathbf{\Lambda}_\delta)\lambda_{\min}(\mathbf{K}_{qe\delta}) + \lambda_{\min}^2(\mathbf{\Lambda}_\delta)\lambda_{\min}(\mathbf{K}_{pD\delta}), \lambda_{\min}(\mathbf{K}_{pD\delta})\}. \quad (1.255)$$

*Proof:* The proof is similar to that of Theorem 1.5 just by letting  $\mathbf{D}_\Delta = \mathbf{O}$ .

a) Firstly, the stability of the whole system will be proven. Consider again the Lyapunov function (1.218). From (1.224) one has

$$\dot{V}(\mathbf{x}) = -\mathbf{x}^T \mathbf{Q} \mathbf{x}. \quad (1.256)$$

Since

$$\lambda_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}, t) \leq \lambda_2 \|\mathbf{x}\|^2, \quad (1.257)$$

$V(\mathbf{x}, t)$  is a decrescent function. On the other hand, since

$$\dot{V}(\mathbf{x}) \leq -\lambda_3 \|\mathbf{x}\|^2 \quad (1.258)$$

with  $\lambda_3$  as in (1.208), system (1.250) and (1.251) is asymptotically stable [Vidyasagar (1978)]. As before, the region of attraction (1.252) can be computed by taking (1.257) and the definition of  $\lambda_3$  into account.

b) To prove the boundedness of the desired trajectory given by (1.248) and (1.249), the existence of a constant  $\hat{v}_m$  such that  $\|\dot{\hat{\mathbf{q}}}\| < \hat{v}_m$  is necessary. Note that  $\hat{v}_m$  must exist in view of part a) of the proof. By employing (1.231)–(1.233) and the Lyapunov function (1.237), and setting  $\mathbf{f}$  to  $\mathbf{0}$ , one gets (see (1.247))

$$\dot{V}_\delta(\mathbf{x}_\delta) \leq -\hat{\lambda}_3 \|\mathbf{x}_\delta\|^2 + \sigma \|\mathbf{x}_\delta\|. \quad (1.259)$$

The proof is accomplished by using Theorem 1.2 with  $\gamma = 0$ .  $\triangle$

As discussed in Section 1.3.3,  $\mathbf{f}$  helps increase the damping of the system

since every element of  $\mathbf{D}_\Delta \dot{\boldsymbol{\delta}}_d + \mathbf{f}$  can be expressed as

$$d_{ij} \dot{\delta}_{ij} \left( 1 - \frac{\dot{\delta}_{dij} s_{xij}}{\|\dot{\delta}_{dij} s_{xij}\| + \epsilon_{ij} e^{-\beta_{ij} t}} \right) \quad i = 1, \dots, n, \quad j = 1, \dots, m_i,$$

so that  $\boldsymbol{\delta}_d$  is damped. When the tracking error becomes negligible,  $\boldsymbol{\delta}$  becomes damped as well. Note that this is true even if  $\boldsymbol{\theta}_d$  is not a constant vector. It is not difficult to show that if  $\boldsymbol{\theta}_d$  is constant and  $\mathbf{g}_\delta(\mathbf{q}) = \mathbf{g}_\delta(\boldsymbol{\theta})$ , then  $\boldsymbol{\delta}_d$  becomes  $-\mathbf{K}^{-1} \mathbf{g}_\delta$ .

It should be noticed that both Theorem 1.5 and Corollary 1.1 assume that  $\hat{\lambda}_3 > 0$ . From its definition, it is obvious that it is always possible to achieve this goal by letting  $\varphi$  be small enough and  $\lambda_{\min}(\mathbf{K}_p)$  large enough. Nevertheless, it is not obvious whether one can enlarge the region of attraction arbitrarily. To show that this is actually possible, let  $\lambda_{\min}(\mathbf{K}_{p\delta})$  be large enough and choose  $k_d$ ,  $\mathbf{K}_q$ ,  $\mathbf{K}_v$  to satisfy conditions (1.199)–(1.201). Note that  $\mathbf{A}$  can be selected freely. From (1.212) it can be seen that

$$z_m = \frac{\lambda_{pd}}{k_c} \quad (1.260)$$

with a proper choice of  $k_d$ ,  $\lambda_q$  and  $\lambda_v$ . Although some of these constants appear not only in the numerator but also in the denominator, it should be pointed out that they appear in the numerator as a product and in the denominator as a summation, so that a parameter selection can always be found so that the assumption (1.260) is valid. Without loss of generality, it can be assumed that

$$\lambda_{\min}(\mathbf{K}_p) = \lambda_{\min}(\mathbf{K}_{p\delta}) \quad (1.261)$$

$$z_m = \frac{\lambda_{\min}(\mathbf{K}_{p\delta})}{k_c}. \quad (1.262)$$

Also without loss of generality, from (1.217) it can be assumed that

$$\lambda_{\min}(\mathbf{P}) = \lambda_{\min}(\mathbf{K}_{p\delta}) + \lambda_{\min}(\mathbf{D}_\Delta), \quad (1.263)$$

since  $\lambda_{\min}(\mathbf{K}_{qe\delta})$  can be tuned arbitrarily large. Taking into account

Eqs. (1.261)–(1.263) together with (1.215) yields

$$\begin{aligned}\hat{\lambda}_3 &= \lambda_{\min}(\mathbf{K}_{p\delta}) + \lambda_{\min}(\mathbf{D}_\Delta) \\ &\quad - (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) (\lambda_{\max}(\mathbf{D}_\Delta)\sqrt{m} + k_{c\delta\delta}\varphi z_m(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))) \\ &= \lambda_{\min}(\mathbf{K}_{p\delta}) + \lambda_{\min}(\mathbf{D}_\Delta) \\ &\quad - (1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) \left( \lambda_{\max}(\mathbf{D}_\Delta)\sqrt{m} + \frac{\lambda_{\min}(\mathbf{K}_{p\delta})}{k_c}\varphi k_{c\delta\delta}(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) \right),\end{aligned}\quad (1.264)$$

which implies that

$$\lambda_{\min}(\mathbf{K}_{p\delta}) \left( 1 - \frac{k_{c\delta\delta}}{k_c}\varphi(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))^2 \right) \quad (1.265)$$

$$- \lambda_{\max}(\mathbf{D}_\Delta)\sqrt{m}(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) + \lambda_{\min}(\mathbf{D}_\Delta) > 0 \quad (1.266)$$

must be satisfied, or in other form

$$\lambda_{\min}(\mathbf{K}_{p\delta}) > \frac{\lambda_{\max}(\mathbf{D}_\Delta)\sqrt{m}(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta)) - \lambda_{\min}(\mathbf{D}_\Delta)}{\left( 1 - \frac{k_{c\delta\delta}}{k_c}\varphi(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))^2 \right)} \quad (1.267)$$

$$\frac{k_c}{k_{c\delta\delta}(1 + \lambda_{\max}(\mathbf{\Lambda}_\delta))^2} > \varphi. \quad (1.268)$$

Since condition (1.268) can always be accomplished, it is possible to choose  $\lambda_{\min}(\mathbf{K}_{p\delta})$  as large as wished and, as a direct consequence,  $z_m$  and the regions of attraction (1.204) and (1.205). The same conclusion can be derived if  $\mathbf{D}_\Delta = \mathbf{O}$ . Conditions (1.202) and (1.203) can also be satisfied by letting either  $\epsilon$  be small enough or  $z_m$  be large enough.

#### 1.4.2 Analysis of matrix $\mathbf{Q}$

In this section, the conditions under which the matrix  $\mathbf{Q}$  (used in the proofs of Section 1.4.1) is positive definite will be studied. A direct analysis is too

difficult. Consider again Eqs. (1.223) and (1.224) to get

$$\begin{aligned}
\dot{V}(\mathbf{x}) \leq & -\tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{K}_{pD} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{K}_{qe} - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{\Lambda}) \tilde{\mathbf{q}} \\
& -\dot{\tilde{\mathbf{q}}}^T (\mathbf{K}_{pD} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}})) \dot{\tilde{\mathbf{q}}} \\
& -\mathbf{z}^T (\mathbf{\Lambda} \mathbf{D} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{K}_{ve} - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) \mathbf{\Lambda}) \mathbf{z} \\
& -\dot{\mathbf{z}}^T (\mathbf{D} + k_d \mathbf{H}(\mathbf{q}) - \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{z}} \\
& +\tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{C}^T(\mathbf{q}, \dot{\mathbf{z}}) + \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}})) \dot{\tilde{\mathbf{q}}} \\
& +\mathbf{z}^T (-k_d \mathbf{\Lambda} \mathbf{H}(\mathbf{q}) + \mathbf{\Lambda} \mathbf{C}^T(\mathbf{q}, \dot{\mathbf{z}}) + \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{z}}) - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{z}} \\
& +\tilde{\mathbf{q}}^T (\mathbf{\Lambda} \mathbf{H}(\mathbf{q}) \mathbf{\Lambda} - \mathbf{\Lambda} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{\Lambda} \mathbf{K}_p) \dot{\mathbf{z}} \\
& +\dot{\tilde{\mathbf{q}}}^T (\mathbf{H}(\mathbf{q}) \mathbf{\Lambda} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_p) \dot{\mathbf{z}} + \epsilon e^{-\beta t}
\end{aligned} \tag{1.269}$$

and take norm bounds into account, so that (1.269) can be written as

$$\begin{aligned}
\dot{V}(\mathbf{x}) \leq & -\|\tilde{\mathbf{q}}\|^2 (\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{\mathbf{z}}\|) \\
& -\|\dot{\tilde{\mathbf{q}}}\|^2 (\lambda_{pd} - k_c \|\dot{\mathbf{z}}\|) \\
& -\|\mathbf{z}\|^2 (\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d - \lambda_M^2 k_c \|\dot{\mathbf{z}}\|) \\
& -\|\dot{\mathbf{z}}\|^2 (\lambda_d + k_d \lambda_h - k_c \|\dot{\mathbf{z}}\| - k_c v_m) \\
& +2\lambda_M k_c \|\dot{\mathbf{z}}\| \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\| \\
& +\|\mathbf{z}\| \|\dot{\mathbf{z}}\| (k_d \lambda_M \lambda_H + \lambda_M k_c v_m + 2\lambda_M k_c \|\dot{\mathbf{z}}\|) \\
& +\|\tilde{\mathbf{q}}\| \|\dot{\mathbf{z}}\| (\lambda_M^2 \lambda_H + \lambda_M k_c v_m + \lambda_M \lambda_P) \\
& +\|\dot{\tilde{\mathbf{q}}}\| \|\dot{\mathbf{z}}\| (\lambda_M \lambda_H + k_c v_m + \lambda_P) + \epsilon e^{-\beta t}.
\end{aligned} \tag{1.270}$$

Define

$$\hat{\mathbf{x}} \triangleq \begin{bmatrix} \|\tilde{\mathbf{q}}\| \\ \|\dot{\tilde{\mathbf{q}}}\| \\ \|\mathbf{z}\| \\ \|\dot{\mathbf{z}}\| \end{bmatrix} \tag{1.271}$$

$$\hat{Q} \triangleq \begin{bmatrix} \lambda_m^2 \lambda_{pd} & -\lambda_M k_c \|\dot{z}\| & 0 & -\frac{1}{2}(\lambda_M^2 \lambda_H \\ +\lambda_m \lambda_{qe} & & & +\lambda_M k_c v_m \\ -\lambda_M^2 k_c \|\dot{z}\| & & & +\lambda_M \lambda_P) \\ \\ -\lambda_M k_c \|\dot{z}\| & \lambda_{pd} & 0 & -\frac{1}{2}(\lambda_M \lambda_H \\ -k_c \|\dot{z}\| & & & +k_c v_m \\ & & & +\lambda_P) \\ \\ 0 & 0 & \lambda_m \lambda_{ve} & -\frac{1}{2}(k_d \lambda_M \lambda_H \\ +\lambda_m^2 \lambda_d & & +\lambda_M k_c v_m) \\ -\lambda_M^2 k_c \|\dot{z}\| & & -\lambda_M k_c \|\dot{z}\| \\ \\ -\frac{1}{2}(\lambda_M^2 \lambda_H & -\frac{1}{2}(\lambda_M \lambda_H & -\frac{1}{2}(k_d \lambda_M \lambda_H & \lambda_d \\ +\lambda_M k_c v_m & +k_c v_m & +\lambda_M k_c v_m) & +k_d \lambda_h \\ +\lambda_M \lambda_P) & +\lambda_P) & -\lambda_M k_c \|\dot{z}\| & -k_c v_m \\ & & & -k_c \|\dot{z}\| \end{bmatrix} \quad (1.272)$$

and rewrite (1.270) as

$$\dot{V}(\mathbf{x}) \leq -\hat{\mathbf{x}}^T \hat{Q} \hat{\mathbf{x}} + \epsilon e^{-\beta t}. \quad (1.273)$$

From (1.273) it can be seen that  $Q > O$  as long as  $\hat{Q} > O$ . However, it is still too hard to say whether or not  $\hat{Q} > O$ . On the other hand, one can express  $\hat{Q}$  as

$$\hat{Q} = \begin{bmatrix} \frac{1}{3}(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{z}\|) & -\lambda_M k_c \|\dot{z}\| & 0 & 0 \\ -\lambda_M k_c \|\dot{z}\| & \frac{1}{3}(\lambda_{pd} - k_c \|\dot{z}\|) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.274)$$

$$+ \begin{bmatrix} \frac{1}{3}(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{z}\|) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d - \lambda_M^2 k_c \|\dot{z}\|) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
& + \begin{bmatrix} \frac{1}{3}(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{z}\|) & 0 & 0 & -\frac{1}{2}(\lambda_M^2 \lambda_H + \lambda_M k_c v_m + \lambda_M \lambda_P) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}(\lambda_M^2 \lambda_H + \lambda_M k_c v_m + \lambda_M \lambda_P) & 0 & 0 & \frac{1}{3}(\lambda_d + k_d \lambda_h - k_c v_m - k_c \|\dot{z}\|) \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}(\lambda_{pd} - k_c \|\dot{z}\|) & 0 & 0 \\ 0 & 0 & \frac{1}{3}(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d - \lambda_M^2 k_c \|\dot{z}\|) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}(\lambda_{pd} - k_c \|\dot{z}\|) & 0 & -\frac{1}{2}(\lambda_M \lambda_H + k_c v_m + \lambda_P) \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}(\lambda_M \lambda_H + k_c v_m + \lambda_P) & 0 & \frac{1}{3}(\lambda_d + k_d \lambda_h - k_c v_m - k_c \|\dot{z}\|) \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{3}(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d - \lambda_M^2 k_c \|\dot{z}\|) & -\frac{1}{2}(k_d \lambda_M \lambda_H + \lambda_M k_c v_m) \\ 0 & -\frac{1}{2}(k_d \lambda_M \lambda_H + \lambda_M k_c v_m) & \frac{1}{3}(\lambda_d + k_d \lambda_h - k_c v_m - k_c \|\dot{z}\|) \end{bmatrix}
\end{aligned}$$

and define the matrices

$$Q_1 \triangleq \begin{bmatrix} \frac{1}{3}(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{z}\|) & -\lambda_M k_c \|\dot{z}\| \\ -\lambda_M k_c \|\dot{z}\| & \frac{1}{3}(\lambda_{pd} - k_c \|\dot{z}\|) \end{bmatrix} \quad (1.275)$$

$$Q_2 \triangleq \begin{bmatrix} \frac{1}{3}(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{z}\|) \\ 0 \end{bmatrix} \quad (1.276)$$

$$\begin{bmatrix} 0 \\ \frac{1}{3}(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d - \lambda_M^2 k_c \|\dot{z}\|) \end{bmatrix}$$

$$Q_3 \triangleq \begin{bmatrix} \frac{1}{3}(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} - \lambda_M^2 k_c \|\dot{z}\|) \\ -\frac{1}{2}(\lambda_M^2 \lambda_H + \lambda_M k_c v_m + \lambda_M \lambda_P) \end{bmatrix} \quad (1.277)$$

$$\begin{bmatrix} -\frac{1}{2}(\lambda_M^2 \lambda_H + \lambda_M k_c v_m + \lambda_M \lambda_P) \\ \frac{1}{3}(\lambda_d + k_d \lambda_h - k_c v_m - k_c \|\dot{z}\|) \end{bmatrix} \quad (1.278)$$

$$\mathbf{Q}_4 \triangleq \begin{bmatrix} \frac{1}{3}(\lambda_{pd} - k_c \|\dot{\mathbf{z}}\|) & 0 \\ 0 & \frac{1}{3}(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d - \lambda_M^2 k_c \|\dot{\mathbf{z}}\|) \end{bmatrix} \quad (1.279)$$

$$\mathbf{Q}_5 \triangleq \begin{bmatrix} \frac{1}{3}(\lambda_{pd} - k_c \|\dot{\mathbf{z}}\|) \\ -\frac{1}{2}(\lambda_M \lambda_H + k_c v_m + \lambda_P) \end{bmatrix} \quad (1.280)$$

$$\begin{bmatrix} -\frac{1}{2}(\lambda_M \lambda_H + k_c v_m + \lambda_P) \\ \frac{1}{3}(\lambda_d + k_d \lambda_h - k_c v_m - k_c \|\dot{\mathbf{z}}\|) \end{bmatrix} \quad (1.281)$$

$$\mathbf{Q}_6 \triangleq \begin{bmatrix} \frac{1}{3}(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d & -\frac{1}{2}(k_d \lambda_M \lambda_H + \lambda_M k_c v_m) \\ -\lambda_M^2 k_c \|\dot{\mathbf{z}}\| & -\lambda_M k_c \|\dot{\mathbf{z}}\| \\ -\frac{1}{2}(k_d \lambda_M \lambda_H + \lambda_M k_c v_m) & \frac{1}{3}(\lambda_d + k_d \lambda_h - k_c v_m \\ -\lambda_M k_c \|\dot{\mathbf{z}}\| & -k_c \|\dot{\mathbf{z}}\|) \end{bmatrix} \quad (1.282)$$

Of course, if every matrix  $\mathbf{Q}_i$  is positive definite, so is  $\hat{\mathbf{Q}}$ . This implies that every term  $\mathbf{Q}_{i11}$  and  $\mathbf{Q}_{i22}$  together with  $\det(\mathbf{Q}_i)$  must be positive.

Regarding  $\mathbf{Q}_1$ , one has

$$\frac{\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe}}{\lambda_M^2 k_c} > \|\dot{\mathbf{z}}\| \quad (1.283)$$

$$\frac{\lambda_{pd}}{k_c} > \|\dot{\mathbf{z}}\| \quad (1.284)$$

$$\left( \frac{\lambda_m^2 \lambda_{pd}^2 + \lambda_m \lambda_{qe} \lambda_{pd}}{8 \lambda_M^2 k_c^2} + \left( \frac{2 \lambda_M^2 \lambda_{pd} + \lambda_m \lambda_{qe}}{16 \lambda_M^2 k_c} \right)^2 \right)^{\frac{1}{2}} - \frac{2 \lambda_M^2 \lambda_{pd} + \lambda_m \lambda_{qe}}{16 \lambda_M^2 k_c} > \|\dot{\mathbf{z}}\|. \quad (1.285)$$

As for  $\mathbf{Q}_2$ , (1.283) must be satisfied together with

$$\frac{\lambda_m^2 \lambda_d + \lambda_m \lambda_{ve}}{\lambda_M^2 k_c} > \|\dot{\mathbf{z}}\|. \quad (1.286)$$

As for  $\mathbf{Q}_3$ , one has condition (1.283) and

$$\lambda_d + k_d \lambda_h - k_c v_m > 0 \quad (1.287)$$

$$\frac{\lambda_d + k_d \lambda_h - k_c v_m}{k_c} > \|\dot{z}\| \quad (1.288)$$

$$(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe})(\lambda_d + k_d \lambda_h - k_c v_m) \quad (1.289)$$

$$- \frac{9}{4}(\lambda_M^2 \lambda_H + \lambda_M k_c v_m + \lambda_M \lambda_P) > 0$$

$$\frac{(\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe})(\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4} \lambda_M (\lambda_M \lambda_H + k_c v_m + \lambda_P)}{k_c (\lambda_m^2 \lambda_{pd} + \lambda_m \lambda_{qe} + \lambda_M^2 (\lambda_d + k_d \lambda_h - k_c v_m))} \quad (1.290)$$

$$> \|\dot{z}\|.$$

Note that  $\mathbf{Q}_4$  is positive definite if conditions (1.284) and (1.286) are satisfied.

As for  $\mathbf{Q}_5$ , one gets (1.284), (1.287), (1.288) together with

$$\lambda_{pd}(\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4}(\lambda_M \lambda_H + k_c v_m + \lambda_P)^2 > 0 \quad (1.291)$$

$$\frac{\lambda_{pd}(\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4}(\lambda_M \lambda_H + k_c v_m + \lambda_P)^2}{k_c (\lambda_{pd} + \lambda_d + k_d \lambda_h - k_c v_m)} > \|\dot{z}\|. \quad (1.292)$$

Finally, as for  $\mathbf{Q}_6$ , one has (1.286)–(1.288) and

$$(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d)(\lambda_d + k_d \lambda_h - k_c v_m) \quad (1.293)$$

$$- \frac{9}{4}(k_d \lambda_M \lambda_H + \lambda_M k_c v_m)^2 > 0$$

$$\left( \frac{(\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d)(\lambda_d + k_d \lambda_h - k_c v_m) - \frac{9}{4}(k_d \lambda_M \lambda_H + \lambda_M k_c v_m)^2}{8 \lambda_M^2 k_c^2} \right) \quad (1.294)$$

$$+ \left( \frac{\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d + \lambda_M^2 (\lambda_d + k_d \lambda_h - k_c v_m) + 9 \lambda_M^2 (k_d \lambda_H + k_c v_m)}{16 k_c \lambda_M^2} \right)^{\frac{1}{2}}$$

$$- \frac{\lambda_m \lambda_{ve} + \lambda_m^2 \lambda_d + \lambda_M^2 (\lambda_d + k_d \lambda_h - k_c v_m) + 9 \lambda_M^2 (k_d \lambda_H + k_c v_m)}{16 k_c \lambda_M^2}$$

$$> \|\dot{z}\|.$$

By sorting Eqs. (1.283) to (1.294), one obtains conditions (1.199)–(1.201) and (1.212).

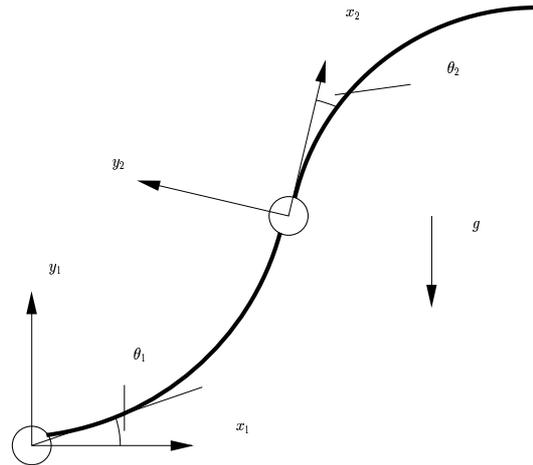


Fig. 1.3 Planar two-link flexible robot arm.

## 1.5 Simulation Results

In order to test the controllers given in Sections 1.3 and 1.4, the planar two-link flexible robot arm modeled in [De Luca and Siciliano (1991)] (Fig. 1.3), and modified in [De Luca and Siciliano (1993b)] to take gravity into account, has been used. A complete description of the arm equations of motion together with the parameters employed in the simulations can be found in [De Luca and Siciliano (1991); De Luca and Siciliano (1993b)].

### 1.5.1 Control Without Observer

The first simulations are aimed at testing the controllers given in Section 1.3. The arm is assumed to be initially in its vertical equilibrium configuration, i.e.

$$\boldsymbol{\theta} = [-90 \ 0]^T [^\circ]$$

and

$$\boldsymbol{\delta} = [0 \ 0 \ 0 \ 0]^T [\text{m}].$$

As for the control goal, the final desired joint angles

$$\boldsymbol{\theta}_d = [-45 \ 45]^T [^\circ]$$

should be reached at a final time  $t_f = 7.5$ [s]. Since control (1.120) has been designed for time-varying trajectories, a linear trajectory interpolation with a fifth-order polynomial is used for the time  $0 \leq t \leq 7.5$ [s]:

$$\boldsymbol{\theta}_d = \begin{bmatrix} -90 + 1.0667t^3 - 0.2133t^4 + 0.0114t^5 \\ 1.0667t^3 - 0.2133t^4 + 0.0114t^5 \end{bmatrix} [^\circ].$$

The matrices  $\mathbf{K}_p$  and  $\mathbf{\Lambda}$  are

$$\mathbf{K}_p = \text{diag}\{10, 10, 0.5, 0.5, 0.5, 0.5\}$$

and

$$\mathbf{\Lambda} = \text{diag}\{1, 1, 1, 1, 1, 1\}.$$

In the first simulation, the desired trajectory  $\boldsymbol{\delta}_d$  is computed from (1.121). The results are depicted in Fig. 1.4 in terms of the joint torques, the joint coordinate errors and the link coordinates with their desired values. As foreseen, the tracking errors converge to zero while  $\boldsymbol{\delta}_d$  and  $\dot{\boldsymbol{\delta}}_d$  remain bounded. The final value for  $\boldsymbol{\delta}_d$  is

$$\boldsymbol{\delta}_d = \text{diag}\{-0.1832, -0.0048, -0.0078, -0.000106\}.$$

In the second simulation, it is assumed that  $\mathbf{D}_\delta \equiv \mathbf{O}$ . The desired trajectory  $\boldsymbol{\delta}_d$  is computed from (1.143), while the elements of  $\mathbf{D}_\Delta$  take on the values of  $\mathbf{D}_\delta$  given in [De Luca and Siciliano (1991); De Luca and Siciliano (1993b)]. Also, with reference to (1.145), it is  $\epsilon_{11} = 0.1$ ,  $\epsilon_{12} = 0.0001$ ,  $\epsilon_{21} = 0.0001$ ,  $\epsilon_{22} = 0.00001$  and  $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0.001$ . The results are depicted in Fig. 1.5. It is worth observing how the overall behavior remains satisfactory; this is because the desired trajectories are damped and control (1.120) guarantees that the tracking error converges to zero. As expected, the final vector  $\boldsymbol{\delta}_d$  is the same as before. Notice that a bad choice of the parameters  $\epsilon_{ij}$  and  $\beta_{ij}$  may cause chattering behavior. More simulation results can be found in [Arteaga (1996c)].

### 1.5.2 Control With Observer

In order to test the observers of Section 1.4, the same control goal as in Section 1.5.1 has to be achieved. The initial conditions for the joint and link coordinates of the arm are

$$\boldsymbol{\theta} = [-90 \ 0]^T [^\circ]$$

and

$$\boldsymbol{\delta} = [0 \ 0 \ 0.05 \ 0.02]^T \text{ [m]},$$

while the velocity vector initial condition is  $\mathbf{0}$ . The gains of the controller and the observer have been selected to be:

$$\mathbf{K}_p = \text{diag}\{5, 5, 0.25, 0.25, 0.25, 0.25\}$$

$$\boldsymbol{\Lambda} = \text{diag}\{0.75, 0.75, 0.75, 0.75, 0.75, 0.75\}$$

$$\mathbf{K}_q = \text{diag}\{1.5, 1.5, 1, 1, 1, 1\}$$

$$\mathbf{K}_v = \text{diag}\{50, 50, 50, 50, 50, 50\}$$

$$k_d = 10.$$

In the first simulation, the desired trajectory  $\boldsymbol{\delta}_d$  is computed from (1.182) by setting  $\mathbf{D}_\Delta$  to  $\mathbf{O}$ . The results for the tracking errors are depicted in Fig. 1.6. Again, they converge to zero while  $\boldsymbol{\delta}_d$  and  $\dot{\boldsymbol{\delta}}_d$  remain bounded. The final value for  $\boldsymbol{\delta}_d$  is the same as in Section 1.5.1. The observation errors are depicted in Fig. 1.7. It can be seen that they converge to zero.

In the second simulation,  $\mathbf{D}_\delta$  is set to  $\mathbf{O}$  and  $\mathbf{D}_\Delta$  takes on the old values of  $\mathbf{D}_\delta$ . The desired trajectory  $\boldsymbol{\delta}_d$  is computed from (1.182). Also, with reference to (1.184), it is  $\epsilon_{11} = 0.1$ ,  $\epsilon_{12} = 0.01$ ,  $\epsilon_{21} = 0.01$ ,  $\epsilon_{22} = 0.05$  and  $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0.001$ . The results for the tracking errors and input torques are shown in Fig. 1.8, while those for the observation errors are shown in Fig. 1.9. As expected, the final vector  $\boldsymbol{\delta}_d$  is the same as before. More simulation results can be found in [Arteaga (2000)].

### 1.5.3 Control with a Reduced-Order Model

In this section, the controllers given in Section 1.3 are used again with the purpose to test their sensitivity to the number of modes taken into account. In particular, it is assumed that only one link coordinate for each link is included in the controller. It is not troublesome to get this reduced-order model from the original one, since it can be obtained by just letting the high-order terms be zero. On the other hand, the simulated model includes two link coordinates for each link as ever before, but no gravitational term is considered so as to obtain somewhat less damped motions. The initial values of the generalized coordinates are

$$\boldsymbol{\theta} = [0 \ 0]^T \text{ [}^\circ\text{]}$$

and

$$\boldsymbol{\delta} = [0 \ 0 \ 0 \ 0]^T \text{ [m].}$$

As for the control goal, the final desired joint angles

$$\boldsymbol{\theta}_d = [45 \ 45]^T \text{ [}^\circ\text{]}$$

should be reached. In order to excite the unmodelled high frequencies, no trajectory for the joint coordinates is assigned. The matrices  $\mathbf{K}_p$  and  $\mathbf{\Lambda}$  are

$$\mathbf{K}_p = \text{diag}\{10, 10, 0.5, 0.5\}$$

and

$$\mathbf{\Lambda} = \text{diag}\{1, 1, 1, 1\}.$$

In the first simulation, the desired trajectory  $\boldsymbol{\delta}_d$  is computed from (1.121). The results are depicted in Fig. 1.10. It can be observed that the tracking errors converge to zero. Notice that there is no desired trajectory for  $\delta_{12}$  nor for  $\delta_{22}$  and that those for  $\delta_{11}$  and  $\delta_{21}$  remain bounded.

The next simulation is aimed at testing the robust term used to increase the damping of the system. As a matter of fact, any mechanical system must be damped, so it makes more sense to show whether a poor damping can be improved. To do this, only the damping of the modeled dynamics is set to zero while that for the unmodeled dynamics is the same as before. The elements of  $\mathbf{D}_\Delta$  are assumed to be the ones of the original system, while  $\epsilon_{11} = 0.1$ ,  $\epsilon_{21} = 0.0001$  and  $\beta_{11} = \beta_{21} = 0.001$ . The results are depicted in Fig. 1.11. It is worth observing how the overall behavior remains satisfactory. Note that  $\delta_{11}$  and  $\delta_{21}$  are damped because  $\delta_{d11}$  and  $\delta_{d21}$  are damped as well.

It is interesting to carry out the same simulation while letting  $\mathbf{D}_\Delta$  be  $\mathbf{O}$ . As can be appreciated in Fig. 1.12,  $\delta_{11}$  and  $\delta_{21}$  are no longer damped, which shows the efficacy of the proposed solution to increase the damping of the system. Nonetheless, the tracking errors become zero in all the cases, as it could have been foreseen.

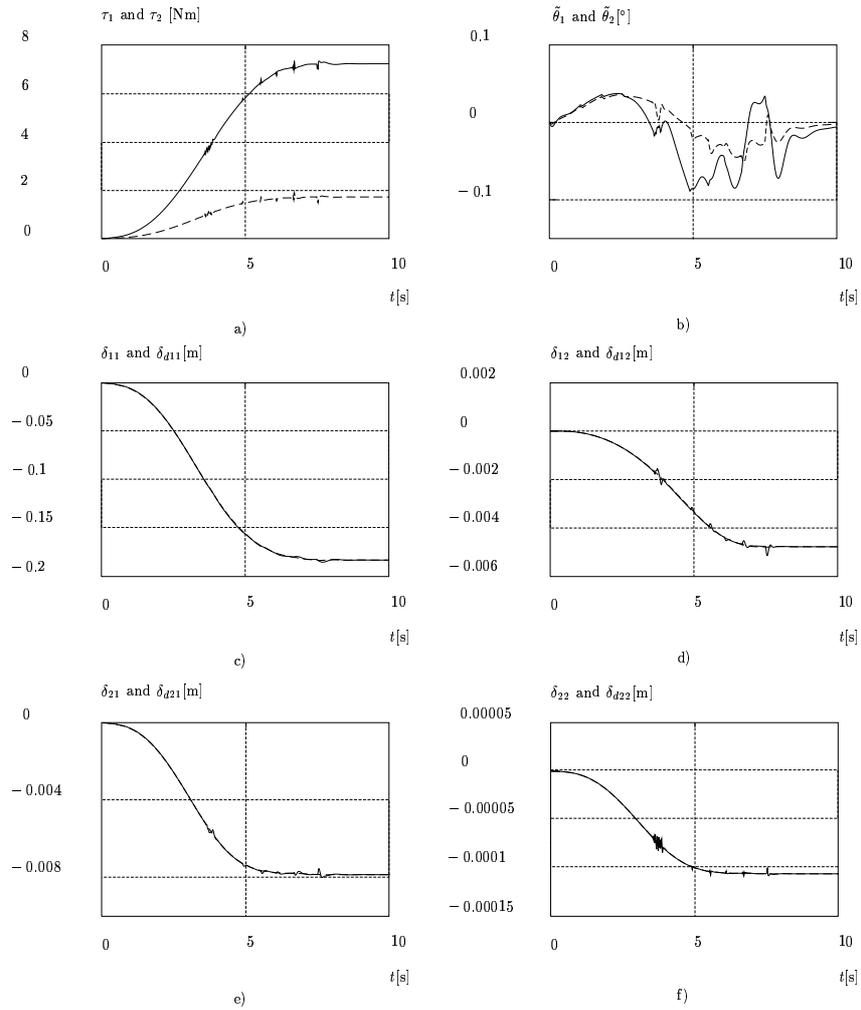


Fig. 1.4 Simulation results for a planar two-link flexible robot arm: Damping case. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$  (—) and its desired value  $\delta_{d12}$  (- - -); e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$  (—) and its desired value  $\delta_{d22}$  (- - -).

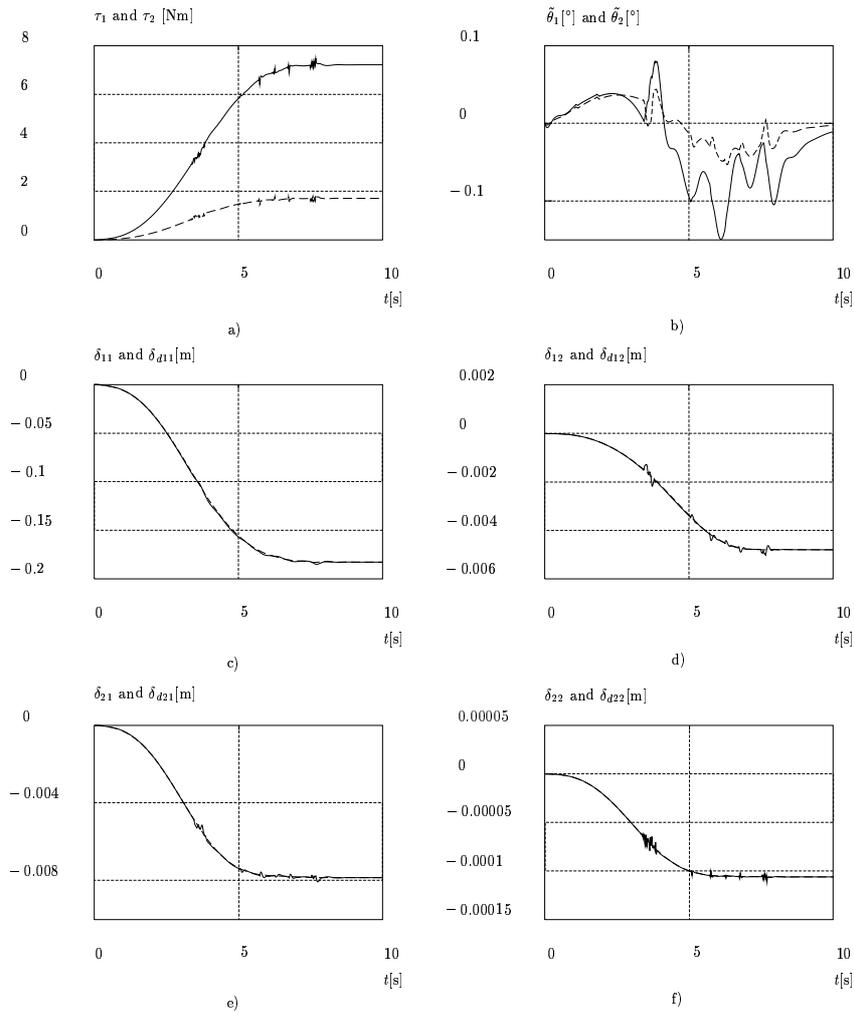


Fig. 1.5 Simulation results for a planar two-link flexible robot arm: No damping case. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$  (—) and its desired value  $\delta_{d12}$  (- - -); e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$  (—) and its desired value  $\delta_{d22}$  (- - -).

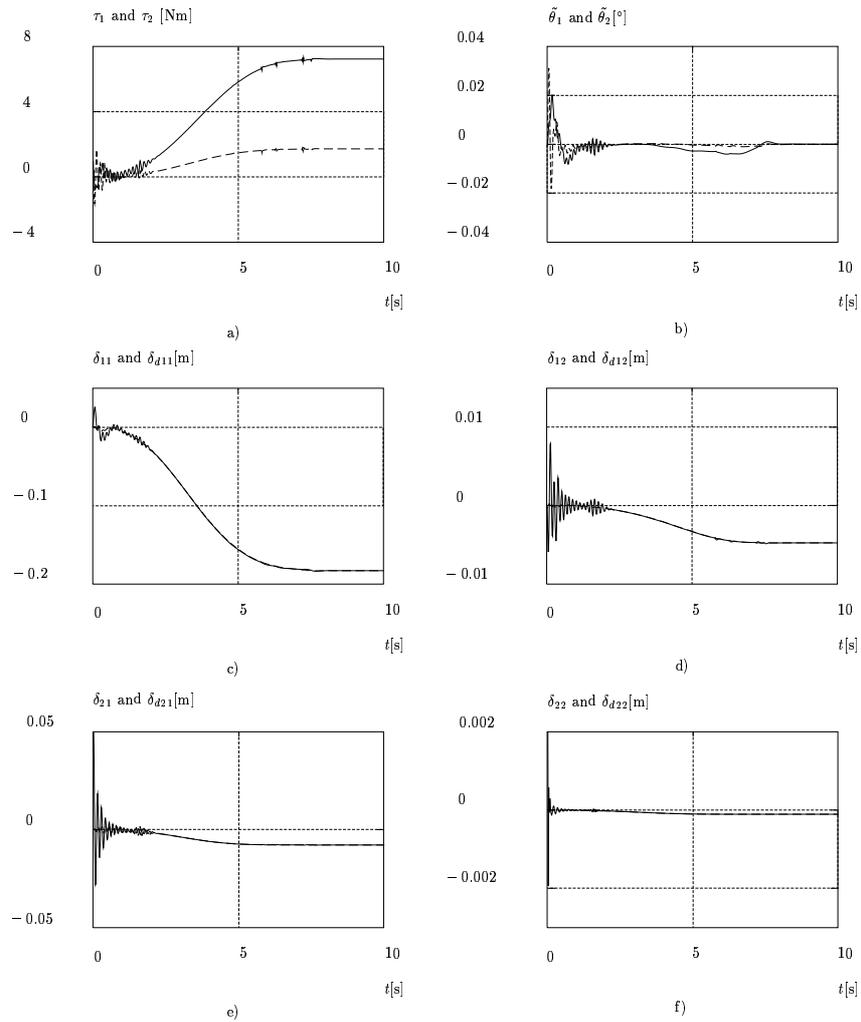


Fig. 1.6 Simulation results for a planar two-link flexible robot arm: Damping case with observer. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$  (—) and its desired value  $\delta_{d12}$  (- - -); e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$  (—) and its desired value  $\delta_{d22}$  (- - -).

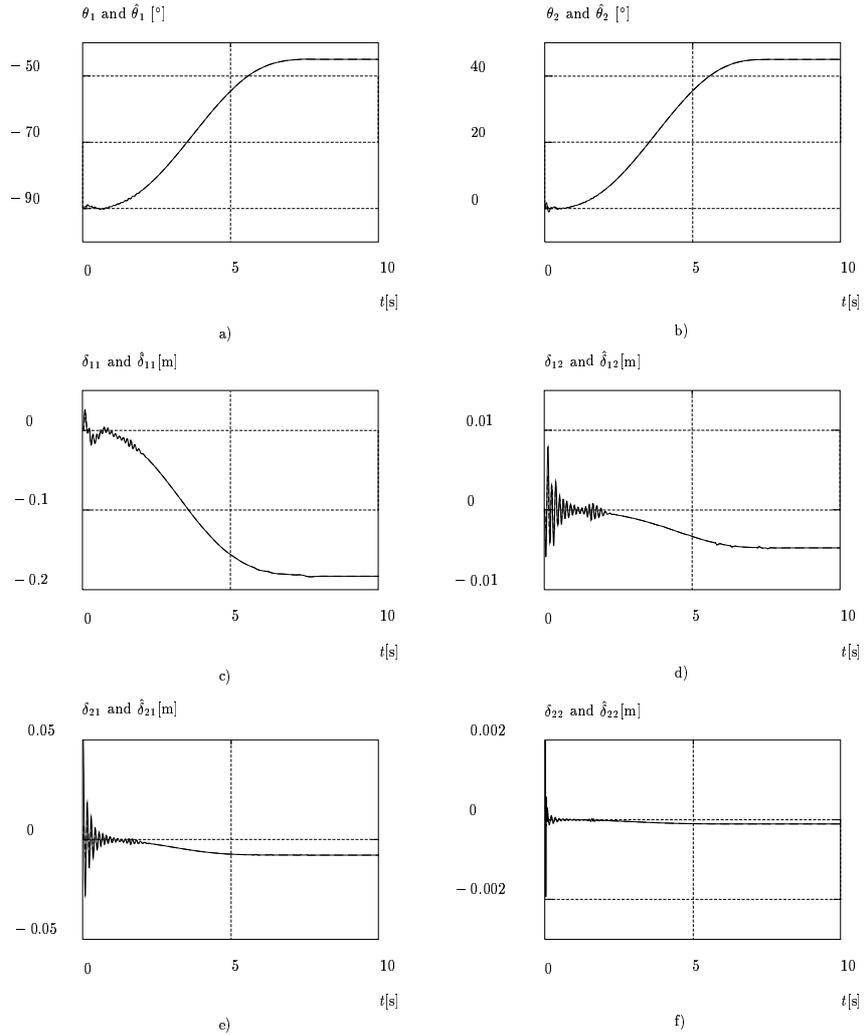


Fig. 1.7 Simulation results for a planar two-link flexible robot arm: Damping case with observer. a) Joint coordinate  $\theta_1$  (—) and its estimate  $\hat{\theta}_1$  (- - -); b) Joint coordinate  $\theta_2$  (—) and its estimate  $\hat{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its estimate  $\hat{\delta}_{11}$  (- - -); d) Link coordinate  $\delta_{12}$  (—) and its estimate  $\hat{\delta}_{12}$  (- - -); e) Link coordinate  $\delta_{21}$  (—) and its estimate  $\hat{\delta}_{21}$  (- - -); f) Link coordinate  $\delta_{22}$  (—) and its estimate  $\hat{\delta}_{22}$  (- - -);

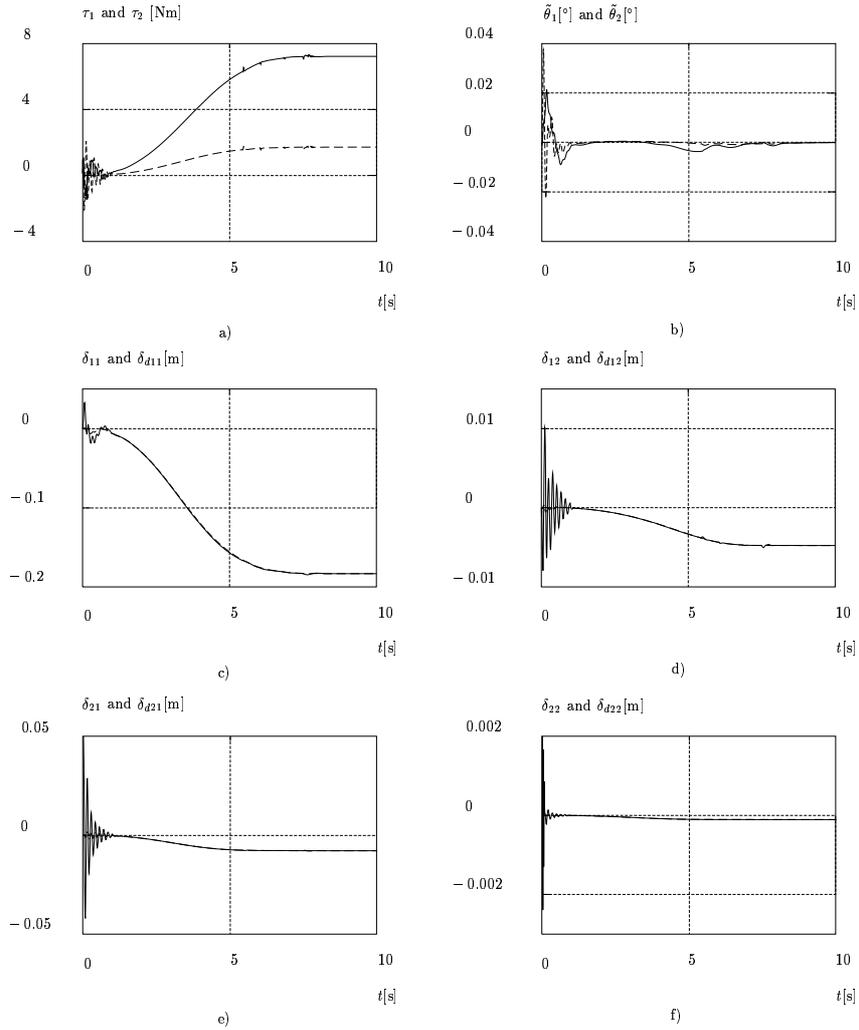


Fig. 1.8 Simulation results for a planar two-link flexible robot arm: No damping case with observer. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$  (—) and its desired value  $\delta_{d12}$  (- - -); e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$  (—) and its desired value  $\delta_{d22}$  (- - -).

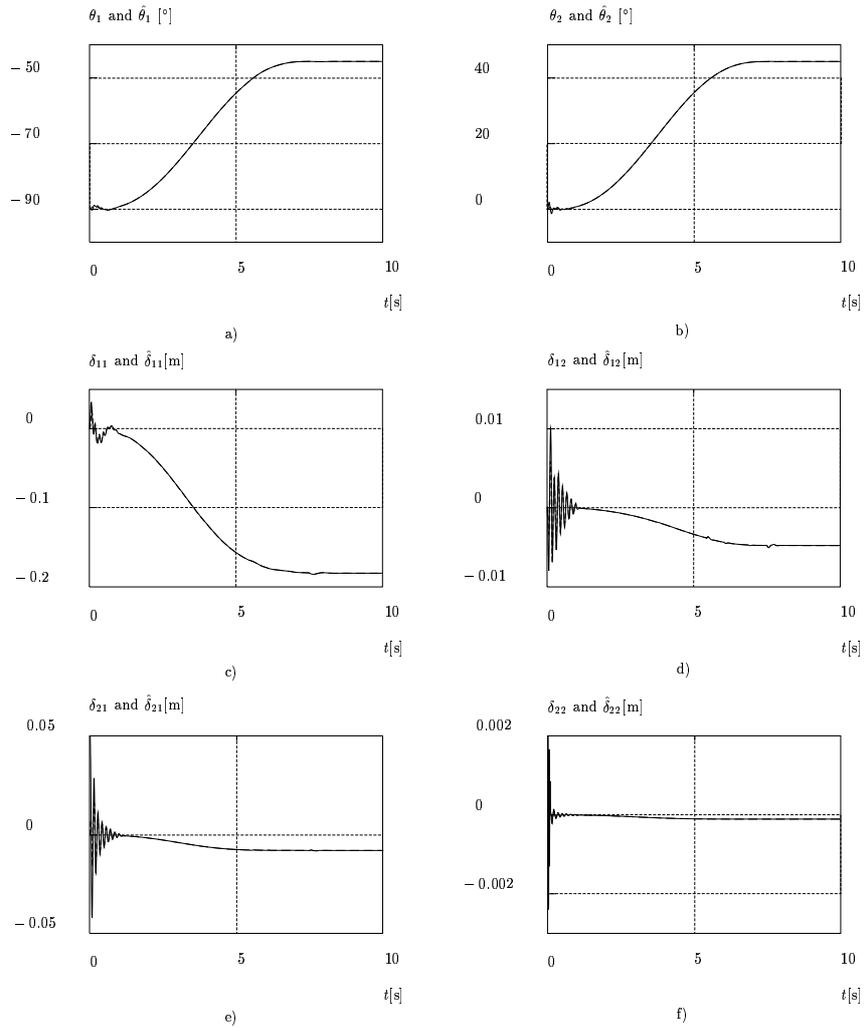


Fig. 1.9 Simulation results for a planar two-link flexible robot arm: No damping case with observer. a) Joint coordinate  $\theta_1$  (—) and its estimate  $\hat{\theta}_1$  (- - -); b) Joint coordinate  $\theta_2$  (—) and its estimate  $\hat{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its estimate  $\hat{\delta}_{11}$  (- - -); d) Link coordinate  $\delta_{12}$  (—) and its estimate  $\hat{\delta}_{12}$  (- - -); e) Link coordinate  $\delta_{21}$  (—) and its estimate  $\hat{\delta}_{21}$  (- - -); f) Link coordinate  $\delta_{22}$  (—) and its estimate  $\hat{\delta}_{22}$  (- - -);

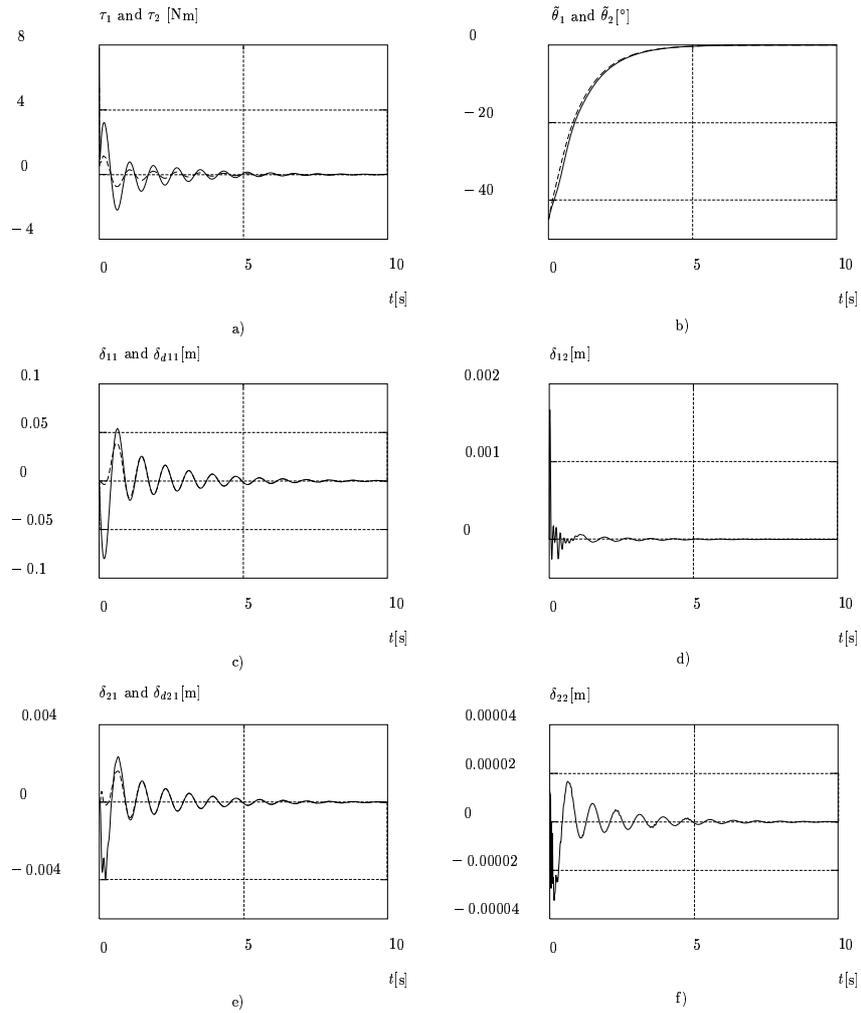


Fig. 1.10 Simulation results for a planar two-link flexible robot arm with reduced-order model: Damping case. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$ ; e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$ .

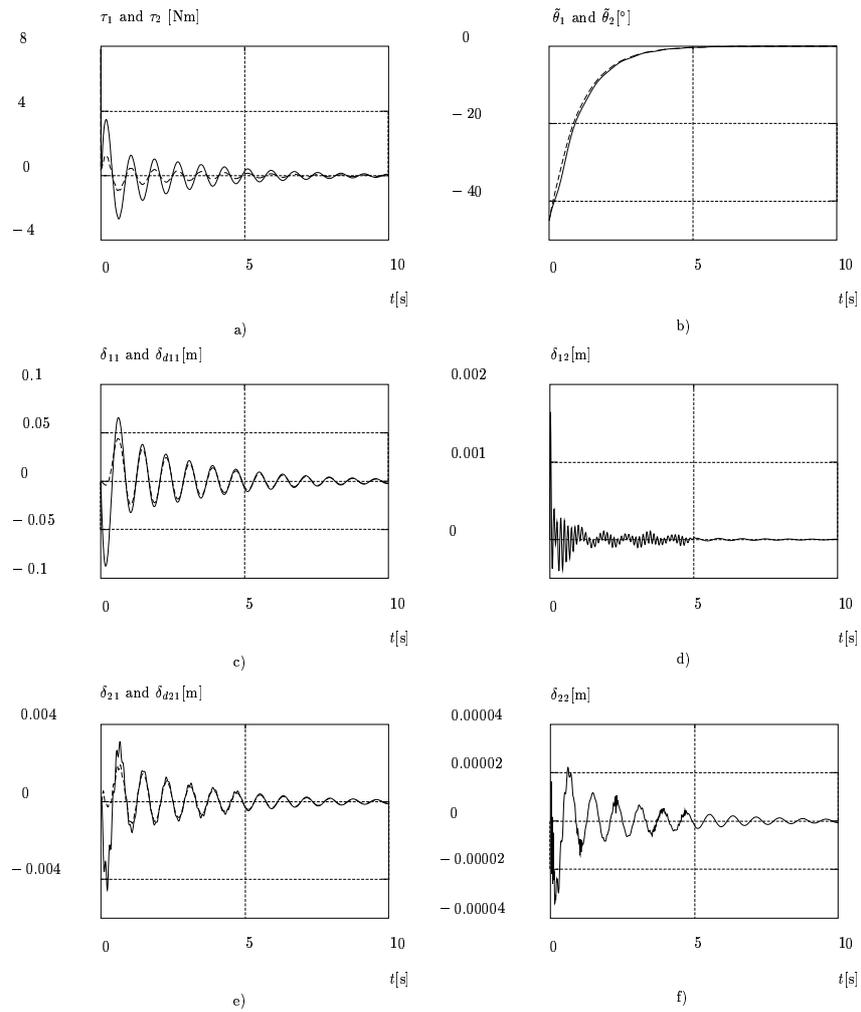


Fig. 1.11 Simulation results for a planar two-link flexible robot arm with reduced-order model: No damping case with desired trajectories  $\delta_{d11}$  and  $\delta_{d21}$  damped. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$ ; e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$ .

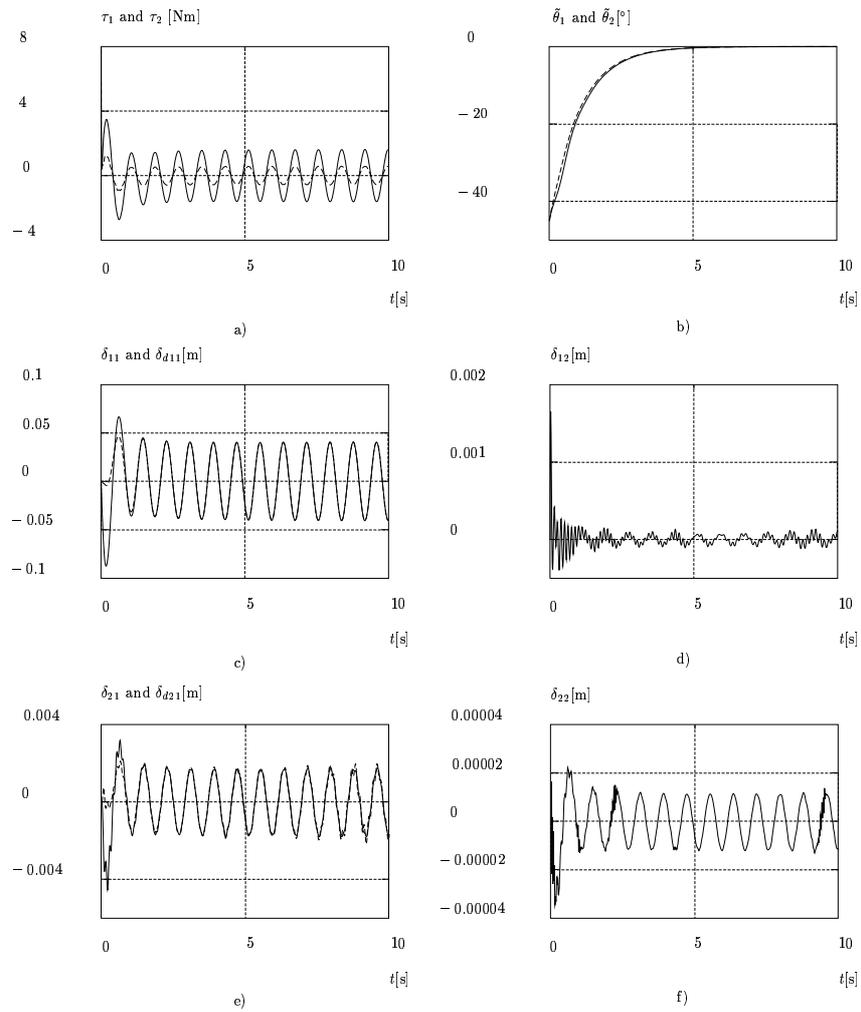


Fig. 1.12 Simulation results for a planar two-link flexible robot arm with reduced-order model: No damping case with desired trajectories  $\delta_{d11}$  and  $\delta_{d21}$  undamped. a) Input torques  $\tau_1$  (—) and  $\tau_2$  (- - -); b) Joint errors  $\tilde{\theta}_1$  (—) and  $\tilde{\theta}_2$  (- - -); c) Link coordinate  $\delta_{11}$  (—) and its desired value  $\delta_{d11}$  (- - -); d) Link coordinate  $\delta_{12}$ ; e) Link coordinate  $\delta_{21}$  (—) and its desired value  $\delta_{d21}$  (- - -); f) Link coordinate  $\delta_{22}$ .

## 1.6 Conclusion

The modeling and control problem of flexible robot manipulators has been studied in the present work. By using Lagrange's equations of motion, a closed-form dynamic model has been obtained where link deflection has been described in terms of assumed modes. For control design purposes, several important properties of the manipulator model have been given and proven. Some of them are physical properties whereas others do arise from the method used to derive the model.

The tracking control problem has been studied. Because a flexible robot has fewer inputs than degrees of freedom, a desired trajectory for the flexible coordinates cannot be selected arbitrarily and has to be computed on-line to accomplish the tracking control goal for the rigid coordinates. It has been proven that this trajectory remains bounded. In order to increase the damping of the system, the equations used for the desired trajectory of the link coordinates have been modified. To ensure global stability of the system, robust control techniques have been employed.

Since the possible lack of measurements of link deflection rates is a drawback of the proposed control scheme, a nonlinear observer has been designed which can still guarantee global stability of the system and boundedness of the desired link coordinate trajectory. It has been shown that the region of attraction for the observer can be enlarged arbitrarily.

In order to test the controller, with and without observer, several simulations have been carried out. It has been shown that the proposed solution does actually increase the damping of the system. To investigate how the controller works in the presence of unmodelled dynamics, some simulations have been accomplished with a reduced-order model yielding satisfactory results.

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