A Closed-Loop Jacobian Transpose Scheme for Solving the Inverse Kinematics of Nonredundant and Redundant Wrists

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The solution of the inverse kinematic problem is of the utmost importance in robotic manipulator control. This article proposes a closed-loop scheme for solving the inverse kinematic problem for nonredundant and redundant wrists based on the computation of the Jacobian transpose. The manipulability measure is suitably introduced as a constraint for redundant wrists, by taking advantage of the null space of the Jacobian matrix. The resulting algorithm provides a computational tool to solve a specified orientation trajectory into a joint trajectory. Numerical results with two spherical wrists show the excellent performance of the scheme.

逆運動学を解くことはロボットマニピュレータの制御において最も重要な問題である。本論文では、冗長および非冗長リストの逆運動学を解くため、ヤコビアン変換に基づく閉ループ解法が提案される。ヤコビアン行列のヌルスペースを利用することにより、冗長リストの拘束条件として可操作度が導入される。結果としてこのアルゴリズムは、関節軌道に対する特定姿勢軌道を解くための計算ツールを供給する。2つの球状リストについての数値計算結果によって本手法の有効性の高さが示される。

I. INTRODUCTION

Robot motion control is usually performed in the joint space, whereas robot motion planning is naturally specified in the task space. The direct mapping from the joint space to the task space is described by a straightforward function, uniquely determined for any robot geometry.[1] Of major importance

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is the inverse mapping from the task space to the joint space, which does not always lead to a closed-form analytical solution. Whenever such a solution is found for a simple geometry,[2] it is not unique.

This issue has stimulated the robotics research community to seek alternative computational inverse kinematic schemes.[3–6] Those are essentially based on the differential kinematics, through the use of the Jacobian matrix, which linearly relates the joint velocities to the task velocities. In particular, the so-called (pseudo)inverse Jacobian control proposed by Whitney,[3] and lately revisited by Klein and Huang,[4] solves for the joint velocities, and the joint displacements are obtained via a simple integration. On the other hand, the method independently proposed by Balestrino et al.[5] and Wolovich and Elliott[6] is based on the computation of the transpose of the Jacobian. The advantages gained with this solution[5,6] versus the common approach[3,4] are a reduced computational burden (no inversion is taken) and the closed-loop fashion which overcomes the drawbacks of the open-loop Jacobian control, such as long-term drifts and initial location errors.

The goal of this article is to establish a computational closed-loop scheme for explicitly solving the orientation inverse kinematic problem. The kinematics of the wrist is suitably described by the orthogonal unit vectors associated with the orientation of the end-effector frame relative to the base frame. Differently from a previous work on the subject,[7] which used the time derivatives of the above unit vectors, the differential kinematics is described in terms of the wrist angular velocities.[8] A computationally fast algorithm to solve a specified wrist trajectory into a joint trajectory is derived.

For redundant wrists, the scheme is suitably modified by adding a manipulability constraint in order to avoid degenerate configurations, performing a so-called task space augmentation.[9] The correctness of the scheme is ensured by projecting the constraint onto the null space of the wrist Jacobian matrix.[10]

It is worth remarking that the previous scheme proposed by De Maria et al.[7] did not allow for a proper task space augmentation for redundant wrists, since a suitable null space of the wrist Jacobian could not be directly extracted.

A set of numerical results are presented which regard the application of the proposed schemes to two different spherical wrists, the three-revolute joint wrist studied by Paul and Stevenson[11] and the four-revolute joint redundant wrist proposed by Yoshikawa.[12]

The paper is organized as follows. Section II gives the necessary background of inverse kinematic techniques. Section III presents the closed-loop Jacobian transpose scheme, and section IV extends it to the case of redundant wrists. Numerical results are discussed in section V, and conclusions are drawn in the final section.

II. BACKGROUND

The direct kinematic equation of an arbitrary manipulator structure with known geometrical dimensions describes the mapping of the $(n \times 1)$ vector of joint coordinates $\mathbf{q}$ into the $(m \times 1)$ vector of robot's end-effector task coordinates $\mathbf{x}$ as[1]

$$\mathbf{x} = \mathbf{f}(\mathbf{q}), \tag{1}$$

where $\mathbf{f}$ is a continuous nonlinear function, whose structure and parameters are known.

It is well-known that the solution of the inverse kinematic problem, i.e., solving Eq. (1) for $\mathbf{q}$, is of fundamental importance for robot control. A typical robot task is specified as a motion assigned to the end-effector. This must be solved into joint motions in order to provide the control servos in the joint space with the reference set points. The most popular approach to the problem relies upon the possibility of finding a closed-form analytical solution to Eq. (1). It is recognized that this is true only for manipulators having simple geometries, such as the spherical wrist, the elbow manipulator etc. Pieper[2] gave a sufficient condition for nonredundant structures $(m = n)$, which establishes that a given kinematical structure is solvable if it contains three consecutive joint rotation axes intersecting at a common point. The spherical wrist does satisfy this condition, while the elbow manipulator does not. Nonetheless, the analytical solution is usually nonunique and sequential, and requires the computation of Atan2 functions.

The two shortcomings of the above technique, namely the solvability of the structure and the computational burden, have inspired the research to finding alternative solution techniques to the inverse kinematic problem, which would be applicable to any kinematical structure as well as be efficient from the computational standpoint. In particular, differentiating Eq. (1) with respect to time yields the mapping of the joint velocity vector $\dot{\mathbf{q}}$ into the end-effector task velocity vector $\dot{\mathbf{x}}$

$$\dot{\mathbf{x}} = J(\mathbf{q})\dot{\mathbf{q}}, \tag{2}$$

where $J(\mathbf{q}) = \partial \mathbf{f}/\partial \mathbf{q}$ is the end-effector Jacobian matrix.

The alternative approach to the inverse kinematic problem, commonly followed in the robotics literature, is based indeed on the use of the inverse of the Jacobian matrix in Eq. (2), that is[3]

$$\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\dot{\mathbf{x}}, \tag{3}$$

where $J(\mathbf{q})$ is assumed to have full rank. In case of redundant manipulators $(m < n)$, a pseudoinverse of J must be adopted in lieu of the inverse.[4] The joint displacements $\mathbf{q}$ are then obtained by integrating (3) over time.

A rather different approach to the inverse kinematic problem, which is applicable to any manipulator structure, was independently proposed by Balestrino et al.[5] and Wolovich and Elliott.[6] The idea is to construct a simple dynamic system whose input is the desired end-effector task vector $\mathbf{x}_d$ and whose outputs are the corresponding joint displacement and velocity vectors, $\mathbf{q}$ and $\dot{\mathbf{q}}$ respectively. The scheme is depicted in Figure 1; K is a positive definite (usually diagonal) matrix.

Theorem. The dynamic system of Figure 1 assures that the error

$$\mathbf{e} = \mathbf{x}_d - \mathbf{x}, \tag{4}$$

can be made arbitrarily small by increasing the minimum eigenvalue of K.[5,6]

Proof. Define the positive definite Lyapunov function

$$v = \frac{1}{2}\mathbf{e}^T K \mathbf{e}. \tag{5}$$

The time derivative of (4) becomes, via (2),

$$\dot{\mathbf{e}} = \dot{\mathbf{x}}_d - J(\mathbf{q})\dot{\mathbf{q}}, \tag{6}$$

which, plugged into the time derivative of (5), directly yields

$$\dot{v} = \mathbf{e}^T K^T \dot{\mathbf{x}}_d - \mathbf{e}^T K^T J(\mathbf{q})\dot{\mathbf{q}}. \tag{7}$$

According to the scheme of Figure 1, the choice

$$\dot{\mathbf{q}} = J^T(\mathbf{q}) K \mathbf{e}, \tag{8}$$

guarantees that $\dot{v}$ is negative definite outside a region in the error space containing $\mathbf{e} = 0$, which is attractive for all trajectories $\dot{\mathbf{x}}_d \in \mathcal{R}(J)$, where $\mathcal{R}(J)$ denotes the range space of J. In particular:

- If $\mathbf{e}$ at time $t = 0$ is null (i.e., the initial configuration $\mathbf{q}(0)$ of the manipulator is known), the tracking error is confined to the above region of the error space; the larger the minimum eigenvalues of K and JJ^T and the inverse of the norm of $\dot{\mathbf{x}}_d$, the smaller the region.
- The steady-state error, i.e., when $\dot{\mathbf{x}}_d = 0$, is null. ■

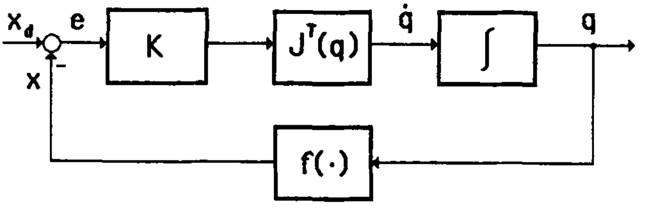

Figure 1. The general Jacobian transpose closed-loop inverse kinematic scheme.

From the above theorem it follows that the application of the scheme of Figure 1 to solve the inverse kinematic problem for a general structure is twofold. It can be used off-line to make $\mathbf{q}(t)$ approach a desired constant solution $\mathbf{q}_d$ to Eq. (1), with $\mathbf{q}(0) \neq \mathbf{q}_d$, arbitrarily fast. It can be adopted on-line to guarantee that $\mathbf{x}(t)$ tracks the desired trajectory $\mathbf{x}_d(t)$ with an arbitrarily fast decaying error. The advantages of this technique can be summarized as follows:

- It is applicable to any robot system since it does not require any special assumption regarding the kinematical structure; the extension to the case of redundant manipulators is discussed by Sciavicco and Siciliano.[9]
- It is computationally efficient since it is based only on direct kinematic functions ($\mathbf{f}$ and J), generating joint velocities at no additional cost.
- The use of the transpose of the Jacobian may avoid problems when kinematic singularities occur; this point has recently been addressed by Chiacchio and Siciliano.[13]

If a null tracking error is desired, the solution (8) can be modified into the more computational demanding solution[5]

$$\dot{\mathbf{q}} = J^{-1}(\mathbf{q})[\dot{\mathbf{x}}_d + K\mathbf{e}], \tag{9}$$

which resembles the resolved rate control proposed by Whitney,[3] but it is inherently closed-loop, i.e., avoids long-term drifts associated with the instantaneous inversion of the mapping J. It is worth remarking that a solution equivalent to (9) has recently been proposed by Tsai and Orin.[14] Similarly to Eq. (3), a pseudoinverse of J must be adopted in lieu of its inverse if the manipulator is redundant.

The above scheme is only general since it assumes that the position and orientation of the end-effector $\mathbf{x}$ on the left-hand side of Eq. (1) is described by means of the usual position vector $\mathbf{p}$ (Fig. 2) and any three orientation angles

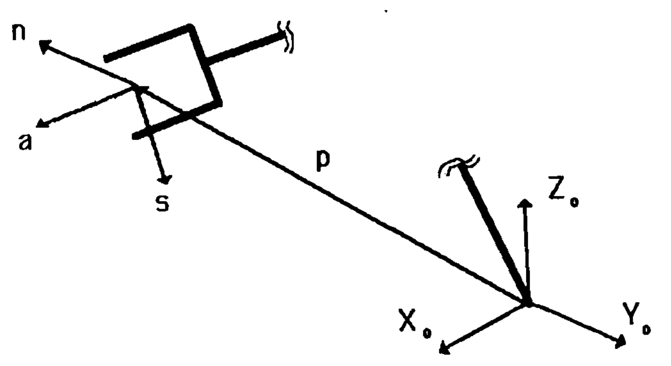

Figure 2. Position and orientation vectors of the end-effector.

(e.g., Euler, RPY). Unfortunately, an explicit dependence of these angles on the joint coordinates is not available,[1] so that the kinematic function in (1) cannot be directly computed for its orientation part.

De Maria et al.[7] adopted a partitioning strategy, according to which the inverse kinematic problem was solved into two sequential stages, the first relative to position and the second relative to orientation. It should be mentioned that this strategy had originally been proposed by Featherstone,[15] Hollerbach and Sahar,[16] and later revisited by Waldron et al.[17] in the context of solving the differential kinematic mapping (3) for spherical wrist manipulators. The solution scheme proposed by De Maria et al.,[7] indeed, was inspired by the above-referenced works, in the context of applying the algorithmic solution scheme of Figure 1 based on the control law (8), though. The case of a spherical wrist manipulator was treated by Balestrino et al.,[18] whereas Sciavicco and Siciliano[19] applied the same strategy to a manipulator having joint revolute axes intersecting two-by-two at the end-effector, which does not have a closed-form analytical solution.

On the basis of the above two-stage strategy, it can easily be shown that the inverse kinematic problem for end-effector position is directly solved by the scheme of Figure 1, with $\mathbf{x}$ replaced by $\mathbf{p}$. On the other hand, the solution of the inverse kinematic problem for end-effector orientation is not trivial and will constitute the subject of the following section.

III. THE JACOBIAN TRANSPOSE SCHEME FOR SOLVING ORIENTATION

As pointed out in the previous section, a direct mapping of the joint variables into the task variables describing orientation is needed to apply the inverse kinematic scheme of Figure 1. A description of wrist orientation in terms of a minimum number of parameters can be obtained through three orientation angles (e.g., Euler, RPY). Such description, however, does not satisfy the above-mentioned requirement.[1] Alternatively, it is well-known that the orientation kinematics of the wrist can be represented by a (3×3) rotation matrix

$$R = [\mathbf{n} \quad \mathbf{s} \quad \mathbf{a}] \tag{10}$$

where $\mathbf{n}$, $\mathbf{s}$, and $\mathbf{s}$ are the normal, slide, and approach unit vectors defining the orientation frame located at the wrist.[1] These unit vectors are defined in the right-hand sense of $\mathbf{n} = \mathbf{s} \times \mathbf{a}$ with respect to the base frame of the manipulator (Fig. 2), and they are orthonormal vectors, i.e.,

$$\mathbf{s}^T\mathbf{s} = \mathbf{a}^T\mathbf{a} = \mathbf{n}^T\mathbf{n} = 1, \tag{11}$$

$$\mathbf{s}^T\mathbf{a} = \mathbf{a}^T\mathbf{n} = \mathbf{n}^T\mathbf{s} = 0, \tag{12}$$

$$R = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix} \quad (13)$$

This relationship relieves the user from specifying the desired orientation in terms of the unit vectors, while guaranteeing the constraints (11)–(12).

If the unit vectors **s** and **a** are adopted to describe the wrist orientation (**n** is redundant, since $\mathbf{n} = \mathbf{s} \times \mathbf{a}$), Balestrino et al.[18] showed that the solution (8) can be suitably modified into

$$\dot{\mathbf{q}} = J_s^T(\mathbf{q})K_s\mathbf{e}_s + J_a^T(\mathbf{q})K_a\mathbf{e}_a, \quad (14)$$

where $J_s = \partial\mathbf{s}/\partial\mathbf{q}$, $J_a = \partial\mathbf{a}/\partial\mathbf{q}$, $\mathbf{e}_s = \mathbf{s}_d - \mathbf{s}$, $\mathbf{e}_a = \mathbf{a}_d - \mathbf{a}$, and K_s, K_a positive matrices.

An alternative scheme based on the usual definition of the differential kinematics for orientation is proposed in the following. It is well-known that the angular wrist velocity vector ω can be expressed in terms of the joint velocity vector $\dot{\mathbf{q}}$ as

$$\omega = J_o(\mathbf{q})\dot{\mathbf{q}}, \quad (15)$$

where J_o is the $(3 \times n)$ wrist Jacobian matrix.[8]

Theorem. The dynamic system of Figure 3 assures that the error

$$\mathbf{e}_o = \frac{1}{2}(\mathbf{n} \times \mathbf{n}_d + \mathbf{s} \times \mathbf{s}_d + \mathbf{a} \times \mathbf{a}_d), \quad (16)$$

between the desired wrist frame $(\mathbf{n}_d, \mathbf{s}_d, \mathbf{a}_d)$ and the actual wrist frame $(\mathbf{n}, \mathbf{s}, \mathbf{a})$ can be made arbitrarily small by increasing the minimum eigenvalue of the positive definite matrix K_o.

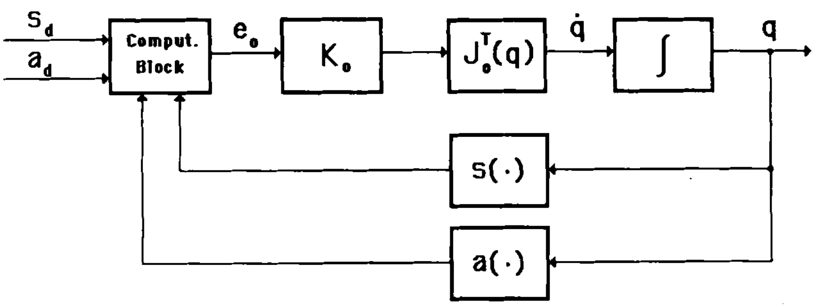

Figure 3. The Jacobian transpose closed-loop inverse kinematic scheme for non-redundant wrists.

Proof. Define the positive definite Lyapunov function

$$v_o = \frac{1}{2} \mathbf{e}_o^T K_o \mathbf{e}_o. \tag{17}$$

Luh, Walker, and Paul[8] showed that the expression (16) is derived from the definition of the orientation error in terms of Euler rotation parameters as

$$\mathbf{e}_o = \mathbf{r} \sin \phi, \tag{18}$$

where the unit vector $\mathbf{r}$ and angle ϕ are illustrated in Figure 4. Note that $\mathbf{n}$ will be aligned with $\mathbf{n}_d$ (and simultaneously $\mathbf{s}$ with $\mathbf{s}_d$ and $\mathbf{a}$ with $\mathbf{a}_d$, although these unit vectors are not evidenced in Figure 4) by a rotation of an angle ϕ about $\mathbf{r}$. It can be shown that[8]

$$\dot{\mathbf{e}}_o = \omega_d - \omega. \tag{19}$$

Substituting (19) in the time derivative of v_o in (17) leads to, via (15),

$$\dot{v}_o = \mathbf{e}_o^T K_o^T \omega_d - \mathbf{e}_o^T K_o^T J_o(\mathbf{q})\dot{\mathbf{q}}. \tag{20}$$

The choice

$$\dot{\mathbf{q}} = J_o^T(\mathbf{q}) K_o \mathbf{e}_o, \tag{21}$$

guarantees that $\dot{v}_o$ is negative definite outside a region in the error space containing $\mathbf{e}_o = 0$, which is attractive for all trajectories $\omega_d \in \mathcal{R}(J_o)$. The remarks for the general scheme in the previous section are still in order. It should be noticed, however, that a singular case occurs when the actual wrist orientation differs from the desired wrist orientation by an Euler rotation of 180°. This corresponds to having $(\mathbf{n}, \mathbf{s}, \mathbf{a}) = (-\mathbf{n}_d, -\mathbf{s}_d, -\mathbf{a}_d)$, with $\mathbf{e}_o = 0$. Therefore, as long as this case does not occur at $t = 0$, Eq. (21) guarantees that the orientation error will uniformly decrease as anticipated above. ■

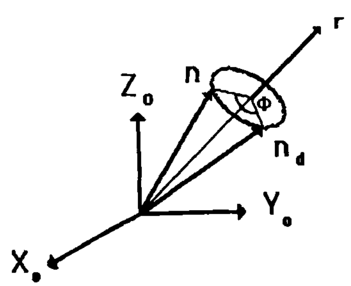

Figure 4. Specification of the orientation error.

IV. EXTENSION TO REDUNDANT WRISTS

It has been recognized in the robotics literature that kinematic redundancy is one key towards the realization of more versatile robots.[4] It is well-known that a manipulator is termed redundant with respect to a given task when more than the minimum number of degrees of freedom to execute that task are available. With regard to general orientation tasks, then, a wrist will be redundant if it possesses more than three degrees of freedom.

Most approaches in the literature solve redundancy according to the differential kinematic mapping (2), or (15), by using the pseudoinverse of the Jacobian matrix,[3,4] which produces the instantaneous minimization of joint velocities. Baillieul et al.[10] pointed out, however, that with this technique singular configurations are not avoided in any practical sense.

The extra degrees of freedom in a redundant structure, indeed, can be conveniently exploited to meet different constraints on the solution of the inverse kinematic problem. Liégeois[20] proposed to modify the pseudoinverse solution by the addition of a term in the space of redundancy—the null space of the Jacobian matrix—which is used for local optimization purposes. One of such constraints is constituted by the so-called manipulability measure introduced by Yoshikawa,[12] which is an index of the ability of arbitrarily translating/orienting the end-effector. A low value of this measure means that the manipulator is close to a singular configuration. With reference to the wrist orientation differential kinematics (15), the manipulability measure can be defined as

$$w(\mathbf{q}) = [\det J_o(\mathbf{q}) J_o^T(\mathbf{q})]^{1/2}. \tag{22}$$

This allows for a so-called task augmentation strategy,[9] i.e., the wrist orientation task space is suitably augmented to include the kinematic constraint expressed by (22). As observed by Baillieul et al.,[10] however, the constraint has to be projected onto the null space of the wrist Jacobian. This will ensure that the constraint is met only on condition that it does not disturb the wrist orientation task which is the primary task, performing thus a true task-priority strategy.[21] It is important to emphasize here that the inverse kinematic scheme for solving orientation based on the control law (14)[7] does not allow to extract a null space of the wrist Jacobian to suitably exploit for fulfilment of the constraint task.

Theorem. The dynamic system of Figure 5 assures that the error $\mathbf{e}_o$ can be made arbitrarily small as above for the scheme of Figure 3, and the error

$$e_w = w_d - w, \tag{23}$$

w_d denoting a desired constant value of manipulability, can be made arbitrarily small provided that the joint velocity solution has a component onto the null space of the wrist Jacobian J_o, by increasing the positive scalar k_w.

Proof. From the scheme of Figure 5, one can easily recognize that the

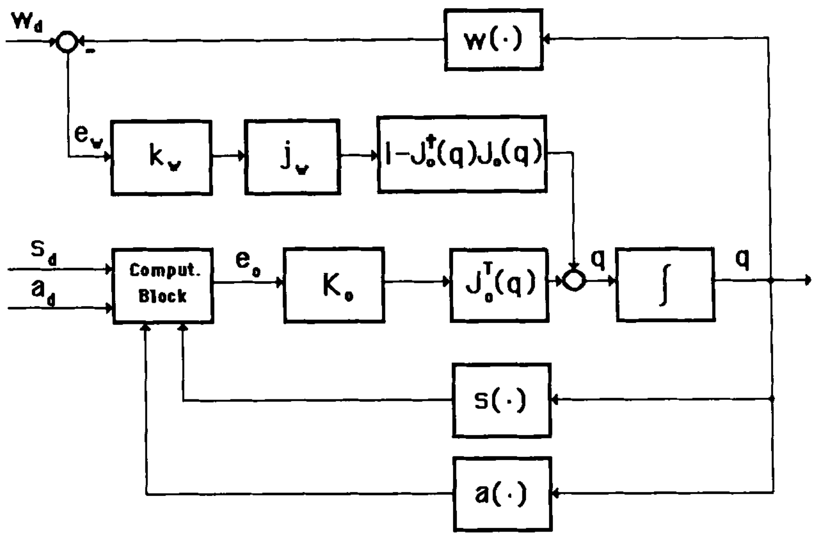

Figure 5. The augmented Jacobian transpose closed-loop inverse kinematic scheme for redundant wrists.

control law (21) has been modified into[22,23]

$$\dot{\mathbf{q}} = J_o^T(\mathbf{q})K_o\mathbf{e}_o + [I - J_o^\dagger(\mathbf{q})J_o(\mathbf{q})]\mathbf{j}_w^T(\mathbf{q})k_w e_w, \tag{24}$$

where $\mathbf{j}_w = \partial w/\partial\mathbf{q}$ is the $(1 \times n)$ constraint Jacobian row vector, $J_o^\dagger$ denotes the pseudoinverse of J_o and $(I - J_o^\dagger J_o)$ is a projection operator onto the null space of J_o. It is easy to show that the choice (24) produces the same results as (21) for the wrist orientation error tracking performance.

Regarding the constraint error e_w, define the positive definite Lyapunov function

$$v_w = \frac{1}{2} k_w e_w^2. \tag{25}$$

Its time derivative turns out, via (23),

$$\dot{v}_w = - k_w e_w \mathbf{j}_w(\mathbf{q})\dot{\mathbf{q}}, \tag{26}$$

which, by virtue of (24), becomes

$$\dot{v}_w = - k_w e_w \mathbf{j}_w(\mathbf{q})J_o^T(\mathbf{q})K_o\mathbf{e}_o - k_w^2 e_w^2 \mathbf{j}_w(\mathbf{q})[I - J_o^\dagger(\mathbf{q})J_o(\mathbf{q})]\mathbf{j}_w^T(\mathbf{q}). \tag{27}$$

At this point, as $\mathbf{e}_o$ is arbitrarily small (null in the limit) from (20) and (24), the first term in $\dot{v}_w$ is negligible. The second term is negative only along the directions of the null space of J_o. Thus, by suitably increasing k_w, w is taken to a locally minimum distance from w_d as long as the required $\dot{\mathbf{q}}$ has a component onto the null space of J_o. ■

Two remarks are in order regarding the solution (24):

- The most appropriate choice for w_d is thought of as the maximum value achievable in the wrist workspace, so as to be independent of the particular trajectory assigned. As a matter of fact, if the wrist cannot achieve that maximum along the given trajectory, the algorithm will guarantee that w is locally maximized as seen from (27).
- As the pseudoinverse of J_o has to be computed to construct the projector $(I - J_o^{\dagger}J_o)$, it may be convenient to modify the first term of the solution (24), according to (9) as

$$\dot{\mathbf{q}} = J_o^{\dagger}(\mathbf{q})[\omega_d + K_o\mathbf{e}_o] + [I - J_o^{\dagger}(\mathbf{q})J_o(\mathbf{q})]\mathbf{j}_w^T(\mathbf{q})k_w e_w, \tag{28}$$

which guarantees a null tracking error $\mathbf{e}_o$.

V. CASE STUDIES

In the following, the inverse kinematic schemes presented in the previous section are tested for the spherical wrists of Paul and Stevenson[11] and of Yoshikawa.[12] It is understood that the purpose of the wrist mechanism is only to change the orientation of the end-effector, while the position is varied by means of the arm supporting the wrist.

Figure 6 shows the three-revolute joint system usually adopted in current industrial robots,[11] while the four-revolute joint redundant system depicted in Figure 7 is constituted by a universal joint with two revolute joints at both ends.[12] The design of Figure 6 has two cones of degeneracy inside the working region where the ability of arbitrarily orienting the end-effector (manipu-

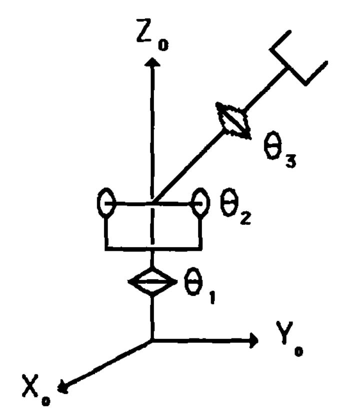

Figure 6. The three-revolute-joint spherical wrist.[11]

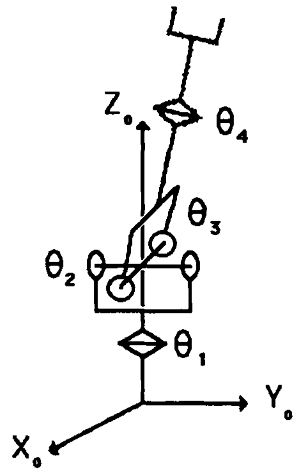

Figure 7. The four-revolute-joint spherical wrist.[12]

lability) becomes poor.[11] This drawback is overcome by the design of Figure 7 which has been argued to possess greater manipulability than the standard design.[12] For reader's convenience, the kinematics of the two wrists are detailed in the Appendix.

Notice that the above two wrists are just schematic systems in that the question of interference between the links is not explicitly addressed here. It should be mentioned that practical designs do exist which are free from interference and singularity problems such as those proposed by Stanišić and Pennock[24] and by Trevelyan et al.[25] Nonetheless, joint limit and singularity avoidance constraints can be systematically embedded in the scheme of Figure 5 according to the task space augmentation with task priority presented above.

Numerical examples have been worked out. The trajectories for the desired orientation frame $(\mathbf{n}_d, \mathbf{s}_d, \mathbf{a}_d)$ are generated from trajectories assigned at the Euler angles of orientation $(\alpha_d, \beta_d, \gamma_d)$, according to Eq. (13). In the space of Euler angles the trajectories are straight lines, with the usual trapezoidal velocity profiles. The initial configuration of either wrist is always assumed to orient the actual end-effector frame $(\mathbf{n}, \mathbf{s}, \mathbf{a})$ as the desired one $(\mathbf{n}_d, \mathbf{s}_d, \mathbf{a}_d)$, i.e., $\mathbf{e}_o(0) = 0$. The desired value of manipulability measure w_d for the wrist of Figure 7 has been chosen as $\sqrt{2}$ (see Eq. (A-7)). A sampling rate of 1 ms is adopted. This is seen to be sufficient to perform on-line all the computations required by the two schemes of Figures 3 and 5 on a 16-bit microprocessor system with floating point co-processor. The elements of the diagonal feedback matrix K_o and the scalar k_w have gradually been increased until before numerical instabilities arise due to the sampling time. For each task trajectory, the resulting joint trajectories will be shown along with the orientation errors, evaluated as the norm of $\mathbf{e}_{\mathrm{Eul}} = (\alpha_d - \alpha \quad \beta_d - \beta \quad \gamma_d - \gamma)^T$, and the manipulability measures in the last example.

A first trajectory goes from $(\alpha_{di}, \beta_{di}, \gamma_{di}) = (0, 45, 0)^\circ$ to $(\alpha_{df}, \beta_{df}, \gamma_{df}) = (90, 90, -90)^\circ$ in a time of 1 s, with a maximum velocity of 180°/s in the space of Euler angles. The results are plotted in Figures 8 and 9 for the scheme of Figure 3 applied to the wrist of Figures 6 and 7, with $K_o = 1200I$ and $K_o = \mathrm{diag}\,(1800\ 1000\ 1000)$ respectively. The tracking performance is seen to be excellent.

For the three-joint wrist of Figure 6, a second trajectory goes from $(\alpha_{di}, \beta_{di}, \gamma_{di}) = (0, 0, 0)^\circ$ to $(\alpha_{df}, \beta_{df}, \gamma_{df}) = (90, 90, -90)^\circ$ in a time of 1 s, with a maximum velocity of 180°/s in the space of Euler angles. Notice that the initial configuration $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)^\circ$ orients the wrist such that a singularity occurs. The relative results are plotted in Figure 10, with $K_o = 1000I$.

For the four-joint wrist of Figure 7, a third trajectory goes from $(\alpha_{di}, \beta_{di}, \gamma_{di}) = (0, 90, 0)^\circ$ to $(\alpha_{df}, \beta_{df}, \gamma_{df}) = (0, 0, -90)^\circ$ in a time of 1 s, with a maximum velocity of 180°/s. Also in this case, the initial configuration of the mechanism $(\theta_1, \theta_2, \theta_3, \theta_4) = (90, 90, 90, 0)^\circ$ is singular. The results are plotted in Figure 11, with $K_o = 1000I$. A common feature of the results in Figures 10(b)–11(b) is that a peak in the tracking error occurs, showing the effort paid to leave the singular configuration. The maximum tracking error, however, is larger for the three-joint wrist, since the four-joint redundant wrist can leave the singularity in different ways.

Finally, in order to test the task space augmentation for the four-joint wrist of Figure 7, a fourth trajectory goes from $(\alpha_{di}, \beta_{di}, \gamma_{di}) = (45, 90, 0)^\circ$ to $(\alpha_{df}, \beta_{df}, \gamma_{df}) = (150, 90, 0)^\circ$ in a time of 1 s, with a maximum velocity of 180°/s. The scheme of Figure 3 is applied first with $K_o = 800I$. From the results of Figure 12 it can be recognized that the trajectory crosses a singularity ($w = 0$ in Fig. 12c), but the tracking performance of the scheme (Fig. 12(b)) is still excellent. The scheme of Figure 5 is applied next with K_o as above and $k_w = 200$. The results of Figure 13 clearly show that the wrist is kept off the singularity (w in Figure 13(c) is locally maximized along the trajectory), while the tracking performance is unchanged (Fig. 13(b)) in force of the task priority strategy.

VI. CONCLUSIONS

A closed-loop computational scheme for solving the inverse kinematics of nonredundant and redundant wrists has been presented in this article. The scheme retains the advantages of a more general inverse kinematic scheme based on the computation of the transpose of the robot's Jacobian. In case of redundant wrists, it has been shown how a manipulability constraint can be suitably embedded into the solution, according to a task augmentation strategy with task priority. Different case studies have been developed which have shown the effectiveness of the proposed scheme, even at singular configurations of the two spherical wrists analyzed.

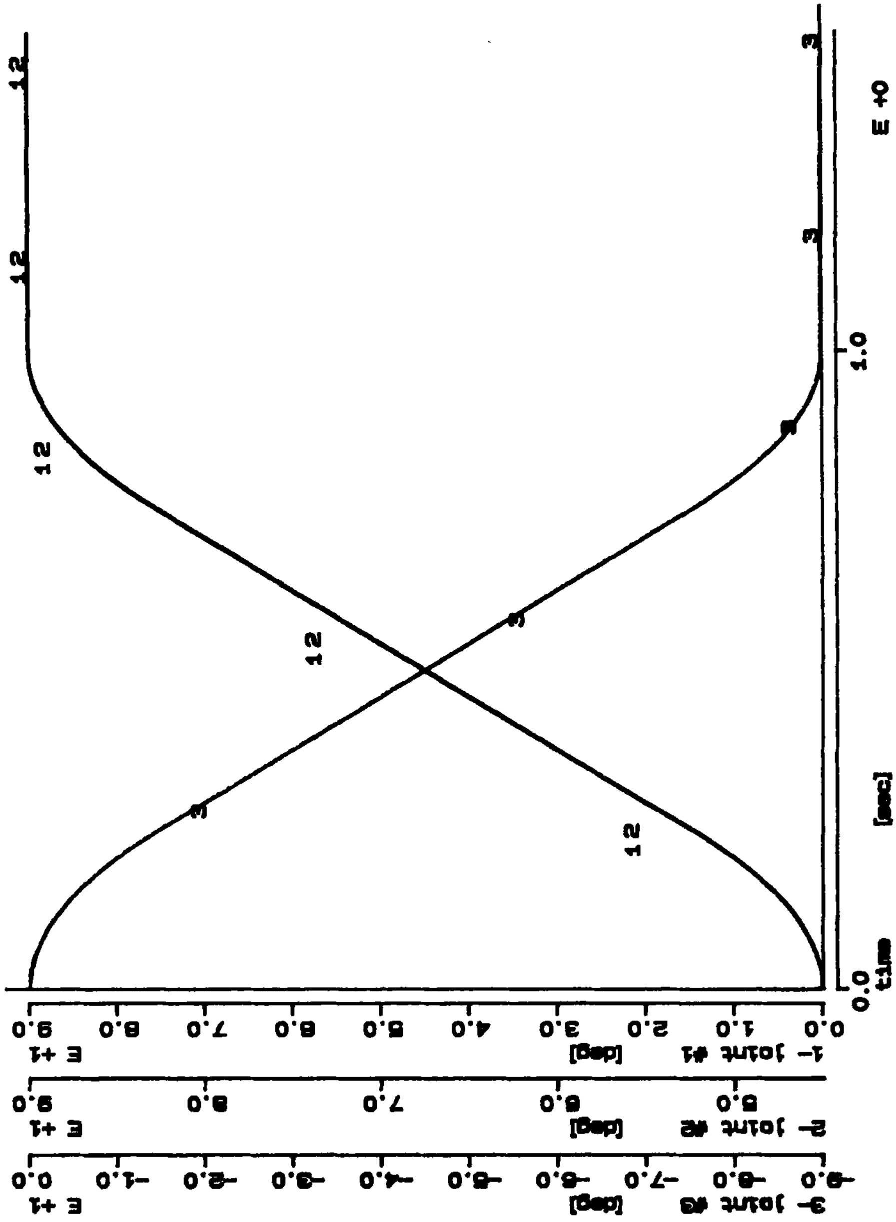

Figure 8a. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 6: joint trajectories.

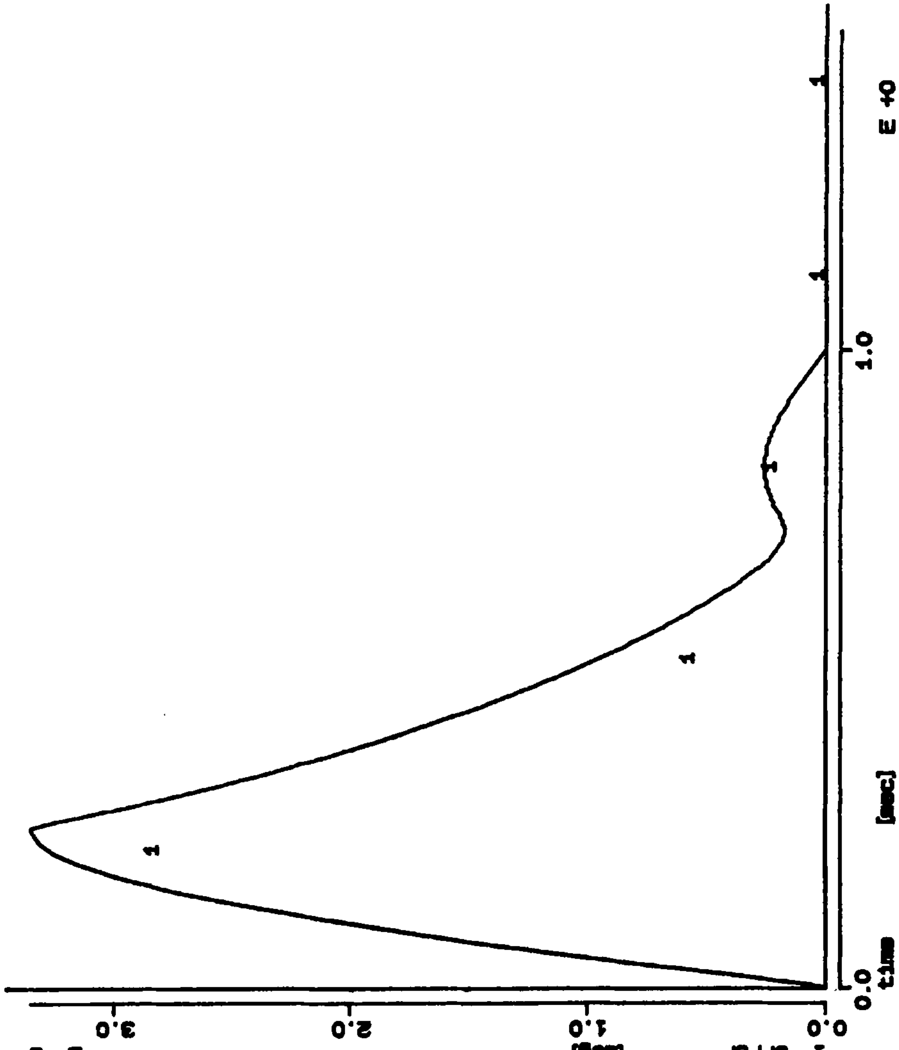

Figure 8b. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 6: orientation error.

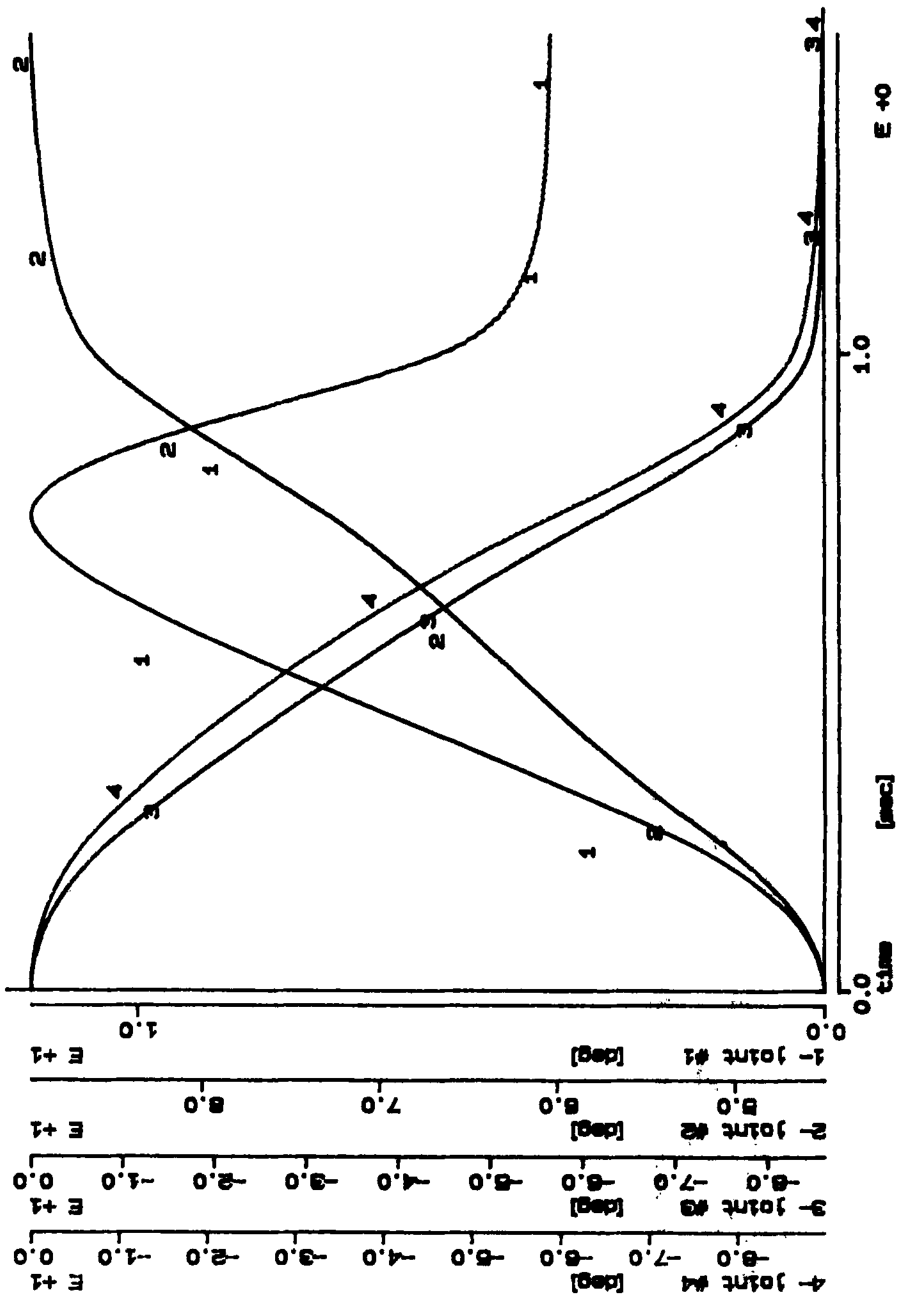

Figure 9a. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7: joint trajectories.

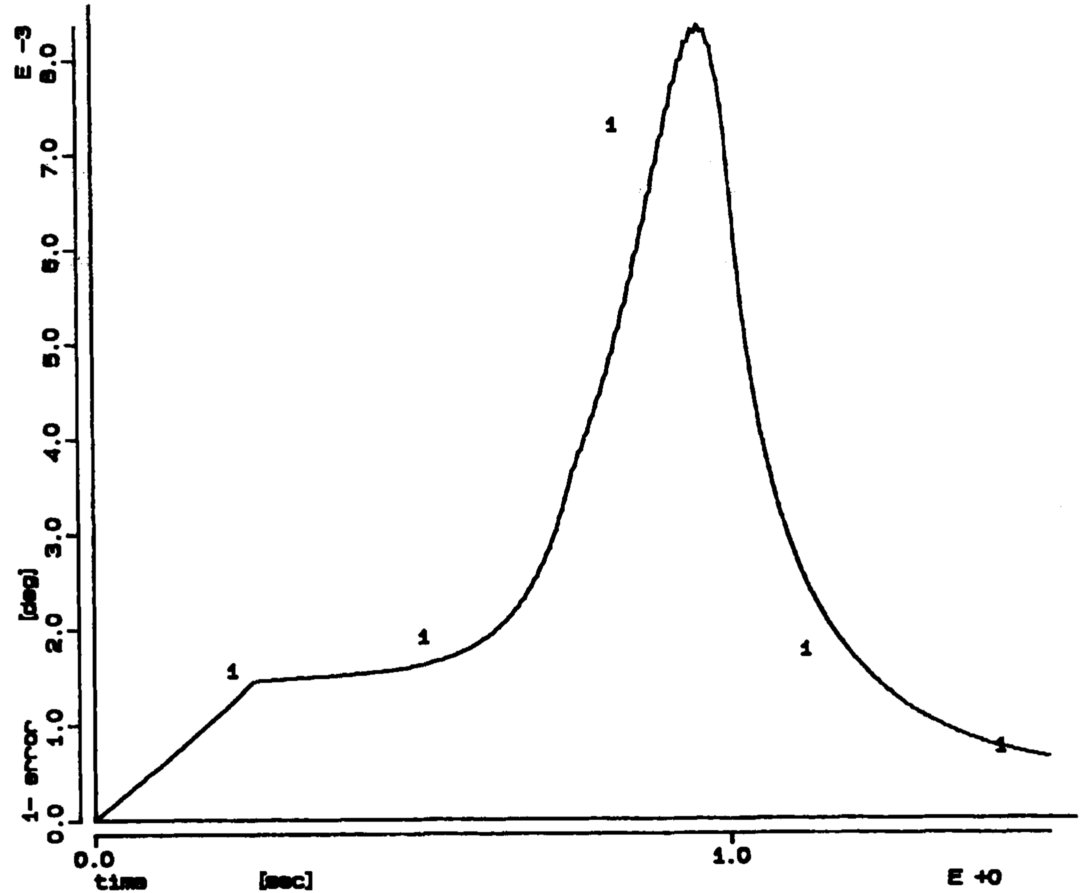

Figure 9b. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7: orientation error.

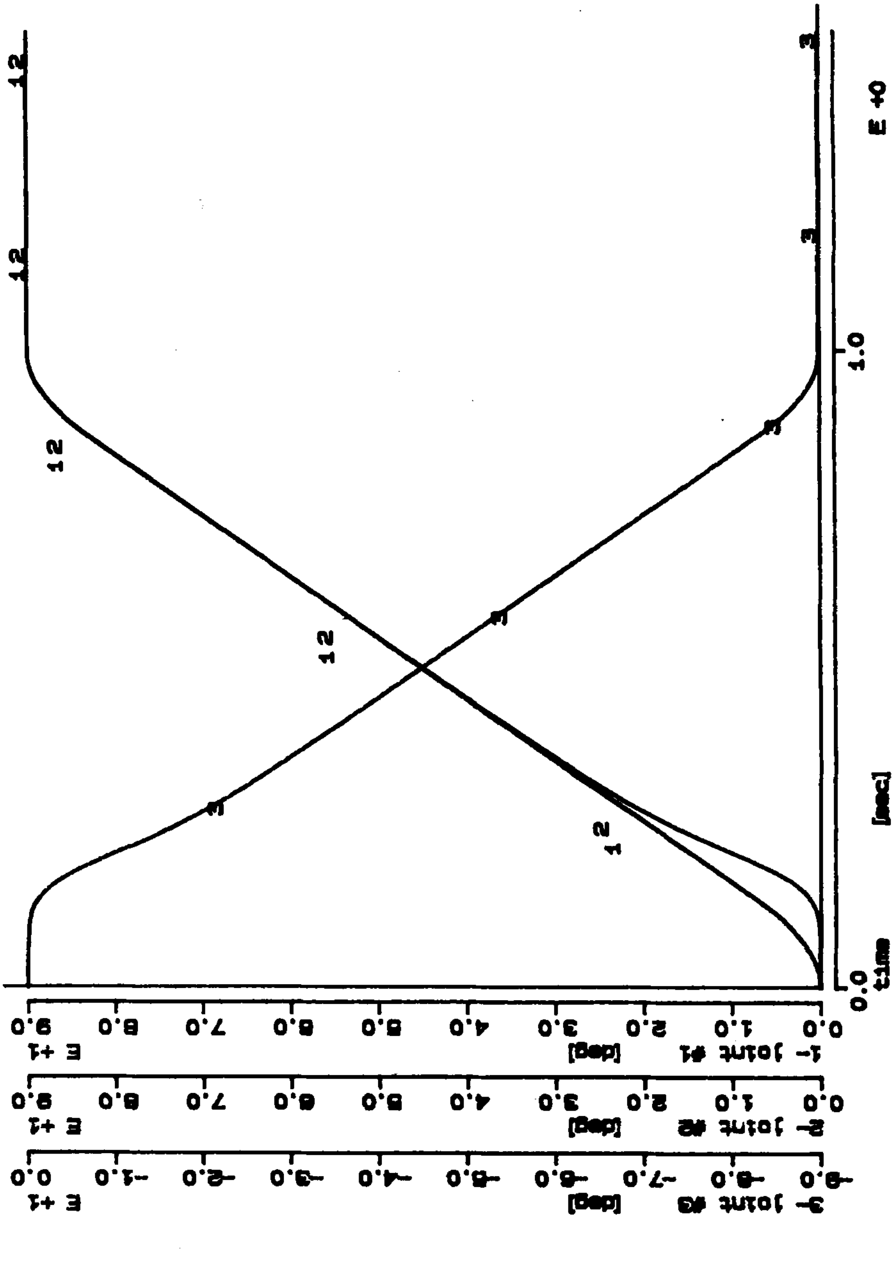

Figure 10a. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 6 (the initial configuration is singular): joint trajectories.

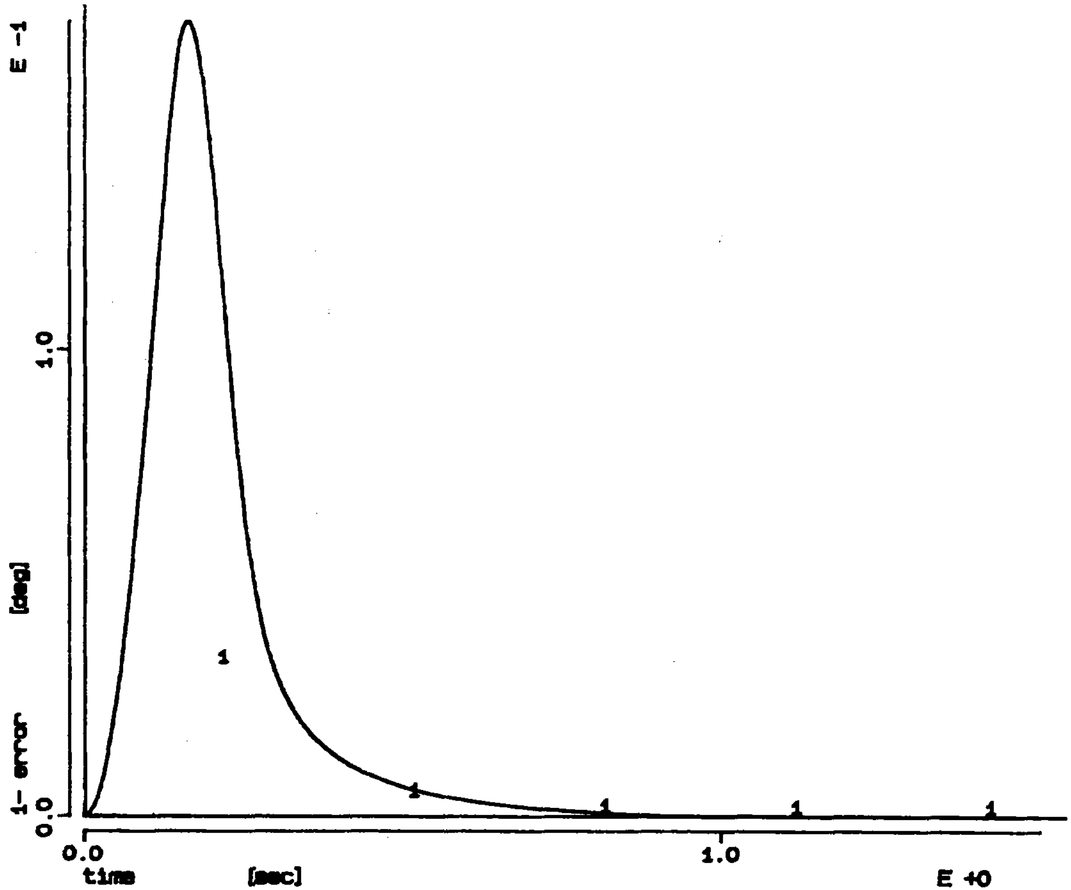

Figure 10b. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 6 (the initial configuration is singular): orientation error.

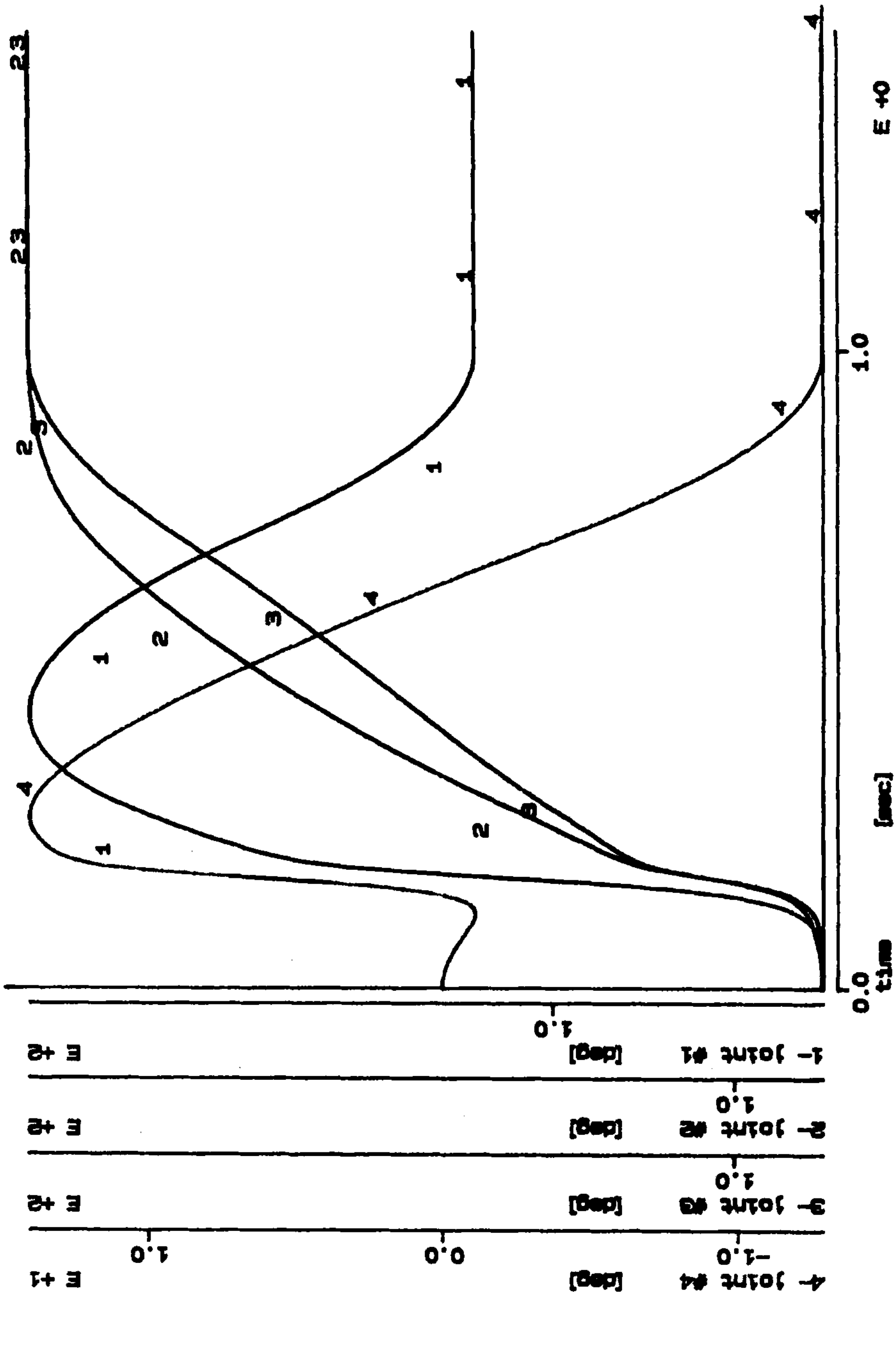

Figure 11a. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7 (the initial configuration is singular): joint trajectories.

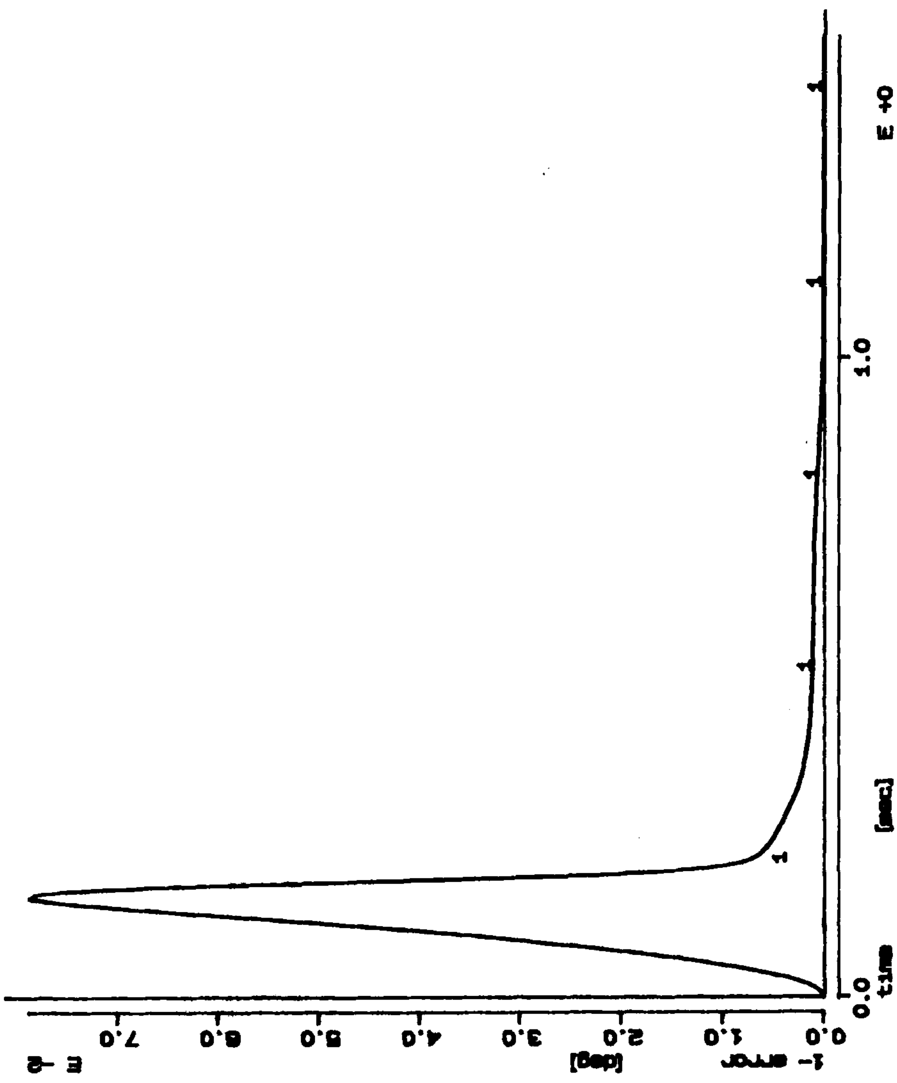

Figure 11b. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7 (the initial configuration is singular): orientation error.

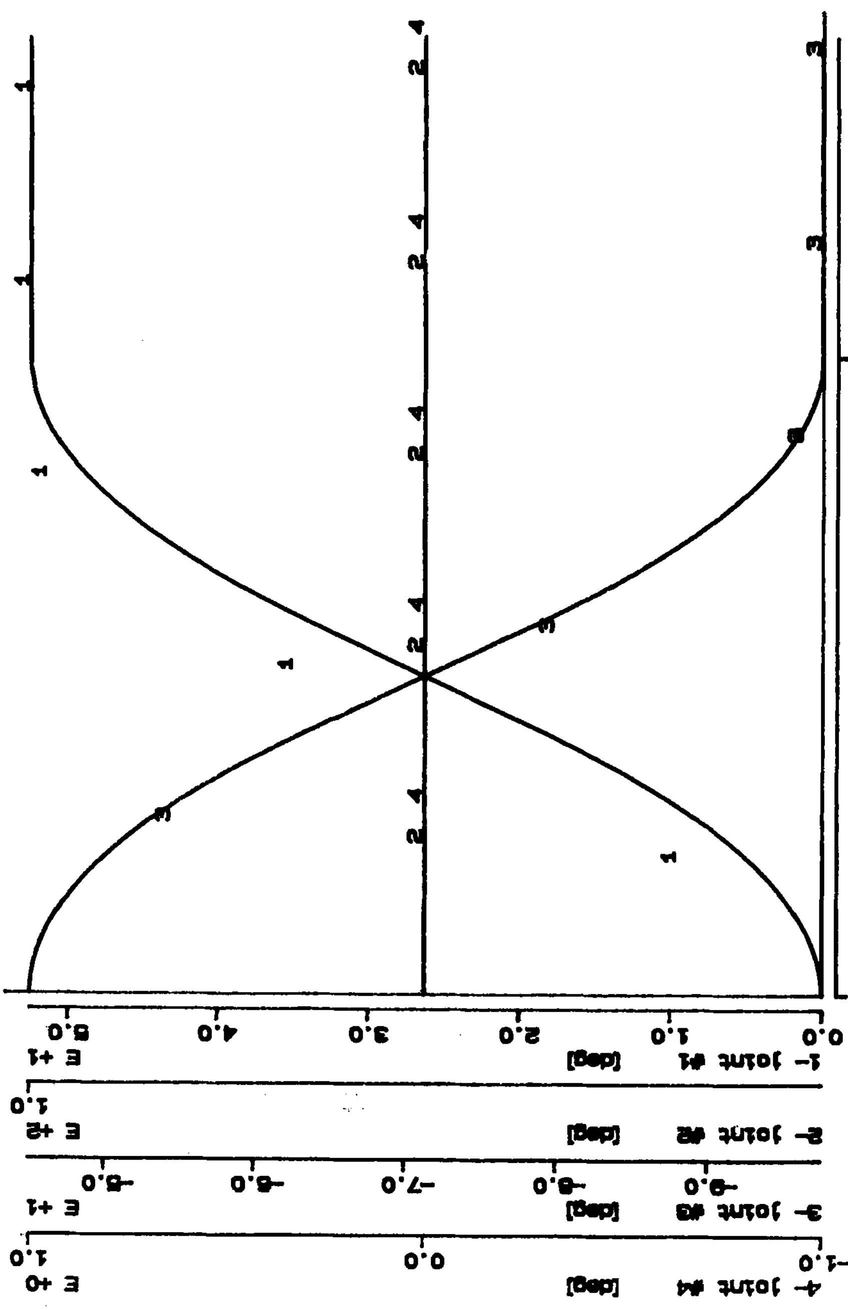

Figure 12a. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7 (the trajectory crosses a singularity): joint trajectories.

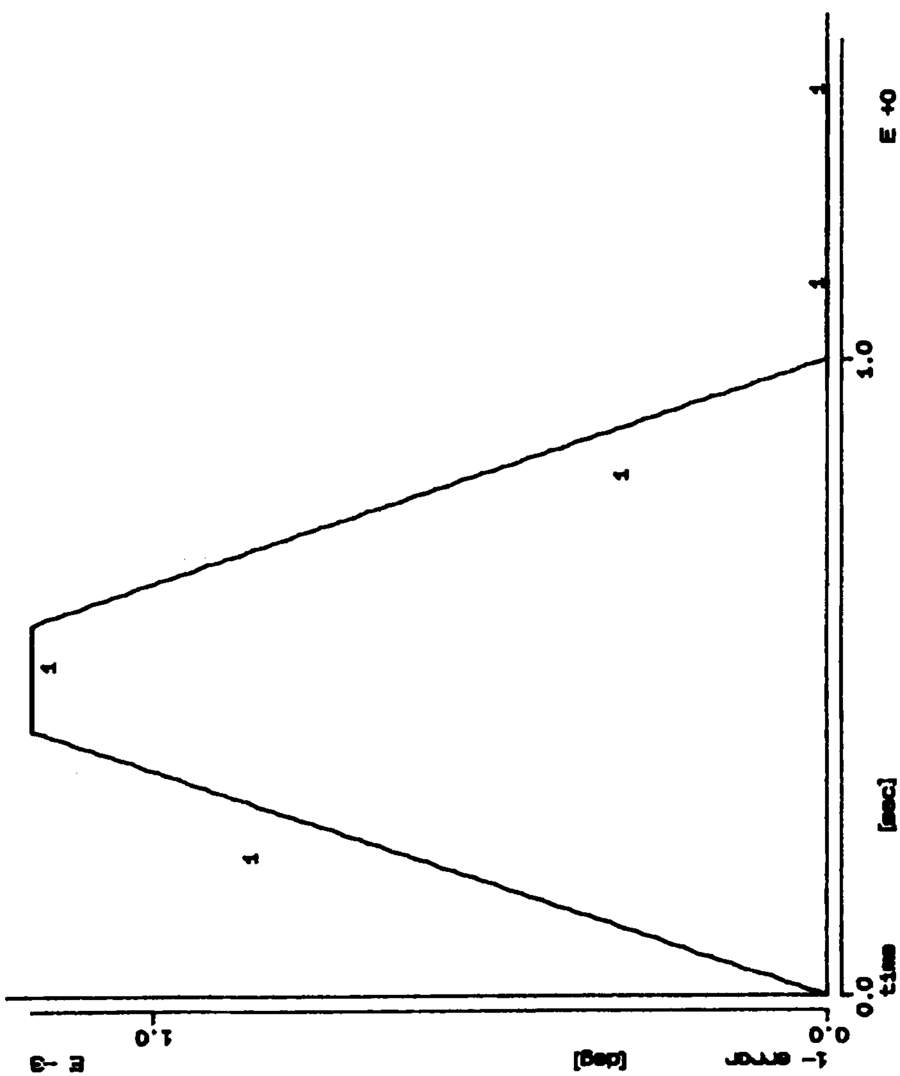

Figure 12b. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7 (the trajectory crosses a singularity): orientation error.

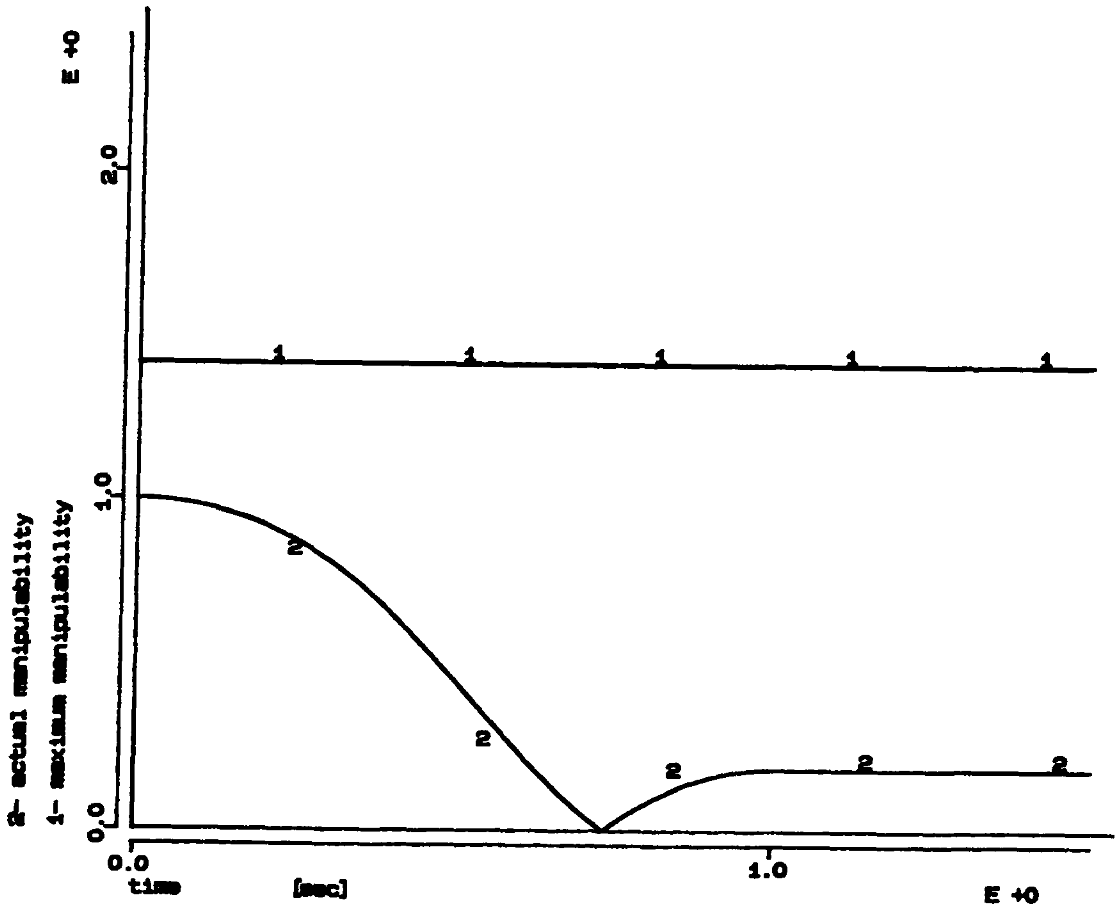

Figure 12c. Tracking performance of the scheme of Figure 3 applied to the wrist of Figure 7 (the trajectory crosses a singularity): manipulability measure.

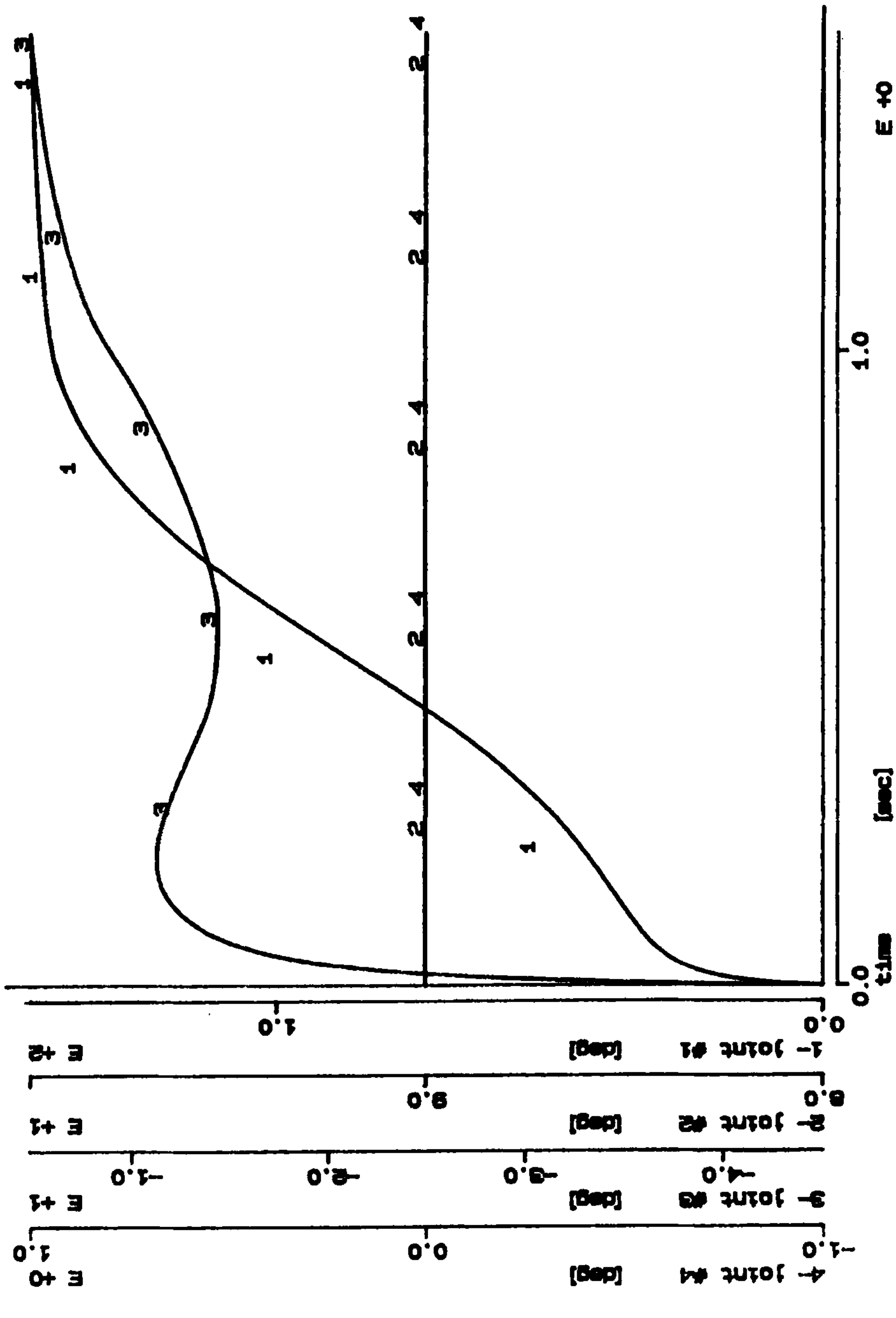

Figure 13a. Tracking performance of the scheme of Figure 5 applied to the wrist of Figure 7 (the trajectory avoids the singularity): joint trajectories.

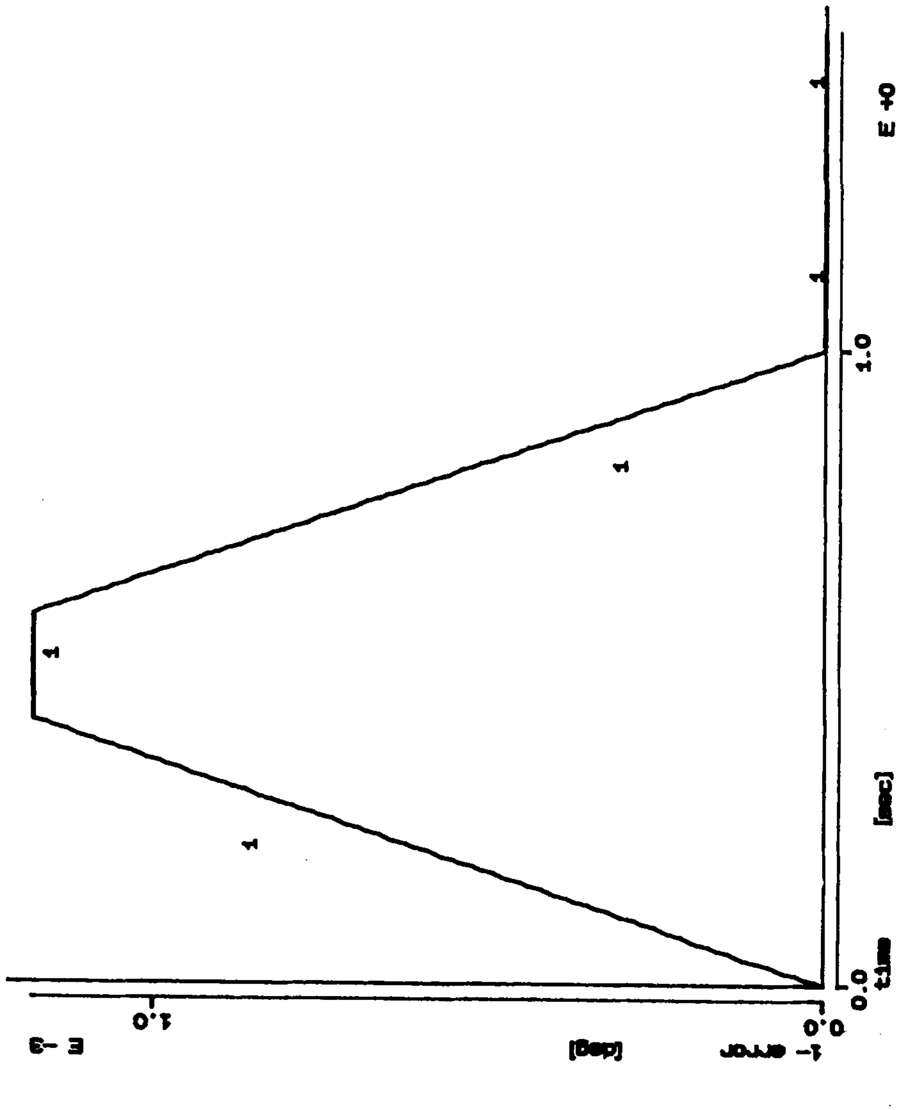

Figure 13b. Tracking performance of the scheme of Figure 5 applied to the wrist of Figure 7 (the trajectory avoids the singularity): orientation error.

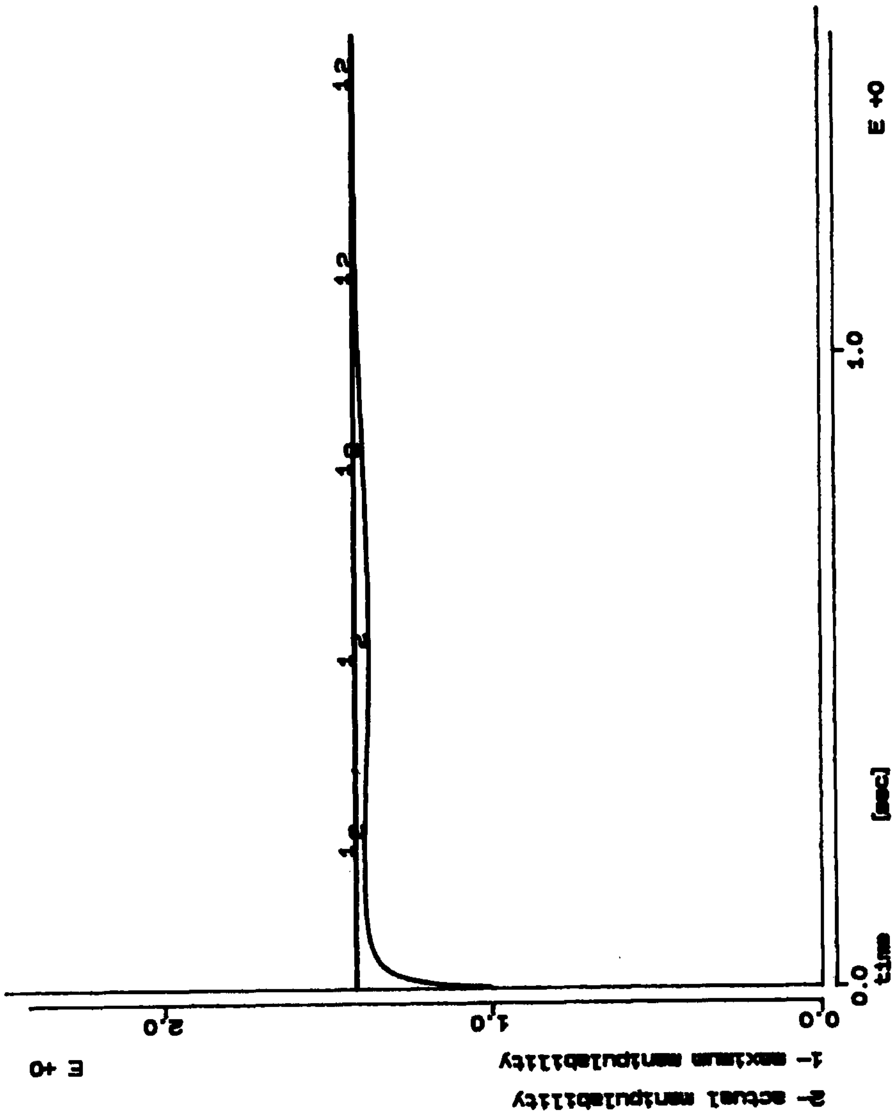

Figure 13c. Tracking performance of the scheme of Figure 5 applied to the wrist of Figure 7 (the trajectory avoids the singularity): manipulability measure.

APPENDIX

Figure 6 shows the three-revolute joint spherical wrist. Using the short notations $\sin\theta_i = s_i$ and $\cos\theta_i = c_i$, the direct kinematic functions of interest are:

$$\mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} -c_1c_2s_3 - s_ic_3 \\ -s_1c_2s_3 + c_1c_3 \\ s_2s_3 \end{bmatrix}, \tag{A-1}$$

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix}, \tag{A-2}$$

$$J_o = \begin{bmatrix} 0 & -s_1 & a_x \\ 0 & c_1 & a_y \\ 1 & 0 & a_z \end{bmatrix} \tag{A-3}$$

The direct kinematic functions for the four-revolute-joint spherical wrist of Figure 7 are:

$$\mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} -c_1c_2s_4 - s_1c_3c_4 + c_1s_2s_3c_4 \\ -s_1c_2s_4 + c_1c_3c_4 + s_1s_2s_3c_4 \\ s_2s_4 + c_2s_3c_4 \end{bmatrix}, \tag{A-4}$$

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} s_1s_3 + c_1s_2c_3 \\ -c_1s_3 + s_1s_2c_3 \\ c_2c_3 \end{bmatrix}, \tag{A-5}$$

$$J_o = \begin{bmatrix} 0 & -s_1 & c_1c_2 & a_x \\ 0 & c_1 & s_1c_2 & a_y \\ 1 & 0 & -s_2 & a_z \end{bmatrix}. \tag{A-6}$$

$$w = [2(1 - s_2^2s_3^2)]^{1/2}, \tag{A-7}$$

$$\mathbf{j}_w = [0 \quad -2s_2c_2s_3^2w^{-1/2} \quad -2s_2^2s_3c_3w^{-1/2} \quad 0]. \tag{A-8}$$

Notice that some handy reductions have been carried out in deriving the above Jacobians since, for each given configuration $\mathbf{q}$, the unit vectors $\mathbf{s}$ and $\mathbf{a}$ are already to be computed by the algorithms in the feedback loop and $\mathbf{n}$ is simply computed as $\mathbf{n} = \mathbf{s} \times \mathbf{a}$.

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