Theorem 2: Assume that the classical feedback structure control problem has been put into the standard problem format of Fig. 2. Assume further that conditions for the existence of a stabilizing controller satisfying bound (6) are met and A is Hurwitz. If, additionally, $D_{12} = I_{p1=m2}$, then the following is true.

i) The first Hamiltonian matrix H_{X_∞} is upper block triangular, and
ii) the solution of the associated Riccati equation is a null matrix

$$X_\infty = 0_n. \tag{25}$$

Proof: Completely analogous to that of Theorem 1.

We might note that similar conclusions can be deduced from the result of Theorem 1′ in [4] provided the augmented plant structure (5) is employed, $\gamma = 1, D_{21} = I$, and $D_{11} = 0$ initially. Since the descriptor form approach taken in [4] requires that both χ_1 and χ_2 be available for controller computation, a natural choice would be $\chi_1 = I_n, \chi_2 = 0_n$.

It is shown in [1] that a necessary and sufficient condition for the existence of a stabilizing controller satisfying the bound (6) is passing the following test on the spectral radius of the product $X_\infty Y_\infty$

$$\rho(X_\infty Y_\infty) < \gamma^2. \tag{26}$$

Hence, meeting the conditions of Theorem 1 trivially satisfies this important prerequisite.

Observe that if conditions of both Theorem 1 and Theorem 2 are met simultaneously, the Riccati equation solution step of the controller synthesis becomes redundant and can be skipped entirely.

There are controller synthesis problems of practical significance in which $D_{21} = I$ does not hold. An example of this is the *two degree of freedom* controller structure where $\dim(y) > \dim(w)$. This difficulty, however, can be readily removed by adding fictitious exogenous input signals of appropriate dimensions at the controller input so that $D_{21} = I$.

IV. Conclusions

Substantial simplification of H_∞ controller synthesis procedure in the case of open-loop stable plant and weight selection is shown to result through the use of the classical cascade controller feedback structure of Fig. 1. In this case the solution to the second associated ARE reduces identically to a null matrix 0_n and this part of the controller computation can be dispensed with.

In addition to reducing the computational intensity of the controller synthesis, this result ensures that the spectral test (28), one of the necessary and sufficient conditions for the existence of a stabilizing controller satisfying the bound (6), is *automatically* satisfied. Furthermore, if $D_{11} = 0$ and $X_\infty \geq 0$ exists, then both conditions of Theorem 1 in [1] are met. Note that the requirement that $D_{11} = 0$ is not at all restrictive and can be easily met in practice by selecting W_1 strictly proper.

Moreover, if conditions of both Theorem 1 and Theorem 2 are met, it is shown that the ARE solution step of the H_∞ controller synthesis becomes redundant and can be skipped altogether. However, the practical implications of $D_{12} = I$ are not as straightforward although the obvious difficulty of having a strictly proper plant may be removed by a proper approximation to sufficiently high frequency [3].

Finally, the results of this note establish a direct link between the classical feedback system structure and the simplified one-Riccati solution of the H_∞-optimal controller synthesis problem, a fact particularly appealing to the practicing control engineer.

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Design of Optimal Output Feedback Compensators in Two-Time Scale Systems

A. J. CALISE, J. V. R. PRASAD, AND B. SICILIANO

Abstract—**A formulation is presented for designing optimal output feedback compensators of fixed order for two-time scale systems. The formulation exploits an observer canonical form to represent the compensator dynamics. The formulation precludes the use of direct feedback of the plant output and achieves spillover suppression. A case study is developed involving the rapid pointing of a flexible robot arm.**

I. Introduction

To date, there exist very limited results that provide a direct design method for two-time-scale output feedback systems. It has been shown [1] that, in the case of full-state feedback, application of singular perturbation theory (SPT) to the LQ regulator problem separates the control design into slow and fast subproblems. In constant gain output feedback problems, this separation occurs naturally only for a very restrictive class of output structures [2], [3]. In general, constraints have to be introduced to suppress control and measurement spillover to achieve a decoupled design [4], otherwise, a single set of feedback gains must be designed to stabilize both the slow and fast subsystems [5]. More recently, an SPT approximation of the LQ optimal constant gain output feedback regulator problem has been obtained [6], which provides an $0(\epsilon^2)$ approximation to optimal closed-loop performance.

It is well known that constant gain feedback presents a severe design limitation. It is generally good practice to avoid direct feedthrough of sensor outputs to improve robustness and to reduce the effect of sensor noise at high frequency. Observer-based control design methods for two-time-scale systems have appeared in [7]–[9], where the separation principle was exploited to achieve a decoupled design. However, the order of the compensator when designed for large scale systems may prove unwarranted. Recently [10], [11], frequency domain results have been obtained for the case of employing strictly proper slow and fast compensators of fixed order, to stabilize strictly proper slow and fast subsystem models. It has been shown that to design a compensator such that the two-frequency-scale structure of the open-loop system is preserved, the compensator itself will have to be two-frequency scale. Moreover, unlike the full order observer case, the design is coupled. A parallel compensator structure is proposed made up of a slow compensator $C_s(s)$ and a fast compensatory $C_f(p)$, where $p = s/\epsilon$. The compensators must

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be designed subject to the constraint $C_s(\infty) = C_f(0)$. Thus, a two-step design is proposed, in which the fast compensator is designed first, followed by a slow compensator design in which $C_f(0)$ acts as an inner feedback loop around the slow plant dynamics. However, no specific methods are given for designing the compensators.

The main contribution in this technical note is to present a direct time domain formulation for designing fixed order dynamic compensators for two-time-scale systems, and to illustrate it by a case study. The formulation is done in an optimal output feedback setting that exploits an observer canonical form to represent the compensator dynamics. It employs a method for penalizing the plant and compensator states to improve the robustness of the design. The canonical structure also permits the constraint $C_f(0) = 0$ to be enforced in a simple form, thus decoupling the fast and slow compensator designs. This can be viewed as a generalization of the constant gain spillover suppression approach [4]. If separate actuators and sensors are provided to control the slow and fast modes, this permits a decentralized controller structure.

The case study involves the rapid pointing of a flexible arm. Results are included which demonstrate the performance and robustness of the controller design.

II. Compensator Design

A. Two-Time Scale Formulations

We consider linear time-invariant two-time scale systems of the form

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \qquad x_1 \in R^{n_1} \qquad u \in R^m \tag{1a}$$

$$\epsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u \qquad x_2 \in R^{n_2} \tag{1b}$$

$$y = C_1 x_1 + C_2 x_2 \qquad y \in R^p \tag{1c}$$

where $0 < \epsilon \ll 1$, and $\det\{A_{22}\} \neq 0$. To control systems of this form, it is desirable to separately design two compensators—one for the slow subsystem that results from formally setting $\epsilon = 0$, and one for the fast subsystem obtained from the time stretching transformation $\tau = t/\epsilon$. In [3] it is shown that a two-frequency scale transfer function matrix can be decomposed in the form

$$C(s, \epsilon) = C_1(s, \epsilon) + C_2(\epsilon s, \epsilon) + D(\epsilon). \tag{2}$$

Thus, a strictly proper two-time scale fixed-order compensator in the observer canonical form of [14] would have the following structure:

$$u = -H_1^\circ \zeta_1 - H_2^\circ \zeta_2 = u_1 + u_2 \tag{3a}$$

$$\dot{\zeta}_1 = P_1^\circ \zeta_1 + u_{c1} \qquad \zeta_1 \in R^{n_{c1}} \tag{3b}$$

$$\epsilon\dot{\zeta}_2 = P_2^\circ \zeta_2 + u_{c2} \qquad \zeta_2 \in R^{n_{c2}} \tag{3c}$$

$$u_{ci} = P_i u - N_i y \qquad u_{ci} \in R^{n_{ci}} \qquad i = 1, 2 \tag{3d}$$

where $n_{ci} \geq m$, P_i and N_i are matrices of free parameters, and

$$H_i^\circ = \text{block diag}\,\{[0 \cdots 0\,1]_{1\times\nu_{ij}} \qquad j = 1, \cdots, m\}, \qquad i = 1, 2 \tag{4a}$$

$$P_i^\circ = \text{block diag}\,[P_{i1}^\circ, \cdots, P_{im}^\circ] \tag{4b}$$

$$P_{ij}^\circ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{\nu_{ij}\times\nu_{ij}} . \tag{4c}$$

The observability indexes of the compensator are chosen to satisfy

$$\text{i) } \sum_{j=1}^{m} \nu_{ij} = n_{ci} \qquad \text{ii) } \nu_{ij} \leq \nu_{i,j+1} \qquad i = 1, 2. \tag{5}$$

In [11] it is shown that the closed-loop poles of the system (1, 3), with the exception of "hidden" and "lost" modes, can be approximated for sufficiently small ϵ by the roots associated with the return difference matrix expressions

$$\det\{I + C_s(s)P_s(s)\} = 0 \tag{6a}$$

$$\det\{I + C_f(p)P_f(p)\} = 0, \qquad p = \epsilon s \tag{6b}$$

where

$$P_s(s) = C_o(sI_{n1} - A_o)^{-1}B_o + D_o \tag{7a}$$

$$P_f(p) = C_2(pI_{n2} - A_{22})^{-1}B_2 \tag{7b}$$

$$C_s(s) = C_1(s, 0) + C_2(0, 0) \tag{7c}$$

$$C_f(p) = C_2(p, 0) \tag{7d}$$

and $A_o = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_o = B_1 - A_{12}A_{22}^{-1}B_2$, $C_o = C_1 - C_2A_{22}^{-1}A_{21}$, $D_o = -C_2A_{22}^{-1}B_2$. The hidden modes within P and C, and the lost modes arising from setting $\epsilon = 0$, are stable if the triples (A_o, B_o, C_o) and (A_{22}, B_2, C_2) are stabilizable-detectable.

Note that (7c) and (7d) imply

$$C_s(\infty) = C_f(0). \tag{8}$$

The compensator transfer function $C(s, \epsilon)$ associated with (3) can then be approximated as

$$C(s, \epsilon) \simeq \hat{C}_s(s) + C_f(p) \tag{9}$$

where $\hat{C}_s(s) = C_s(s) - C_s(\infty)$. Comparison of (9) with (3) reveals that

$$\hat{C}_s(s) = H_1^\circ(sI_{n1} - P_1^\circ + P_1H_1^\circ)N_1 \tag{10a}$$

$$C_f(p) = H_2^\circ(pI_{n2} - P_2^\circ + P_2H_2^\circ)N_2. \tag{10b}$$

B. Output Feedback

The difficulty that arises here is that the expression for the slow poles given in (6a) is in terms of $C_s(s)$. This implies that $\hat{C}_s(s)$ should be designed for the reduced plant $P_s(s)$ with $C_f(0)$ as an inner loop feedback, where from (10b)

$$C_f(0) = H_2^\circ(-P_2^\circ + P_2H_2^\circ)N_2. \tag{11}$$

Thus, the fast subsystem design should be performed first using the augmented system matrices

$$\tilde{A}_2 = \begin{bmatrix} A_{22} & -B_2H_2^\circ \\ 0 & P_2^\circ \end{bmatrix} \qquad \tilde{B}_2 = \begin{bmatrix} 0 \\ I_{n_{c2}} \end{bmatrix}$$

$$\tilde{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & H_2^\circ \end{bmatrix} \qquad G_2 = [N_2 \quad P_2]. \tag{12}$$

This is followed by a slow subsystem design where the augmented system matrices, including the inner loop feedback through $C_f(o)$, have the following structures:

$$\tilde{A}_1 = \begin{bmatrix} A_o' & -B_o'H_1^\circ \\ 0 & P_1^\circ \end{bmatrix} \qquad \tilde{B}_1 = \begin{bmatrix} 0 \\ I_{n_{c1}} \end{bmatrix}$$

$$\tilde{C}_1 = \begin{bmatrix} C_o & -D_oH^\circ \\ 0 & H^\circ \end{bmatrix} \qquad G_1 = [N_1 \quad P_1] \tag{13}$$

where $A_o' = A_o - B_o'C_f(0)C_o$, $B_0' = B_o[I_m + C_f(0)D_o]^{-1}$, $C_o = [I_p - D_o'C_f(0)]C_o$, $D_o' = D_o[I_m + C_f(0)D]^{-1}$.

At this point, we can define two optimal output feedback problems, for the subsystems (12) and (13). For each subsystem, define

$$J_i = E_{\tilde{x}_{io}}\left\{\int_0^\infty [\tilde{x}_i^T Q_i \tilde{x}_i + u_{ci}^T R_i u_{ci}]\,dt\right\} \qquad i = 1, 2 \tag{14}$$

where the augmented state vector is

$$\tilde{x}_i^T = [x_i^T \quad \xi_i^T] \qquad i = 1, 2 \tag{15}$$

and the control (used to design each compensator) is defined as

$$u_{ci} = -G_i\tilde{C}_i\tilde{x}_i \qquad i = 1, 2. \tag{16}$$

In what follows, the subscript i is suppressed to simplify the notation.

The necessary conditions for optimality require the solution of the triple $\{G, K, L\}$ satisfying

$$A_c^T K + KA_c + Q + C^T G^T RGC = 0 \tag{17a}$$

$$A_c L + LA_c^T + X_o = 0 \tag{17b}$$

$$RGCLC^T - B^T KLC^T = 0 \tag{17c}$$

for a stable closed-loop system matrix

$$A_c = \tilde{A} - \tilde{B}\tilde{G}\tilde{C}. \tag{18}$$

In (17b), $X_o = E\{x_o x_o^T\}$ is the variance matrix associated with the distribution assumed for the initial conditions.

One advantage to the above formulation is that each compensator is represented by a minimum number of parameters, and these are compactly placed in the equivalent constant gain matrices G_1 and G_2 in (12) and (13). A convergent numerical method for calculating G is given in [15]. Also, because we have precluded the use of direct feedback of the output, the design carries the same advantage of a full order observer in reducing the effect of sensor noise, and improving robustness to high-frequency unmodeled dynamics by guaranteeing additional rolloff (over that of the open loop plant) at high frequencies.

C. Spillover Suppression

The effect of $C_f(0)$ on the slow subsystem design, as described above, may be viewed as a combination of measurement and control spillover. In general, it should be avoided if possible since the fast subsystem control design could seriously destabilize a previously stable slow subsystem. Also, it has the same effect as a direct feedthrough of the output to the input, which is undesirable from a robustness and sensor noise viewpoint. In any case, since the primary objective of the fast controller is simply to stabilize the high-frequency dynamics, there should be no need for significant gain at low frequencies in the fast compensator. Therefore, it is of interest to examine the constraint $C_f(0) = 0$, when performing the fast subsystem control design. In general, such a constraint is related to the compensator parameters in a complicated fashion. However, in this case, it can be shown that the observer canonical form in (3) possesses a unique left matrix fraction description

$$C(s) = D(s)^{-1}N(s) \tag{19}$$

where

$$D(s) = L(s)P + S(s), \qquad N(s) = L(s)N \tag{20a}$$

$$L(s) = \text{block diag }\{[1 \; s \cdots s^{\nu_j - 1}] \qquad j = 1, \cdots, m\} \tag{20b}$$

$$S(s) = \text{diag }\{s^{\nu_j - 1} \qquad j = 1, \cdots, m\}. \tag{20c}$$

Thus, for the fast compensator ($P = P_2$, $N = N_2$, $s \to p$), the constraint $C_f(0) = 0$ becomes

$$L(0)N_2 = 0. \tag{21}$$

This amounts to zeroing selected rows in N_2 depending on the observability indexes (ν_{2j}) of the compensator. In order to avoid a trivial solution ($N_2 = 0$), it is necessary that $(\nu_{2j}) \geq 2$ for at least one value of j. In general, the number of constraint equations in (21) can be kept to a minimum by balancing the indexes. That is, the indexes should be selected as large as possible while satisfying the constraints in (5). For a single input plant, (21) reduces to constraining the first row in N_2 to zero. The numerical algorithm in [15] permits constraints of a very general class on the output feedback gain matrix by using a penalty function approach.

D. Loop Recovery Formulation

A major objection to the design of constrained dynamic compensators is that there are no guarantees on stability margins, and there are few guidelines for selecting the distribution of initial conditions, and for penalizing the plant states and compensator states to improve either performance or robustness. Full-state feedback using Linear Quadratic Regulator Theory on the other hand is quite easy by comparison to carry out. In [16], a loop transfer recovery procedure for fixed-order compensators is outlined which uniquely defines the state and compensator weighting matrix, and the initial state distribution matrix. This is briefly summarized below.

Let K^* be the gain matrix resulting from a full-state feedback design. The return signal in the case of full-state design is $-K^*x$. Referring to (3), the return signal in the case of fixed-order compensator design is $-H^o\zeta$. Thus, the objective in designing the compensator should be to minimize

$$y_1 = K^*x - H^o\zeta \tag{22}$$

for a suitably chosen input and for *zero* initial conditions. This naturally leads to selecting the following index of performance:

$$H = E_{x_o}\left\{\int_0^\infty [(y_1)^2 + u_c^T u_c]\, dt\right\}. \tag{23}$$

Substituting for y_1 from (22) and rewriting (23) in the form of (14) leads to the following expressions for the weighting matrices:

$$Q = \begin{bmatrix} K^{*T}K^* & -K^{*T}H^o \\ -H^{oT}K^* & H^{oT}H^o \end{bmatrix} \quad R = \rho 1_{n_c}. \tag{24}$$

Selecting the input waveforms as impulses with magnitudes uniformly distributed on the unit sphere results in the following expression for X_o:

$$X_o = \begin{bmatrix} BB^T & 0 \\ 0 & 0 \end{bmatrix}. \tag{25}$$

Equations (24) and (25) uniquely define the structure of the weighting matrices needed for the fixed-order compensator design. Note that, unlike the design of a full-order observer, the design of a fixed-order controller depends on the gain matrix from the full-state design step. Moreover, this gain matrix is not implemented as a part of the final controller.

III. A Case Study

A. System Description

A flexible slewing arm with a rigid body rotation and flexible "clamped-mass" modes is considered as depicted in Fig. 1. The actual flexible arm chosen is a prototype in the Flexible Automation Laboratory at the Georgia Institute of Technology [12]. For this arm modeled as an Euler-Bernoulli beam, the first two flexible modes have measured modal frequencies of 13 rad/s and 87.5 rad/s. The next higher mode has a measured frequency of 260 rad/s, which is too large to be actively controlled by the available actuator. In order to derive a two-time scale formulation, the quantity $\epsilon = (1/k_2)^{1/2}$ was selected as the perturbation parameter, where $k_2 = 200$ is the stiffness parameter associated with the second flexible mode. A similar formulation can be found in [13], which treats the case of full-state feedback. The motivation behind this choice is that the modal frequency associated with the first mode is considerably

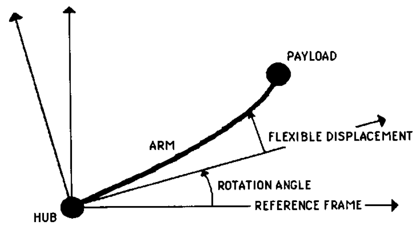

Fig. 1. Flexible slewing arm.

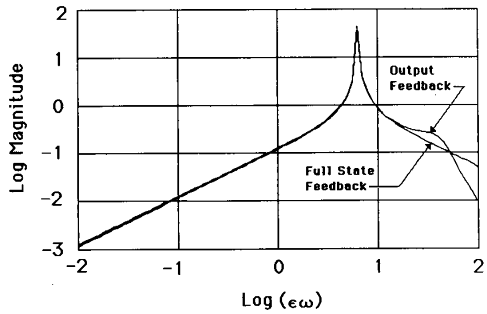

Fig. 2. Comparison of magnitude plots for the fast subsystem design.

lower, and within the range of the bandwidth of the desired closed-loop rigid-body motion. Thus, the slow subsystem states are the joint angle, the first flexible modal displacement, and their respective rates. The fast subsystem states are the second flexible modal displacement and its rate. The available measurements are the joint angle, the joint angle rate, and two strain gauge measurements. The first is located close to the base to ensure a good measurement of the first flexible mode. The second is placed at the midpoint, close to a peak in the second mode shape. Also, the third mode shape is nearly zero at this location.

B. Controller Design

Two second-order compensators were separately designed to stabilize the slow and fast dynamics. First, a full-state feedback design was carried out using the fast subsystem dynamics to damp the second flexible mode. This produced a damping ratio of 0.4. This was followed by an output feedback design using the loop transfer recovery formulation in [16]. Since the control is scalar, the spillover suppression constraint in (21) was enforced by constraining the first row of N_2 to zero. This resulted in approximately the same damping as the full-state design. The robustness of this design is shown in Fig. 2, which compares the magnitude Bode plots for the full-state design and the constrained output feedback design in the fast time scale frequency. Note that the output feedback controller satisfies the requirement $C_f(0) = 0$, and the loop properties of full-state feedback are adequately recovered. More important, however, is the fact that the output feedback controller provides an additional 40 dB/decade rolloff at high frequency. The phase margins for both controllers are close to 90°. The full-state design also resulted in zero gain at low frequency, which substantiates the point made earlier that low-frequency gain is not needed to damp high-frequency dynamics.

The full-state design for the slow subsystem was performed to achieve a 3 rad/s bandwidth for the rigid-body response with 0.7 damping of the first flexible mode. The output feedback design with loop transfer recovery achieved essentially the same bandwidth, with 0.67 damping of the flexible mode and an additional 40 dB/decade rolloff. Fig. 3 illustrates the robustness of the composite design by showing the magnitude and phase plots for the loop broken at the plant input. The composite system has about 12 dB of gain margin and 85° of phase margin. Comparison of Fig. 3 with Fig. 2 illustrates the close agreement between the fast frequency scale transfer function $C_f(p)P_f(p)$ and the composite system high-frequency response. A similar agreement was obtained at low frequency. Figs. 4 and 5 are simulation results for a unit step command in joint angle. The responses are shown both with and without the fast compensator implemented. In Fig. 4, the joint angle response demonstrates that the fast controller has essentially no effect on the rigid-body rotation and first flexible mode. In Fig. 5, the tip responses of the flexible arm are compared in terms of the resulting total tip deflection. This clearly shows the effect of the fast compensator, and the fact that the two compensators operate in a decoupled manner over separate frequency ranges.

IV. Summary

A formulation has been presented for designing output feedback compensators for two-time scale systems. The formulation results in a design that avoids direct feedback of outputs to inputs, and minimizes the number of free parameters needed in the compensator representation. A detailed example was used to demonstrate the robustness of the design approach.

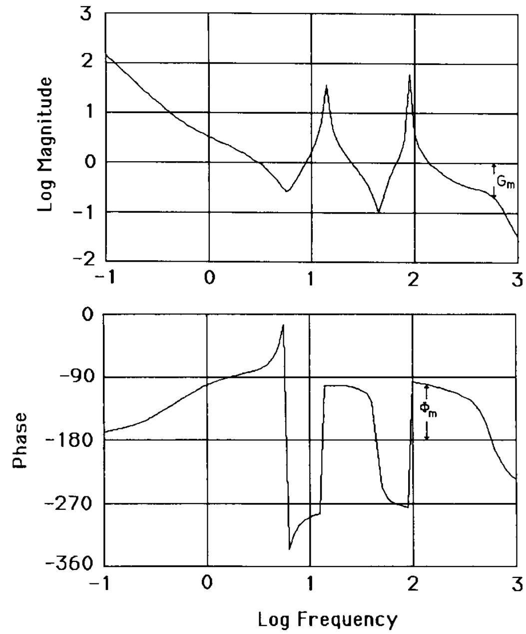

Fig. 3. Bode plot of the composite system for the loop broken at the plant input.

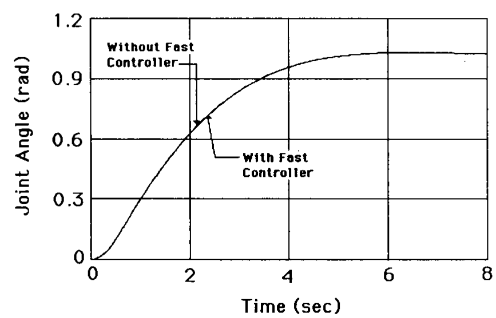

Fig. 4. Joint angle response for a step joint angle command input.

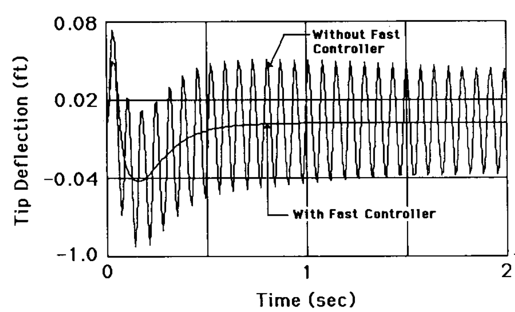

Fig. 5. Flexible arm tip response for a step joint angle command input.

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A New Approach for Designing a Reduced-Order Controller of Linear Singular Systems

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Abstract—**This note presents a novel approach for designing a multivariable reduced-order controller of a linear time-invariant singular system. The approach is based on decomposing the original system into slow and fast subsystems using the Drazin inverse technique. The resulting subsystems are then used to obtain an observer of reduced order. This, in turn, is used in designing a new reduced-order controller which is capable of placing the dominant eigenvalues of the system to arbitrary locations. The present technique is shown to overcome some difficulties inherent in other treatments of reduced-order controllers of singular systems and assures that the known corresponding controller of the regular state space systems is merely a special case of the developed results.**

I. Introduction

It is well known that there are advantages to using linear controllers for improving systems performance. The traditional approach to realizing such controllers in practical situations, where all the internal variables are not available for direct measurements, is to incorporate an observer into the system design. The observer proved to offer practical solutions to various regular control problems as well as a fascinating research area. In the linear time-invariant control systems, a host of papers regarding the design procedures of the minimal order observers have appeared in the literature in recent years, since the pioneering paper of Luenburger [1]-[3]. A long-standing problem in control system theory is still the question of designing observers of reduced order for singular systems. In [4], we have introduced two different approaches in designing both full- and reduced-order observers of linear singular systems. These observers have been used in the design of linear controllers to eliminate the impulsive modes and place the dominant eigenvalues of the original singular system to arbitrarily chosen locations under a certain condition. In this note we present a novel approach that gives a direct access to design a reduced-order controller for a linear multivariable singular system with no constraints on the system parameters. The system considered here is characterized by

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $x \in R^n$, $U \in R^r$, $y \in R^m$, and E, A, B, and C are real constant matrices with appropriate dimensions, and C is of full row rank, and E is possibly singular. It is known that systems of the form (1) are of practical importance since they appear in many areas such as electrical networks, singularly perturbed systems, composite systems, Leontieff models in multisector economy, Leslie population models in biology, etc. [5]. These kinds of systems also offer many advantages over the regular state space systems since they are used in obtaining general descriptions of many control problems in which the resulting mathematical models cannot be obtained in the state space forms [5]. System (1) is sometimes referred to as a generalized state space system or a descriptor system. In the development to follow, it will be assumed throughout that system (1) is solvable, i.e., there exists a scalar $\lambda \in R$ such that $(A + \lambda E)^{-1}$ exists for some scalar λ. For more about solvability, see [5]. The approach proposed here is based on decomposing the original system (1) into slow and fast subsystems [6]. It is well known that the fast subsystem reduces to an algebraic equation for all $t \geq 0+$. This property motivates us to extend the design procedure of the well-known controller from state space systems to the part which contains the dominant eigenvalues of system (1) with some modifications. Such a controller is shown to give easy and simple computations with certain desirable properties. Throughout, it is assumed that the slow subsystem is controllable and observable in the sense of regular state space systems. The structure of the note is as follows. In Section II we present the basic observer structure of system (1). Before presenting the general design procedures of a reduced-order controller in Section III, we also provide in Section II a heuristic discussion of the method to be used in deriving a reduced-order observer. An example to support the usefulness of our approach is presented in Section IV. Section V presents our conclusion.

II. Basic Observer Structure

In singular systems, it is convenient to work with a standard form that makes the analysis and computations much easier. To get this form, first recall from the theory of singular systems [5] that if system (1) is solvable, then there always exists a nonsingular matrix $(A + \lambda E)^{-1}$ for some scalar $\lambda \in R$ such that (1) can be rewritten in another form as

$$\tilde{E}\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) \tag{2a}$$

$$y(t) = Cx(t) \tag{2b}$$

where

$$\tilde{E} = (A + \lambda E)^{-1}E; \qquad \tilde{A} = (A + \lambda E)^{-1}A$$

$$\tilde{B} = (A + \lambda E)^{-1}B; \qquad \tilde{E}\tilde{A} = \tilde{A}\tilde{E}.$$

It is important to note that system (2) preserves all properties of system (1) and sometimes is referred to as an alternative form of (1). Throughout, we will be concerned with system (2), unless otherwise stated. Using the Drazin inverse matrices, one can easily decompose system (2) into slow and fast subsystems of dimensions $n - \nu$ and ν, respectively, and ν denotes the index of E, which is defined as the smallest nonnegative integer such that the rank (E^ν) = rank $(E^{\nu+1})$. The slow part takes the

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