Modeling of multi-link rigid-soft manipulator

Stanislao Grazioso

Thursday 17th May, 2018
The goal

Modeling of rigid and soft links connected through

- all low-pair joints
- active/passive joints
- rigid/flexible joints
Introduction
The coupling approach

Proposition

The relative transformation between two mappings $a$ and $b$ can be represented by a right multiplication on $SE(3)$, i.e.

$$H^b(t) = H^a(t)H^r(t)$$

where $H^r(t)$ represents the relative transformation expressed with respect to the frame $H^a(t)$

Notice that this mapping can be inverted as

$$H^a(t) = H^b(t)(H^r(t))^{-1}$$

such that $(H^r(t))^{-1}$ takes the meaning of the relative transformation between $a$ and $b$ expressed with respect to $b$. 
Rigid joint

\[ H_B = H_A H_{J,I} \]

- \( H_{J,I} \) is the *relative frame* which represents the relative transformation according to the behavior of joint \( I \)
- \( H_{J,I} \) has a restricted number of degrees of freedom \( k_I \leq 6 \) \( \rightarrow \) it belongs to a subgroup of \( SE(3) \)
Rigid joint (cont’d)

\[
\begin{align*}
\delta(H_{J,I}) &= H_{J,I} \tilde{\delta} h_{J,I} = H_{J,I}(\tilde{A}_I \delta h_{j,I}) \\
H_{J,I}' &= H_{J,I} \tilde{\eta}_{J,I} = H_{J,I}(\tilde{A}_I \eta_{j,I})
\end{align*}
\]
## Rigid joint (cont’d)

The table below defines various rigid joints with their respective subgroups, dimensions, and associated matrices.

<table>
<thead>
<tr>
<th>Joint</th>
<th>Subgroup</th>
<th>Dimension</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Joint 1" /></td>
<td>$SO(2)$</td>
<td>1</td>
<td>$\begin{bmatrix} 0_{3 \times 1} \ n \end{bmatrix}$</td>
</tr>
<tr>
<td><img src="image2.png" alt="Joint 2" /></td>
<td>$\mathbb{R}$</td>
<td>1</td>
<td>$\begin{bmatrix} n \ 0_{3 \times 1} \end{bmatrix}$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Joint 3" /></td>
<td>$H_p$</td>
<td>1</td>
<td>$\begin{bmatrix} p_n &amp; n \ n \ n \end{bmatrix}$</td>
</tr>
<tr>
<td><img src="image4.png" alt="Joint 4" /></td>
<td>$SO(2) \times \mathbb{R}$</td>
<td>2</td>
<td>$\begin{bmatrix} 0_{3 \times 1} &amp; n \ n &amp; 0_{3 \times 1} \end{bmatrix}$</td>
</tr>
<tr>
<td><img src="image5.png" alt="Joint 5" /></td>
<td>$\mathbb{R}^2$</td>
<td>2</td>
<td>$\begin{bmatrix} n_1 &amp; n_2 \ 0_{3 \times 1} &amp; 0_{3 \times 1} \end{bmatrix}$</td>
</tr>
<tr>
<td><img src="image6.png" alt="Joint 6" /></td>
<td>$SO(3)$</td>
<td>3</td>
<td>$\begin{bmatrix} 0_{3 \times 1} &amp; 0_{3 \times 1} &amp; 0_{3 \times 1} \ n_1 &amp; n_2 &amp; n_3 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

**Table:** Joint definition
Kinematic constraints

Constraint equations

\[
I_{4 \times 4} = H_B^{-1}H_AH_{J,I}
\]
\[
I_{4 \times 4} = H_\phi(H_A, H_B, H_{J,I})
\]

\[\downarrow\] six-dimensional vector from an SE(3) element

\[
\phi(H_\phi) = \text{vect}_{SE(3)}(H_\phi) = 0_{6 \times 1}
\]

\[
\text{vect}_{SE(3)}(\mathcal{H}(R, u)) = \begin{bmatrix} u \\ \phi(R) \end{bmatrix}
\]

such that \( \widetilde{\phi}(R) = (R - R^T)/2 \).

Vectorial map \(\rightarrow\) 6 constraint equations!
Kinematic constraints

Constraint gradient

\[ \delta(\phi) = D\phi \cdot \mathbf{A} \delta \mathbf{h} = \phi_q \mathbf{A} \delta \mathbf{h} \]

where

\[ \mathbf{A} = \text{diag}(\mathbf{I}_{6 \times 6}, \mathbf{I}_{6 \times 6}, \mathbf{A}_I) \]

and

\[ \delta \mathbf{h} = \begin{bmatrix} \delta h_A \\ \delta h_B \\ \delta h_{j,l} \end{bmatrix} \]
Flexible joint

Joint deflection

\[ \delta_I = \alpha_I - \alpha_I^0 \]
Flexible joint: Internal forces

**Elastic forces**

\[ g_{int, I, K}(\alpha_I) = K_I \delta_I \]

where \( K_I = \text{diag}(K_{I,1}, \ldots, K_{I,k_I}) \) is the diagonal stiffness matrix of the kinematic joint \( I \)

**Dissipation forces**

\[ g_{int, I, D}(\alpha_I) = D_I \dot{\alpha}_I \]

where \( D_I = \text{diag}(D_{I,1}, \ldots, D_{I,D_I}) \) is the diagonal damping matrix of the kinematic joint \( I \)
The multi-link rigid-soft manipulator
piece-wise constant deformation soft arm

kinematic joint

rigid constraint

boundary conditions

constant deformation soft arm

rigid arm
Kinematic configuration

Generic multi-link manipulator:

- $n$ nodal frames $H_i \in SE(3)$
  \[
  \delta(H_i) = H_i \delta h_i \\
  \dot{H}_i = H_i \eta_i
  \]

- $m$ relative frames $H_{J,I} \in$ subgroup of $SE(3)$
  \[
  \delta(H_{J,I}) = H_{J,I} \delta h_{J,I} = H_{J,I}(A_I \delta h_{J,I}) \\
  \dot{H}_{J,I} = H_{J,I} \eta_{J,I} = H_{J,I}(A_I \eta_{J,I})
  \]
Kinematic configuration

Kinematic configuration of the generic multi-link manipulator

\[ H = \text{diag}(H_1, \ldots, H_n, H_{J,1}, \ldots, H_{J,m}) \]

\[
\delta(H) = HA\delta h \\
\dot{H} = HA\eta
\]

\[
\delta h = \begin{bmatrix}
\delta h_1 \\
\vdots \\
\delta h_n \\
\delta h_{j,1} \\
\vdots \\
\delta h_{j,m}
\end{bmatrix}; \quad \eta = \begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_n \\
\delta \eta_{j,1} \\
\vdots \\
\delta \eta_{j,m}
\end{bmatrix}
\]

\[ A = \text{diag}(I_{6 \times 6}, \ldots, I_{6 \times 6}, A_1, \ldots, A_m) \]
Hamilton's principle

$$\delta \left( \int_{t_0}^{t_1} \left( K(H, \eta) - V_{int}(H) + V_{ext}(H) - \lambda^T \phi(H) \right) dt \right) = 0 .$$

$$\delta \left( \int_{t_0}^{t_1} K(H, \eta) dt \right) = - \int_{t_0}^{t_1} g_{ine}^T (H, \eta, \dot{\eta}) dt$$

$$\delta \left( \int_{t_0}^{t_1} V_{int}(H) dt \right) = + \int_{t_0}^{t_1} g_{int}^T (H) dt$$

$$\delta \left( \int_{t_0}^{t_1} V_{ext}(H) dt \right) = - \int_{t_0}^{t_1} g_{ext}^T (H) dt$$

$$\delta \left( \int_{t_0}^{t_1} \lambda^T \phi(H) dt \right) = + \int_{t_0}^{t_1} \left( h^T A^T \phi_q^T (H) \lambda + \delta \lambda^T \phi(H) \right) dt$$
Hamiltonian formulation

Equations of motion (DAE)

\[
\begin{align*}
\dot{H} &= H \tilde{A} \eta \\
\mathbf{g}_{ine}(H, \eta, \dot{\eta}) + \mathbf{g}_{int}(H) + \mathbf{A}^T \phi_q^T(H) \lambda - \mathbf{g}_{ext}(H) &= \mathbf{0}_{(6n+k) \times 1} \\
\phi(H) &= \mathbf{0}_{6m \times 1}
\end{align*}
\]
Time integration

Equations of motion (DAE)

\[ \dot{H} = H \tilde{A} \eta \]  
\[ g(H, \eta, \dot{\eta}) + A^T \phi_q^T(H) \lambda = 0_{(6n+k) \times 1} \]  
\[ \phi(H) = 0_{6m \times 1} \]

\[ g(H, \eta, \dot{\eta}) = g_{\text{ine}}(H, \eta, \dot{\eta}) + g_{\text{int}}(H) - g_{\text{ext}}(H) \]
Time integration

The Crounch and Grossman geometric methods: the exponential map is the solution of a differential equation of a frozen velocity field

Discretized equations of motion

\[ H_{n+1} = H_n \exp_{SE(3)}(\tilde{A}n_{n+1}) \]  
\[ g(H_{n+1}, \eta_{n+1}, \dot{\eta}_{n+1}) + A^T \phi_q(H_{n+1})\lambda_{n+1} = 0_{(6M+k)\times1} \]  
\[ \phi(H_{n+1}) = 0_{6m\times1} \]

where in (4):

\[ H_{l,n+1} = H_{l,n} \exp_{SE(3)}(\tilde{\eta}_{l,n+1}) \]
\[ H_{J,l,n+1} = H_{J,l,n} \exp_{SE(3)}(\tilde{A}l_{n,j,l,n+1}) \]
Time integration

Integration formulae

\[ n_{n+1} = h \eta_n + (0.5 - \beta) h^2 a_n + \beta h^2 a_{n+1} \]  \hspace{1cm} (7)

\[ \eta_{n+1} = \eta_n + (1 - \gamma) h a_n + \gamma h a_{n+1} \]  \hspace{1cm} (8)

\[ a_{n+1} = \frac{1}{1 - \alpha_m} \left( (1 - \alpha_f) \dot{\eta}_{n+1} + \alpha_f \dot{\eta}_n - \alpha_m a_n \right) \]  \hspace{1cm} (9)

- \( n \): vector of incremental motions
- \( a \): vector of pseudo-accelerations
- \( n, h \): time step, time step size
- \( \alpha_m, \alpha_f, \gamma, \beta \): numerical parameter such that a desired spectral radius \( \rho \in [0, 1] \) is achieved

\[ \alpha_m = \frac{2\rho - 1}{\rho + 1}; \quad \alpha_f = \frac{\rho}{\rho + 1}; \quad \gamma = \frac{3 - \rho}{2(\rho + 1)}; \quad \beta = \frac{1}{(\rho + 1)^2} \]
∀ time step $n$: Eq. 4-6 are solved for the unknowns $n_{n+1}, \eta_{n+1}, \dot{\eta}_{n+1}, a_{n+1}$

Discretized equations of motion are nonlinear $\rightarrow$ Newton-Raphson!
Time integration

Linearization of (4)

\[ \Delta(H_{n+1}) = H_{n+1} \tilde{A} \Delta H_{n+1} \]  
(10)

\[ \Delta(H_{n+1}) = H_{n+1} (T(n_{n+1}) A \Delta n_{n+1}) \tilde{} \]  
(11)

\[ \Downarrow \]

\[ A \Delta H_{n+1} = T(n_{n+1}) A \Delta n_{n+1} \]  
(12)

\[ T(n_{n+1}) = \text{diag}(T_{SE(3)}(n_{1,n+1}), \ldots, T_{SE(3)}(n_{M,n+1}), \ldots, T_{SE(3)}(A_{1nJ,1,n+1}), \ldots, T_{SE(3)}(A_{m,nJ,m,n+1})) \]  
(13)
Time integration

Linearization of (5) $r(H_{n+1}, \eta_{n+1}, \dot{\eta}_{n+1}, \lambda_{n+1}) = 0$

\[
\begin{align*}
    Dr/A\Delta H_{n+1} &= K_T A\Delta H_{n+1} = K_T T(n_{n+1}) A\Delta n_{n+1} \\
    Dr/\Delta \eta_{n+1} &= C_T \Delta \eta_{n+1} \\
    Dr/\Delta \dot{\eta}_{n+1} &= M_T \Delta \dot{\eta}_{n+1} \\
    Dr/\Delta \lambda_{n+1} &= A^T \phi_q^T \Delta \lambda_{n+1}
\end{align*}
\]
Time integration

Linearization of (6)

\[
D\phi/(A\Delta H_{n+1}) = \phi_q A\Delta H_{n+1} = \phi_q T(n_{n+1})A\Delta n_{n+1}
\]  

(18)
Time integration

Linearization of integration formulae (7)-(9)

\[
\Delta \eta_{n+1} = \gamma / (\beta h) \Delta n_{n+1} = \gamma' \Delta n_{n+1} \tag{19}
\]
\[
\Delta \dot{\eta}_{n+1} = (1 - \alpha_m) / (\beta h^2 (1 - \alpha_f)) \Delta n_{n+1} = \beta' \Delta n_{n+1} \tag{20}
\]
Time integration

(19) and (20) in (14)–(18)

Linear system to be solved at each iteration step

\[
\begin{bmatrix}
K_T T(n_{n+1}) A + C_T \gamma' + M_T \beta' \\
n_{n+1} \phi_q T(n_{n+1}) A \\
\phi_q T(n_{n+1}) A
\end{bmatrix}
\begin{bmatrix}
A^T \phi_q^T \\
0_{3 \times 1}
\end{bmatrix}
\begin{bmatrix}
\Delta n_{n+1} \\
\Delta \lambda_{n+1}
\end{bmatrix}
= - \begin{bmatrix}
r^* \\
\phi^*
\end{bmatrix}
\]

(21)

Solve (21), then compute the finite variations of velocities \( \Delta \eta_{n+1} \) and accelerations \( \Delta \dot{\eta}_{n+1} \) from \( \Delta n_{n+1} \) using (19) and (20)
Time integration: Summary

- Prediction phase: Implicit generalized $\alpha$ scheme (outer)
- Correction phase: Netwon-Raphson (inner)
Modeling of a multi-link rigid-soft manipulator