BEST CONSTANT AND EXTREMALS FOR A VECTOR POINCARÉ INEQUALITY WITH WEIGHTS

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Abstract. We provide the best constant C as well as all the extremals for the generalized Poincaré inequality

$$\int_0^T a|u|^p \leq C \int_0^T a|u'|^p$$

where $a \in L^\infty([0,T])$ satisfies $1 \leq a(t) \leq L$, $u \in W_0^{1,p}([0,T],\mathbb{R}^N)$, $N \geq 1$, $p > 1$ and $T > 0$.

1 Introduction In the seminal article [15], Piccinini and Spagnolo computed the best Hölder exponent for weak solutions to the elliptic equation in divergence form: $\operatorname{div}(A(x) \cdot \nabla v) = 0$ in Ω, where $A = (a_{ij})$, $i, j = 1, 2$ is a 2×2 positive definite matrix-valued function satisfying $\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^2$, $x \in \Omega$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain. More precisely, they proved that any solution $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is α-Hölder continuous with $\alpha \geq L^{-1/2}$, where $L = \Lambda/\lambda$ denotes the ellipticity constant of A. Furthermore, they showed that if A has the isotropic form $A = aI$ for some measurable function a satisfying $1 \leq a \leq L$, where I denotes the identity matrix, then the best Hölder exponent is improved, namely $\alpha \geq 4\pi^{-1} \arctan L^{-1/2}$. A key ingredient used in order to obtain the second sharp estimate is the best constant C in the following weighted Wirtinger-type inequality

$$\int_0^{2\pi} au^2 \leq C \int_0^{2\pi} a(u')^2, \tag{1}$$

where u satisfies $\int_0^{2\pi} au = 0$ and a is a measurable weight function satisfying $1 \leq a \leq L$. The extremals for (1) are also characterized.

Motivated by various problems in analysis and geometry, several extensions and variations of (1) have been obtained in recent years. In particular, for the homogeneous case $a \equiv 1$, inequalities with general powers of u and various integral constraints have been considered, in connection with the Wulff theorem in geometry, by Dacorogna, Gangbo and Subía [2], Croce and Dacorogna [1]. The vectorial case of (1) in the homogeneous case $a \equiv 1$ with general powers of u has been studied, among others, by Manásevich and Mawhin [13, 14], Del Pino [3], under various boundary conditions, in connection with $p-$Laplace equations. Variations of (1) with different weight functions were studied in [7, 16, 17], particularly in connection to quasiharmonic maps and Beltrami equations. See also Iwaniec and Sbordone [10], Hencl, Moscariello, Passarelli di Napoli and Sbordone [8] for further applications and developments in this direction. Extensions of (1) with general powers of u were derived in [5, 6].

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In consideration of these results, it is natural to investigate the following weighted vector inequality of Poincaré type

$$\int_0^T a|u|^p \leq C \int_0^T a|u'|^p, \tag{2}$$

where u belongs to the space $W_0^{1,p}([0,T],\mathbb{R}^N)$. Here and in what follows we set

$$W_0^{1,p}([0,T],\mathbb{R}^N) = \left\{u \in W^{1,p}([0,T],\mathbb{R}^N) : u(0) = 0 \text{ and } u(T) = 0\right\},$$

where $N \geq 1$, $T > 0$, $p > 1$. Furthermore, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$, $a \in L^\infty(0,T)$ satisfies $1 \leq a \leq L$. Our aim is to estimate the best constant C in (2), see Theorem 2.2 below for the precise statement. Let

$$\mathcal{A} = \{a \in L^\infty(0,T) : \inf a = 1 \text{ and } \sup a = L\},$$

and let

$$\frac{1}{C_p(a)} = \inf\left\{\frac{\int_0^T a(t)|u'|^p}{\int_0^T a(t)|u|^p} : u \in W_0^{1,p}([0,T],\mathbb{R}^N)\setminus\{0\}\right\}, \tag{3}$$

for every given function $a \in \mathcal{A}$. By standard arguments it follows that the infimum in (3) is achieved for some $u \in W_0^{1,p}([0,T],\mathbb{R}^N)\setminus\{0\}$. We prove that if

$$\frac{1}{C_p} = \inf_{a\in\mathcal{A}} \frac{1}{C_p(a)},$$

then the infimum is achieved for a unique piecewise constant function $\tilde{a} \in \mathcal{A}$. Moreover, we characterize the set of functions $\tilde{u} \in W_0^{1,p}([0,T],\mathbb{R}^N)$ for which (2) becomes an equality when $a = \tilde{a}$, in terms of generalized trigonometric functions, see Theorem 2.3 below.

We note that the Euler-Lagrange equation for (3) corresponds to the nonlinear eigenvalue problem

$$\begin{cases} (a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \\ u(0) = 0, \qquad u(T) = 0, \end{cases}$$

where $\lambda = C_p(a)^{-1}$ and $\psi_p : \mathbb{R}^N \to \mathbb{R}^N$ is the continuous function defined by

$$\psi_p(x) = \begin{cases} |x|^{p-2}x & \text{if } x \in \mathbb{R}^N\setminus\{0\}, \\ 0 & \text{if } x = 0. \end{cases} \tag{4}$$

We denote by ϕ_p the function ψ_p when $N = 1$. By homogeneity, any solution w to the scalar problem:

$$\begin{cases} (a(t)\phi_p(w'))' + \lambda a(t)\phi_p(w) = 0, \\ w(0) = 0, \qquad w(T) = 0, \end{cases}$$

yields a "one-dimensional" solution to (1) by setting $u = wd$, for any $d \in \mathbb{R}^N$. Using a uniqueness result of García-Huidobro, Manásevich and Ôtani [4], we begin by showing that, if a is smooth, then all solutions to (1) are one-dimensional. Hence, the technique of Piccinini and Spagnolo [15], as extended in [6], may be applied to obtain the sharp estimate for the best constant C. The optimal piecewise constant form $\tilde{a}$ of a also follows by such arguments.

Our second task is to characterize all extremals. We recall that in the scalar case with $a \equiv 1$ the extremals may be written in terms of generalized trigonometric functions, see [9, 11, 12]. By suitably gluing generalized trigonometric functions, we contruct an extremal for (2) with $a = \tilde{a}$, thus showing that our estimate is sharp. However, it is not a priori clear from our previous arguments whether or not such an extremal is unique, since $\tilde{a}$ is piecewise constant and therefore the above mentioned uniqueness argument does not apply. Nevertheless, we are able to show that all extremals for (3) with $a = \tilde{a}$ are indeed one-dimensional. We note that such a characterization of extremals is new even for the scalar case of (2). The weighted vectorial analog of the Piccinini-Spagnolo inequality (1) is new even for $p = 2$.

The remaining part of this paper is organized as follows. In Section 2 we clarify notation and we recall the basic definitions and properties of generalized trigonometric functions. With such notation at hand, we state our main results. In Section 3 we use the afore mentioned uniqueness result from [4] in order to reduce our problem to the one-dimensional case when a is smooth. Finally, in Section 4 we complete the proofs of our main results.

2 Notation, generalized trigonometric functions and statement of the main results For every $p > 1$, we denote by p^* the conjugate exponent of p, i.e. $p^* = p/(p-1)$. It is readily seen that the function ψ_p defined by (4) is a continuous function and it has an inverse given by ψ_{p^*}. We will denote by ϕ_p the function ψ_p when $N = 1$.

For later use, we now briefly define the generalized trigonometric functions and outline their main properties. See, e.g., [9], [11], [12], [14], for more details. Let $p > 1$. The function $\arcsin_p : [0,1] \to \mathbb{R}$ is defined by

$$\arcsin_p(\sigma) = \int_0^\sigma \frac{dy}{(1-y^p)^{1/p}}.$$

We set

$$\frac{\pi_p}{2} = \arcsin_p(1) = \frac{1}{p} B\left(\frac{1}{p}, \frac{1}{p^*}\right),$$

where $B(\cdot,\cdot)$ denotes the Beta function defined by

$$B(h,k) = \int_0^1 t^{h-1}(1-t)^{k-1} dt = B(k,h),$$

for every $h, k > 1$. The function $\arcsin_p : [0,1] \to [0, \frac{\pi_p}{2}]$ is strictly increasing and its inverse function is denoted by $\sin_p$. The function $\sin_p$ is extended as an odd function to the interval $[-\pi_p, \pi_p]$ by setting $\sin_p(t) = \sin_p(\pi_p - t)$ in $[\pi_p/2, \pi_p]$, $\sin_p(t) = -\sin_p(-t)$ in $[-\pi_p, 0]$, and to the whole real axis as a $2\pi_p$−periodic function. Furthermore, it holds that $\sin_p(\pi_p + t) = -\sin_p(t)$. The function $\sin_p$ is the unique global solution of the initial value problem

$$\begin{cases} (\phi_p(w'))' + \frac{p}{p^*}\phi_p(w) = 0, \\ w(0) = 0, \qquad w'(0) = 1. \end{cases} \tag{5}$$

The function $\cos_p$ is defined by

$$\cos_p(t) = \phi_p(\sin_p'(t)). \tag{6}$$

It is $2\pi_p$-periodic and satisfies:

$$\begin{aligned}\cos_p(-t) &= \cos_p(t),\\ \cos_p(\pi_p - t) &= -\cos_p(t),\\ \cos_p(\pi_p + t) &= -\cos_p(t).\end{aligned}$$

The following identity holds, which generalizes the fundamental identity for trigonometric functions:

$$|\cos_p(t)|^{p^*} + |\sin_p(t)|^p \equiv 1. \tag{7}$$

For later purposes, we also note the following identity:

$$\cos_p\left(\frac{\pi_p}{2} - t\right) = \sin_{p^*}\left(\frac{p}{p^*}t\right). \tag{8}$$

¿From (5) we derive

$$\cos_p'(t) = -\frac{p}{p^*}\phi_p(\sin_p(t)). \tag{9}$$

On the other hand, from (6) we have:

$$\sin_p'(t) = \phi_{p^*}(\cos_p(t)). \tag{10}$$

Finally, we define $\tan_p$ as follows:

$$\tan_p(t) = \frac{\sin_p(t)}{\phi_{p^*}(\cos_p(t))}.$$

The function $\tan_p$ is π_p-periodic, with singularities at the zeros of $\cos_p$. The inverse of $\tan_p$ restricted to the interval $[-\pi_p/2, \pi_p/2]$, denoted by $\arctan_p$, is given by

$$\arctan_p(\sigma) = \int_0^\sigma \frac{dy}{1 + |y|^p},$$

for every $\sigma \in \mathbb{R}$. It results that

$$\lim_{\sigma \to +\infty} \arctan_p(\sigma) = \frac{\pi_p}{2}. \tag{11}$$

The next lemma generalizes to the case $p \neq 2$ a well known identity.

Lemma 2.1. *For every $p > 1$ and for every $\sigma > 0$ the following identity holds*

$$\arctan_p(\sigma^{-p^*/p}) + \frac{p^*}{p}\arctan_{p^*}(\sigma) = \frac{\pi_p}{2}. \tag{12}$$

Proof. In view of (11) we have

$$\frac{\pi_p}{2} = \int_0^{+\infty} \frac{dy}{1 + y^p} = \arctan_p(\sigma^{-p^*/p}) + \int_{\sigma^{-p^*/p}}^{+\infty} \frac{dy}{1 + y^p}.$$

Performing the change of variables $y = z^{-p^*/p}$ we obtain

$$\int_{\sigma^{-p^*/p}}^{+\infty} \frac{dy}{1 + y^p} = \frac{p^*}{p}\int_0^\sigma \frac{dz}{1 + z^{p^*}}.$$

Hence, the asserted identity follows. □

Now, we can state our main results. Our first result provides an estimate for the best constant C in inequality (2). It is convenient to define:

$$\beta(L) = \left[\frac{L^{p^*/p}(L-1)}{L^{p^*/p}-1} \right]^{1/p^*} . \tag{13}$$

With this notation, we have:

Theorem 2.2. *Let $N \geq 1$, $p > 1$ and $T > 0$. Let $a : [0,T] \to \mathbb{R}$ be a measurable function such that $1 \leq a(t) \leq L$. Then, the following inequality holds:*

$$\int_0^T a(t)|u(t)|^p dt \leq C_p \int_0^T a(t)|u'(t)|^p dt \tag{14}$$

for every $u \in W_0^{1,p}([0,T], \mathbb{R}^N)$, where

$$C_p = \left(\frac{T}{2}\right)^p \frac{(p/p^*)^{p/p^*}}{\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right]^p}.$$

We note that in view of identity (12) we may write:

$$\begin{aligned}\frac{\pi_{p^*}}{2} &- \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L} \\ &= \frac{p}{p^*} \arctan_p \left[L^{-p^*/p^2} \left(\frac{L^{p^*/p}-1}{L-1} \right)^{1/p} \right] + \arctan_{p^*} \left[L^{-1/p^*} \left(\frac{L-1}{L^{p^*/p}-1} \right)^{1/p^*} \right].\end{aligned}$$

Therefore, in the special case $p = 2$ and $T = \pi$, the best constant C_p takes the value

$$C_2 = \left(\frac{\pi}{4 \arctan L^{-1/2}} \right)^2,$$

in agreement with Piccinini and Spagnolo's result [15].

Our next result shows that Theorem 2.2 is sharp, and characterizes all extremals.

Theorem 2.3. *Inequality (14) reduces to an equality if and only if $a = \tilde{a}$, where $\tilde{a}$ is defined by*

$$\tilde{a}(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < \tilde{\tau}, \ \ T - \tilde{\tau} \leq t \leq T, \\ L & \text{for} \quad \tilde{\tau} \leq t < T - \tilde{\tau}, \end{cases}$$

with

$$\tilde{\tau} = \frac{T}{2} \left(1 - \frac{\arctan_{p^*} \frac{\beta(L)}{L}}{\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}} \right), \tag{15}$$

and $u = \tilde{u} = \tilde{w}d$ for some $d \in \mathbb{R}^N$, where $\tilde{w}$ is the scalar function defined by

$$\tilde{w}(t) = \begin{cases} \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} \sin_p \left[\left(\frac{\tilde{\lambda} p^*}{p}\right)^{1/p} t \right] & \text{for } 0 \leq t \leq \tilde{\tau}, \\ \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} L^{-1/p} \cos_{p^*} \left[\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p} \left(t - \frac{T}{2}\right) \right] & \text{for } \tilde{\tau} \leq t \leq T - \tilde{\tau}, \\ \left(\frac{\tilde{\lambda} p^*}{p}\right)^{-1/p} \sin_p \left[\left(\frac{\tilde{\lambda} p^*}{p}\right)^{1/p} (T-t) \right] & \text{for } T - \tilde{\tau} \leq t \leq T. \end{cases} \tag{16}$$

with

$$\tilde{\lambda} = C_p^{-1} = \left(\frac{2}{T}\right)^p \left(\frac{p^*}{p}\right)^{p/p^*} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L} \right]^p.$$

3 Reduction to the one-dimensional case In this section we consider the nonlinear eigenvalue problem:

$$\begin{cases} (a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \\ u(0) = 0, \qquad u(T) = 0 \end{cases} \tag{17}$$

corresponding to the Euler-Lagrange equation for (3). Our aim is to show that if a is smooth, then solutions to (17) are necessarily one-dimensional, see Proposition 3.2 below. We shall need the following uniqueness result.

Proposition 3.1 ([4]). *Suppose that $\beta \in L^1_{\mathrm{loc}}(\mathbb{R})$ with $\beta > 0$ a.e. Then, for any $\xi, \eta \in \mathbb{R}^N$ and $s_0 \in \mathbb{R}$, the problem*

$$\begin{cases} (\psi_p(v'))' + \beta(s)\psi_p(v) = 0, \\ v(s_0) = \xi, \qquad v'(s_0) = \eta \end{cases} \tag{18}$$

has a unique C^1 solution globally defined on $\mathbb{R}$.

The existence of a local solution is a direct application of Schauder's fixed point theorem. The main idea to prove the uniqueness is to write the equation in (18) in the equivalent form

$$v'(s) = \psi_{p^*}\left[\psi_p(\eta) - \int_{s_0}^{s} \beta(\theta)\psi_p(v(\theta))d\theta\right].$$

Then, a careful use of the properties of β allows to overcome the possible lack of Lipschitz continuity of the function ψ_p.

Proposition 3.2. *Let $a : [0,T] \to \mathbb{R}$ be a smooth function such that $1 \le a(t) \le L$ for any $t \in [0,T]$. If $u \in W_0^{1,p}([0,T],\mathbb{R}^N)$ is a weak solution of the vector eigenvalue problem*

$$(a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \tag{19}$$

then $u \in C^1$ and it follows that

$$u(t) = w(t)d, \tag{20}$$

where $d = u'(0)$ and w is a solution of the scalar eigenvalue problem

$$\begin{cases} (a(t)\phi_p(w'))' + \lambda a(t)\phi_p(w) = 0, \\ w(0) = 0, \qquad w(T) = 0 \end{cases} \tag{21}$$

satisfying $w'(0) = 1$.

Proof. We first prove that if u is a solution of (19) then $u \in C^1$. By continuity of a, ψ_p, u and using equation (19), we have that $(a(t)\psi_p(u'))'$ is continuous. Therefore, $h(t) = a(t)\psi_p(u')$ belongs to $C^1([0,T],\mathbb{R}^N)$ and $\psi_p(u') = a(t)^{-1}h(t)$ is continuous. Now the claim follows by continuity of $\psi_{p^*} = \psi_p^{-1}$.
By a change of variables, we first reduce the equation in (19) to an equation of the form (18). Let us first consider the function $G : [0,T] \to [0,T]$ defined by

$$G(t) = \frac{T}{\int_0^T a^{-\frac{1}{p-1}}} \int_0^t a^{-\frac{1}{p-1}}.$$

Since $1 \leq a(t) \leq L$ the function G is well defined. It is easily seen that G is a nondecreasing differentiable function whose derivative is given by

$$G'(t) = \frac{T}{\int_0^T a^{-\frac{1}{p-1}}} a(t)^{-\frac{1}{p-1}}.$$

Now, suppose that u is a solution of (19) with $u(0) = 0$ and $u'(0) = d$; we claim that the function $v : [0, T] \to \mathbb{R}^N$ defined by

$$v(s) = u(G^{-1}(s)),$$

is a C^1 solution of the initial value problem

$$\begin{cases} (\psi_p(v'))' + \mu\alpha(s)\psi_p(v) = 0, \\ \\ v(0) = 0, \qquad v'(0) = \gamma a(0)^{\frac{1}{p-1}} d, \end{cases} \tag{22}$$

where

$$\alpha(s) = a(G^{-1}(s))^{p^*}, \qquad \mu = \gamma^p \lambda \qquad \gamma = \frac{1}{T}\int_0^T a^{-\frac{1}{p-1}}. \tag{23}$$

Indeed, it results that $u(t) = v(G(t))$ and consequently the derivative of u is given by

$$\frac{du}{dt}(t) = \gamma^{-1} a(t)^{-\frac{1}{p-1}} \frac{dv}{ds}(G(t)). \tag{24}$$

From (24) it follows that

$$\frac{d}{dt}\Big[a(t)\psi_p(u'(t))\Big] = \gamma^{-p} a(t)^{-\frac{1}{p-1}} \left[\frac{d}{ds}\psi_p(v'(s))\right]_{s=G(t)},$$

and therefore we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[a(t)\psi_p(u'(t))\Big] + \lambda a(t)\psi_p(u(t)) = \\ &\qquad = \gamma^{-p} a(t)^{-\frac{1}{p-1}} \left[\frac{d}{ds}\psi_p(v'(s)) + \mu\alpha(s)\psi_p(v(s))\right]_{s=G(t)}, \end{aligned}$$

with α, γ and μ given by (23). On the other hand, the function $s \in [0, T] \mapsto \gamma a(0)^{\frac{1}{p-1}} g(s) d \in \mathbb{R}^N$, where g is the unique solution of the scalar initial value problem (see again Proposition 3.1 for $N = 1$)

$$\begin{cases} (\phi_p(g'))' + \mu\alpha(s)\phi_p(g) = 0, \\ g(0) = 0, \qquad g'(0) = 1, \end{cases}$$

is a solution of the problem (22). Therefore, $v(s) = \gamma a(0)^{\frac{1}{p-1}} g(s) d$. Consequently, the vector initial value problem

$$\begin{cases} (a(t)\psi_p(u'))' + \lambda a(t)\psi_p(u) = 0, \\ u(0) = 0, \qquad u'(0) = d. \end{cases}$$

has a unique C^1 solution given by $u(t) = v(G(t)) = w(t)d$ where $w(t) = \gamma a(0)^{\frac{1}{p-1}} g(G(t))$. Moreover, w is the unique C^1 solution of the scalar initial value problem

$$\begin{cases} (a(t)\phi_p(w'))' + \lambda a(t)\phi_p(w) = 0, \\ w(0) = 0, \qquad w'(0) = 1. \end{cases}$$

Since u in (20) also satisfies $u(T) = 0$ it must be that $w(T) = 0$; thus w is a solution to the scalar eigenvalue problem (21) and this completes the proof. □

Remark 3.1. Proposition 3.2 shows that the problems (19) and (21) share the same eigenvalues; moreover, it is possible to prove that they form a sequence $\lambda_n = \lambda_n(a)$ such that $0 < \lambda_1(a) < \lambda_2(a) < \cdots < \lambda_n(a) < \cdots$. Indeed, we recall that (see Section 3 in [4] when $N \geq 1$ and Section 2 in [18] when $N = 1$) for any $\alpha \in L^1(0,T)$ with $\alpha > 0$ a.e. and for any $\mu > 0$, a problem of the type

$$\begin{cases} (\psi_p(v'))' + \mu\alpha(s)\psi_p(v) = 0, \\ v(0) = 0, \qquad v(T) = 0, \end{cases} \tag{25}$$

has a strictly monotone sequence of eigenvalues. On the other hand, the proof of Proposition 3.2 implies that λ is an eigenvalue of (19) if and only if $\mu = \gamma^p\lambda$ is an eigenvalue of (25) with α and γ as in (23). This proves the asserted property.

4 Proofs of Theorem 2.2 and Theorem 2.3

Proof of Theorem 2.2. By a standard approximation argument it is sufficient to prove Theorem 2.2 in the special case where $a \in \mathcal{A}$ is a smooth function. It is well known that $C_p^{-1}(a) = \lambda_1(a)$, hence the following estimate holds

$$\int_0^T a(t)|u(t)|^p dt \leq \frac{1}{\lambda_1(a)} \int_0^T a(t)|u'(t)|^p dt,$$

for every u. Therefore, in order to prove (14) it is sufficient to show that, if $\lambda \neq 0$ and $u \not\equiv 0$ satisfy (19), then necessarily

$$\lambda \geq \left(\frac{2}{T}\right)^p \left(\frac{p^*}{p}\right)^{p/p^*} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right]^p.$$

In view of Proposition 3.2 there exists a vector $d \in \mathbb{R}^N$ such that $u(t) = w(t)d$ where w is a solution of the scalar problem (21). Now we apply the arguments of Piccinini and Spagnolo [15], as extended by one of the authors [6], to problem (21). By standard properties of eigenfunctions any solution w of (21) in $[0,T]$ has at least two zeros, and between any pair of zeros of w there is exactly one zero of its derivative w'. Let t_0 and t_2 be two consecutive zeros of w and let t_1 be a zero of w' in such a way that $t_0 < t_1 < t_2$. Without loss of generality we may suppose that $w(t_1) > 0$. It is obvious that

$$t_2 - t_0 \leq T. \tag{26}$$

We define, for $t_0 < t \leq t_1$, the function

$$f(t) = \frac{a(t)\phi_p(w'(t))}{\phi_p(w(t))}.$$

In view of (21) it results that f satisfies the following first order differential equation

$$f'(t) = -\lambda a(t) - \frac{p}{p^*}\frac{|f(t)|^{p^*}}{a(t)^{p^*/p}}.$$

We remark that f is strictly decreasing, since $f'(t) < 0$. Furthermore $\lim_{t\to t_0^+} f(t) = +\infty$, $f(t_1) = 0$. Hence, there is exactly one point, say τ, in the interval (t_0, t_1) such that $f(\tau) = (\lambda p^*/p)^{1/p^*}\beta(L)$, where $\beta(L)$ is defined in (13). Now we prove that the following inequalities hold:

$$\begin{cases} -\lambda a(t) - \frac{p}{p^*}\frac{|f|^{p^*}}{a^{p^*/p}} \geq -\lambda - \frac{p}{p^*}|f|^{p^*} & \text{for} \quad t_0 < t \leq \tau \\ -\lambda a(t) - \frac{p}{p^*}\frac{|f|^{p^*}}{a^{p^*/p}} \geq -\lambda L - \frac{p}{p^*}\frac{|f|^{p^*}}{L^{p^*/p}} & \text{for} \quad \tau \leq t \leq t_1. \end{cases} \tag{27}$$

Indeed, it is readily checked that the first inequality in (27) is equivalent to

$$f(t)^{p^*} \geq \frac{\lambda p^*}{p}\beta^{p^*}(a(t)) \qquad \text{for} \quad t_0 < t \leq \tau$$

where the function β is defined in (13). Since f is decreasing and β is increasing in $(1, L)$, for $t \leq \tau$ we obtain

$$f(t)^{p^*} \geq f(\tau)^{p^*} = \frac{\lambda p^*}{p}\beta^{p^*}(L) \geq \frac{\lambda p^*}{p}\beta^{p^*}(a(t)).$$

Hence, the first inequality in (27) is established. On the other hand, the second inequality in (27) is equivalent to

$$f(t)^{p^*} \leq \frac{\lambda p^*}{p}L^{p^*/p}\gamma(a(t)) \qquad \text{for} \quad \tau \leq t \leq t_1$$

where γ is the function defined for $1 \leq a \leq L$ by

$$\gamma(a) = \frac{a^{p^*/p}(L-a)}{L^{p^*/p} - a^{p^*/p}}.$$

Since f is decreasing and γ is increasing, we have for $t \geq \tau$:

$$f(t)^{p^*} \leq f(\tau)^{p^*} = \frac{\lambda p^*}{p}\beta^{p^*}(L) = \frac{\lambda p^*}{p}L^{p^*/p}\gamma(1) \leq \frac{\lambda p^*}{p}L^{p^*/p}\gamma(a(t)).$$

Hence, the second inequality in (27) is also established.

Now, we prove that the Cauchy problem

$$\begin{cases} f_0'(t) \;=\; \begin{cases} -\lambda - \frac{p}{p^*}|f_0|^{p^*} & \text{for} \quad t_0 < t \leq \tau \\ -\lambda L - \frac{p}{p^*}\frac{|f_0|^{p^*}}{L^{p^*/p}} & \text{for} \quad \tau \leq t < t_1 \end{cases} \\ f_0(\tau) \;=\; (\lambda p^*/p)^{1/p^*}\beta(L) \end{cases} \tag{28}$$

has a unique solution. Indeed, note that f_0 is strictly decreasing. Denoting by g_0 its inverse, it results that

$$
\text{(29)} \qquad \begin{cases} g_0'(s) & = \begin{cases} -\left(\lambda L + \frac{p}{p^*} L^{-p^*/p} s^{p^*}\right)^{-1} & \text{for} \quad f_0(t_1) < s \le f_0(\tau) \\ -\left(\lambda + \frac{p}{p^*} s^{p^*}\right)^{-1} & \text{for} \quad f_0(\tau) \le s < f_0(t_0) \end{cases} \\ g_0(f_0(\tau)) & = \tau. \end{cases}
$$

Hence, there exists a unique solution for (29). It follows that uniqueness holds for (28) and that f_0 is given by:

$$
\text{(30)} \qquad f_0(t) = \begin{cases} (\frac{\lambda p^*}{p})^{1/p^*} \tan_{p^*} \left[\lambda^{1/p} \left(\frac{p}{p^*}\right)^{1/p^*} (\tau - t) + \arctan_{p^*} \beta(L)\right] \\ \qquad\qquad\qquad\qquad \text{for} \quad t_0 < t \le \tau \\ L(\frac{\lambda p^*}{p})^{1/p^*} \tan_{p^*} \left[\lambda^{1/p} \left(\frac{p}{p^*}\right)^{1/p^*} (\tau - t) + \arctan_{p^*} \frac{\beta(L)}{L}\right] \\ \qquad\qquad\qquad\qquad \text{for} \quad \tau \le t \le t_1. \end{cases}
$$

In particular, we obtain

$$
\text{(31)} \qquad \begin{cases} f_0(t) \ge f(t) & \text{for} \quad t \le \tau \\ f_0(t) \le f(t) & \text{for} \quad t \ge \tau. \end{cases}
$$

Since

$$
\lim_{t \to \frac{\pi_{p^*}}{2}} \tan_{p^*}(t) = +\infty
$$

we have that

$$
f_0(t) \to +\infty \text{ as } t \to \tau - \frac{1}{\lambda^{1/p} \left(\frac{p}{p^*}\right)^{1/p^*}} \left(\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L)\right)
$$

and vanishes for $t = \tau + \frac{1}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}} \arctan_{p^*} \frac{\beta(L)}{L}$. It follows:

$$
t_1 - t_0 \ge \frac{1}{\lambda^{1/p} \left(\frac{p}{p^*}\right)^{1/p^*}} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right].
$$

In a similar way we can prove that

$$
t_2 - t_1 \ge \frac{1}{\lambda^{1/p} \left(\frac{p}{p^*}\right)^{1/p^*}} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right];
$$

hence by the relations above we derive

$$
t_2 - t_0 \ge \frac{2}{\lambda^{1/p} \left(\frac{p}{p^*}\right)^{1/p^*}} \left[\frac{\pi_{p^*}}{2} - \arctan_{p^*} \beta(L) + \arctan_{p^*} \frac{\beta(L)}{L}\right].
$$

So recalling (26), we can state

$$T \geq \frac{2}{\lambda^{1/p}\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right],$$

that is

$$\lambda \geq \left\{\frac{2}{T\left(\frac{p}{p^*}\right)^{1/p^*}}\left[\frac{\pi_{p^*}}{2} - \arctan_{p^*}\beta(L) + \arctan_{p^*}\frac{\beta(L)}{L}\right]\right\}^p. \tag{32}$$

The proof of Theorem 2.2 is complete. □

In order to characterize the extremals as in Theorem 2.3 we shall need the following.

Lemma 4.1. *Let $u \in W_0^{1,p}([0,T],\mathbb{R}^N)$ be a weak solution of the equation*

$$(\tilde{a}(t)\psi_p(u'))' + \tilde{\lambda}\tilde{a}(t)\psi_p(u) = 0. \tag{33}$$

with $\tilde{\lambda}$ and $\tilde{a}$ as in Theorem 2.3. Let

$$\lim_{t\to\tilde{\tau}^-} u'(t) = u'(\tilde{\tau}^-), \qquad \lim_{t\to\tilde{\tau}^+} u'(t) = u'(\tilde{\tau}^+).$$

Then

$$u'(\tilde{\tau}^-) = L^{p^*-1}u'(\tilde{\tau}^+). \tag{34}$$

Proof. Since $\tilde{a}(t) \equiv 1$ in $[0,\tilde{\tau}]$ and $\tilde{a}(t) \equiv L$ in $[\tilde{\tau}, T/2]$, from (33) we conclude that the restrictions of u respectively to the intervals $[0,\tilde{\tau}]$ and $[\tilde{\tau}, T/2]$ are both C^1 functions. Now, we prove that $u'(\tilde{\tau}^+)$ is completely determined by $u'(\tilde{\tau}^-)$. Since u is a weak solution of (33) we have, for any function $\varphi \in W^{1,p}([0,T/2],\mathbb{R}^N)$

$$-\int_0^{T/2}\tilde{a}(t)\langle\psi_p(u');\varphi'\rangle = \tilde{\lambda}\int_0^{T/2}\tilde{a}(t)\langle\psi_p(u);\varphi\rangle, \tag{35}$$

Let $1 \leq j \leq N$ and $\varepsilon > 0$. In (35) we first choose vector valued piecewise linear test function $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_N(t))$ defined by

$$\varphi_k(t) = 0 \qquad \text{if } k \neq j, \quad \varphi_j(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \tilde{\tau} - \varepsilon, \\ \frac{1}{\varepsilon}(t - \tilde{\tau} + \varepsilon) & \text{if } \tilde{\tau} - \varepsilon \leq t \leq \tilde{\tau}, \\ 1, & \text{if } \tilde{\tau} \leq t \leq T/2. \end{cases}$$

The derivative of φ_j is given by

$$\varphi_j'(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } \tilde{\tau} - \varepsilon \leq t \leq \tilde{\tau}, \\ 0, & \text{if } 0 \leq t \leq \tilde{\tau} - \varepsilon,\ \ \tilde{\tau} \leq t \leq T/2. \end{cases}$$

Let us set for every $1 \leq j \leq N$

$$\psi_{p,j}(x) = \begin{cases} |x|^{p-2}x_j & \text{if } x \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Hence,

$$\int_0^{T/2} \tilde{a}(t)\langle\psi_p(u');\varphi'\rangle = \frac{1}{\varepsilon}\int_{\tilde{\tau}-\varepsilon}^{\tilde{\tau}} \psi_{p,j}(u'), \tag{36}$$

and in a similar way

$$\int_0^{T/2} \tilde{a}(t)\langle\psi_p(u);\varphi\rangle = \frac{1}{\varepsilon}\int_{\tilde{\tau}-\varepsilon}^{\tilde{\tau}} (t-\tilde{\tau}+\varepsilon)\psi_{p,j}(u), +L\int_{\tilde{\tau}}^{T/2} \psi_{p,j}(u). \tag{37}$$

By substituting (36) and (37) in (35) and letting $\varepsilon \to 0^+$ we obtain

$$-|u'(\tilde{\tau}^-)|^{p-2}u_j'(\tilde{\tau}^-) = \tilde{\lambda}L\int_{\tilde{\tau}}^{T/2} \psi_{p,j}(u), dt, \tag{38}$$

A second choice of φ, namely

$$\varphi_k(t) = 0 \qquad \text{if } k \neq j, \quad \varphi_j(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tilde{\tau}, \\ \frac{1}{\varepsilon}(t-\tilde{\tau}) & \text{if } \tilde{\tau} \le t \le \tilde{\tau}+\varepsilon, \\ 1, & \text{if } \tilde{\tau}+\varepsilon \le t \le T/2 \end{cases}$$

and an argument similar to the one that yields (38) leads to

$$-|u'(\tilde{\tau}^+)|^{p-2}u_j'(\tilde{\tau}^+) = \tilde{\lambda}\int_{\tilde{\tau}}^{T/2} \psi_{p,j}(u), dt. \tag{39}$$

Thus, from (38) and (39) we have, for every $1 \le j \le N$

$$-L|u'(\tilde{\tau}^+)|^{p-2}u_j'(\tilde{\tau}^+) = -|u'(\tilde{\tau}^-)|^{p-2}u_j'(\tilde{\tau}^-),$$

and therefore

$$L\psi_p(u'(\tilde{\tau}^+)) = \psi_p(u'(\tilde{\tau}^-)).$$

From the above and from the fact that $\psi_p^{-1} = \psi_{p^*}$ we obtain (34). □

Proof of Theorem 2.3. The inequalities (26), (27), (31), (32) in the proof of Theorem 2.2 hold strictly unless $t_0 = 0$, $t_2 = T$, $f(t) = f_0(t)$ and $a(t) = \tilde{a}(t)$. In this case the function f_0 satisfies

$$\lim_{t\to 0^+} f_0(t) = +\infty. \tag{40}$$

Since $\tan_{p^*}(\theta) \to +\infty$ as $\theta \to (\pi_{p^*}/2)^-$, in view of (30) there is a unique value of τ, denoted by $\tilde{\tau}$, such that (40) holds. Thus $\tilde{\tau}$ satisfies

$$\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p}\tilde{\tau} + \arctan_{p^*}\beta(L) = \frac{\pi_{p^*}}{2}, \tag{41}$$

and this yields (15). By requiring that $f_0(t_1) = 0$ we obtain

$$\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p}(\tilde{\tau}-t_1) + \arctan_{p^*}\frac{\beta(L)}{L} = 0, \tag{42}$$

and this implies $t_1 = T/2$. It remains to prove that all extremals of inequality (14) with $a = \tilde{a}$ are of the form $u = \tilde{u} = \tilde{w}\, d$, where $\tilde{w}$ is defined by (16). Hence, we seek all non-trivial solutions of the equation

$$(\tilde{a}(t)\psi_p(u'))' + \tilde{\lambda}\tilde{a}(t)\psi_p(u) = 0, \tag{43}$$

such that $u(0) = 0$ and $u(T) = 0$. Since $\tilde{a}(t) \equiv 1$ in $[0, \tilde{\tau}]$, in view of Proposition 3.1 (see also Lemma 3.1 in [13]) we have that, for any given $d \in \mathbb{R}^N$, there exists a unique solution $\tilde{u}$ defined in the interval $[0, \tilde{\tau}]$ of equation (43) satisfying the initial conditions

$$u(0) = 0, \qquad u'(0) = d.$$

Recalling the definition of $\sin_p$, we may write $\tilde{u}$ in the form

$$\tilde{u}(t) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p} \sin_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p} t\right] d \qquad \forall t \in [0, \tilde{\tau}].$$

Observe that

$$\tilde{u}(\tilde{\tau}^-) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p} \sin_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p} \tilde{\tau}\right] d. \tag{44}$$

In order to simplify the above expression for $\tilde{u}(\tilde{\tau}^-)$ we note that, using identity (8), we may write

$$\begin{aligned}\sin_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p} \tilde{\tau}\right] &= \sin_p\left[\frac{p^*}{p}\left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p}\tilde{\tau}\right] \\ &= \cos_{p^*}\left(\frac{\pi_{p^*}}{2} - \left(\frac{p}{p^*}\right)^{1/p^*} \tilde{\lambda}^{1/p}\tilde{\tau}\right) = \cos_{p^*}(\arctan_{p^*} \beta(L))\end{aligned}$$

where we used (41) in order to derive the last equality. In turn, from identity (7) we derive

$$|\cos_{p^*}(t)|^p = \frac{1}{1 + |\tan_{p^*}(t)|^{p^*}}$$

and therefore we may write

$$\cos_{p^*}(\arctan_{p^*} \beta(L)) = \left(\frac{1}{1 + \beta^{p^*}(L)}\right)^{1/p} = \left[\frac{L^{p^*/p} - 1}{L^{p^*} - 1}\right]^{1/p}.$$

We conclude from (44) and the arguments above that

$$\tilde{u}(\tilde{\tau}^-) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p} \left[\frac{L^{p^*/p} - 1}{L^{p^*} - 1}\right]^{1/p} d.$$

We still denote by $\tilde{u}$ the restriction of the solution of equation (43) to the interval $[\tilde{\tau}, T - \tilde{\tau}]$. By continuity of $\tilde{u}$,

$$\tilde{u}(\tilde{\tau}^+) = \tilde{u}(\tilde{\tau}^-) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p} \left[\frac{L^{p^*/p} - 1}{L^{p^*} - 1}\right]^{1/p} d. \tag{45}$$

Now we compute derivatives. Using (10), we have

$$\tilde{u}'(\tilde{\tau}^-) = \phi_{p^*}\left(\cos_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p}\tilde{\tau}\right]\right). \tag{46}$$

On the other hand, similarly as before, using (8) and (41) we compute:

$$\begin{aligned}\cos_p\left[\left(\frac{\tilde{\lambda}p^*}{p}\right)^{1/p}\tilde{\tau}\right] &= \cos_p\left[\frac{p^*}{p}\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\tilde{\tau}\right]\\ &= \sin_{p^*}\left(\frac{\pi_{p^*}}{2}-\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\tilde{\tau}\right) = \sin_{p^*}(\arctan_{p^*}\beta(L)).\end{aligned}$$

¿From the basic identity (7) we derive

$$|\sin_p(t)|^p = \frac{|\tan_p(t)|^p}{1+|\tan_p(t)|^p}$$

and consequently

$$\sin_{p^*}(\arctan_{p^*}\beta(L)) = \left[\frac{L^{p^*}-L^{p^*/p}}{L^{p^*-1}}\right]^{1/p^*}.$$

We conclude from (46) and the arguments above that

$$\tilde{u}'(\tilde{\tau}^-) = \left[\frac{L^{p^*}-L^{p^*/p}}{L^{p^*}-1}\right]^{1/p} d.$$

Now, in view of Lemma 4.1 we have

$$\tilde{u}'(\tilde{\tau}^+) = L^{-p^*/p}\left[\frac{L^{p^*}-L^{p^*/p}}{L^{p^*}-1}\right]^{1/p} d = \left[\frac{L-1}{L(L^{p^*}-1)}\right]^{1/p} d. \tag{47}$$

Since $\tilde{a}(t)\equiv L$ in $[\tilde{\tau}, T-\tilde{\tau}]$, again by Proposition 3.1, $\tilde{u}$ coincides in $[\tilde{\tau}, T-\tilde{\tau}]$ with the unique solution of (43) satisfying the initial conditions

$$u(\tilde{\tau}) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p}\left[\frac{L^{p^*/p}-1}{L^{p^*}-1}\right]^{1/p} d, \tag{48}$$

$$u'(\tilde{\tau}) = \left[\frac{L-1}{L(L^{p^*}-1)}\right]^{1/p} d. \tag{49}$$

according to (45) and (47). We claim that

$$\tilde{u}(t) = \left(\frac{\tilde{\lambda}p^*}{p}\right)^{-1/p}\frac{1}{L^{1/p}}\cos_{p^*}\left[\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\left(t-\frac{T}{2}\right)\right] d \quad \forall t\in[\tilde{\tau}, T-\tilde{\tau}].$$

Indeed, using (42) it follows that $\tilde{u}$ satisfies (48). Moreover, recalling that (see (9)) $p\cos_{p^*}'(t) = -p^*\phi_{p^*}(\sin_{p^*}(t))$ we have

$$\tilde{u}'(t) = -\frac{1}{L^{1/p}}\phi_{p^*}\left\{\sin_{p^*}\left[\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\left(t-\frac{T}{2}\right)\right]\right\} d. \tag{50}$$

By similar arguments as above, we compute

$$\sin_{p^*}(\arctan_{p^*}\frac{\beta(L)}{L}) = \left(\frac{L-1}{L^{p^*}-1}\right)^{1/p^*}.$$

Hence, $\tilde{u}$ satisfies (49). From (50) we have

$$\phi_p\left(\tilde{u}'(t)\right) = -\frac{1}{L^{1/p^*}}\sin_{p^*}\left[\left(\frac{p}{p^*}\right)^{1/p^*}\tilde{\lambda}^{1/p}\left(t-\frac{T}{2}\right)\right]d. \tag{51}$$

Differentiating (51) we obtain

$$\left(\phi_p\left(\tilde{u}'(t)\right)\right)' = -\tilde{\lambda}\phi_p\left(\tilde{u}(t)\right),$$

and thus we check that $\tilde{u}$ solves (43) in $[\tilde{\tau}, T-\tilde{\tau}]$. By similar arguments we evaluate $\tilde{u}$ in the interval $[T-\tilde{\tau}, T]$. The proof is complete. $\square$

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