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Blow-up analysis for an elliptic equation describing stationary vortex flows with variable intensities in 2D-turbulence

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A B S T R A C T

We consider the mean field equation arising in the high-energy scaling limit of point vortices with a general circulation constraint, when the circulation number density is subject to a probability measure. Mathematically, such an equation is a non-local elliptic equation containing an exponential nonlinearity which depends on this probability measure. We analyze the behavior of blow-up sequences of solutions in relation to the circulation numbers. As an application of our analysis we derive an improved Trudinger–Moser inequality for the associated variational functional.

1. Introduction

Mean field equations for many point vortices have been extensively studied in recent years from both the physical and the mathematical points of view, see [14,22,11,9,15,4,5,23,17]. Following ideas introduced by Onsager [21], the vortex system is first formulated as a Hamilton system, and then a mean field equation is derived by making use of tools from equilibrium statistical mechanics theory. The propagation of chaos is achieved furthermore, if this mean field equation admits a unique solution. Various mean field equations have been obtained according to different constraints, such as the

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mono- or the opposite-signed circulations. The mathematical analysis concerning the existence and the uniqueness of solutions has also been widely performed, see [16,28,1,7,6].

If general constraints are considered assuming that the circulation number density is subject to a probability measure, then a new mean field equation arises in the high-energy scaling limit, that is as the number of vortices goes to infinity, the statistical energy remains bounded, and the statistical inverse temperature is proportional to the number of vortices. In this article we are interested in the mathematical analysis of this new equation, which in the case of zero boundary conditions is given by

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}\,dx}\mathcal{P}(d\alpha) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $\mathcal{P} = \mathcal{P}(d\alpha)$, $\alpha \in [-1,1]$, is a probability measure determining the relative circulation number density, $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $v = v(x)$ is the mean field limit stream function, $\lambda \geqslant 0$ is a constant associated with the inverse temperature. A formal derivation of (1) is provided in [24]. If $\mathcal{P} = \delta_1$, that is in the case where every vortex has the same circulation, we obtain from (1)

$$\begin{cases} -\Delta v = \lambda \dfrac{e^v}{\int_\Omega e^v\,dx} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Eq. (2) is mathematically justified by the minimizing free energy method in the canonical formulation [4,15], and its mathematical analysis has revealed the quantized blow-up mechanism of sequences of solutions, see, e.g., [29–31] and the references therein. In the other case where $\mathcal{P}$ is given by

$$\mathcal{P} = n_+\delta_1 + n_-\delta_{-1}, \tag{3}$$

we obtain

$$\begin{cases} -\Delta v = \lambda \left(\dfrac{n_+ e^v}{\int_\Omega e^v\,dx} - \dfrac{n_- e^{-v}}{\int_\Omega e^{-v}\,dx} \right) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Thus each vortex has the circulation ± 1, and $n_\pm \in [0,1]$, $n_+ + n_- = 1$ indicate the ratios of the point vortex numbers, see [14,22].

In the case where the relative circulations of the vortices are independent and identically distributed random variables subject to a common probability measure $\mathcal{N} = \mathcal{N}(d\gamma)$, $\gamma \in [-1,1]$, the corresponding mean field equation takes the form

$$\begin{cases} -\Delta v = \lambda \dfrac{\int_{[-1,1]} \gamma e^{\gamma v}\,\mathcal{N}(d\gamma)}{\int_{[-1,1]}\int_\Omega e^{\gamma v}\,dx\,\mathcal{N}(d\gamma)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

It is derived in [17] using the minimizing free energy method in the canonical formulation. The difference between (5) and (1) becomes evident by substituting

$$\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$$

for $\mathcal{P}$ and $\mathcal{N}$. Indeed, essentially different equations

$$\begin{cases} -\Delta v = \lambda \cdot \dfrac{e^v - e^{-v}}{\int_\Omega e^v + e^{-v}\, dx} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

and

$$\begin{cases} -\Delta v = \dfrac{\lambda}{2}\left(\dfrac{e^v}{\int_\Omega e^v\, dx} - \dfrac{e^{-v}}{\int_\Omega e^{-v}\, dx}\right) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

are derived from (5) and (1), respectively. Eq. (7) is the neutral mean field equation derived in [14,22]. The variational functionals associated to (1) and (5), on the other hand, are given by

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_{[-1,1]} \log\left(\int_\Omega e^{\alpha v}\, dx\right) \mathcal{P}(d\alpha) \tag{8}$$

and

$$K_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega dx \int_{[-1,1]} e^{\gamma v}\, \mathcal{N}(d\gamma)\right)$$

defined for $v \in H_0^1(\Omega)$, respectively. A rigorous derivation of Eq. (1) will be carried out in a forthcoming article.

Throughout this paper we shall consider the analog of (1) in the case where Ω is a compact orientable Riemannian surface without boundary. That is, we study

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_{[-1,1]} \alpha\left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}\, dx} - \frac{1}{|\Omega|}\right) \mathcal{P}(d\alpha) & \text{in } \Omega, \\ \displaystyle\int_\Omega v\, dx = 0, \end{cases} \tag{9}$$

where (Ω, g) is a two-dimensional compact orientable Riemannian manifold, $\mathcal{P}(d\alpha)$ is a Borel probability measure on $[-1, 1]$ and dx denotes the volume element on Ω. We note that Eq. (9) is the Euler–Lagrange equation of the functional

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_{[-1,1]} \log\left(\int_\Omega e^{\alpha v}\, dx\right) \mathcal{P}(d\alpha)$$

defined on the space

$$\mathcal{E} = \left\{ v \in H^1(\Omega) \;\Big|\; \int_\Omega v = 0 \right\},$$

equipped with the norm $\|v\|_{\mathcal{E}} = \|\nabla v\|_2$. As already mentioned, here we are concerned with the blow-up analysis for (9). Such an analysis is motivated by the results in [20] for the special case where $\mathcal{P}$ is given by (3). In this case, it was noticed in [20] that the blow-up masses satisfy a quadratic identity. See also [13,8] for further results in this direction. From such a property, an improved sharp Trudinger–Moser inequality was derived. Our blow-up analysis for (9) provides the natural analog of such a quadratic identity, see Theorem 2.2(iii) below. However, due to the presence of the general probability measure $\mathcal{P}$, in order to carry out our blow-up analysis we need to consider measures defined on the *product space* $I \times \Omega$, taking an approach which appears to be new. Similarly as in [20], our analysis combined with arguments from [12] yields as an application an improved Trudinger–Moser inequality involving $\mathcal{P}$, which is also sharp in some special cases not contained in [20].

This paper is organized as follows. In Section 2 we outline our main results. In Section 3 we provide a preliminary blow-up analysis, showing that the blow-up set is finite. In Section 4 we refine such a blow-up analysis on the product space $I \times \Omega$. In Section 5 we derive the above mentioned quadratic identity for blow-up masses. In Section 6 we apply our blow-up analysis in order to prove a Trudinger–Moser inequality. Finally, in Section 7 we conclude with some remarks on sharpness.

2. Main results

We consider solution sequences $\{v_n\}_{n\in\mathbb{N}}$, $\lambda_n \to \lambda_0$, to

$$-\Delta v_n = \lambda_n \int\limits_{[-1,1]} \alpha\left(\frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} - \frac{1}{|\Omega|}\right)\mathcal{P}(d\alpha), \qquad \int\limits_\Omega v_n = 0. \tag{10}$$

As usual, we define the blow-up sets

$$\mathcal{S}_\pm = \left\{p \in \Omega \;\middle|\; \text{there exists } p_{\pm,n} \in \Omega,\ p_{\pm,n} \to p \text{ such that } v_n(p_{\pm,n}) \to \pm\infty\right\}$$

and we denote $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$. We define the measures $\nu_{\pm,n} \in \mathcal{M}(\Omega)$ by setting

$$\nu_{\pm,n} = \lambda_n \int\limits_{I_\pm} \frac{|\alpha| e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}\,\mathcal{P}(d\alpha), \tag{11}$$

where we denote $I_+ = (0,1]$ and $I_- = [-1,0)$. Since $\int_\Omega \nu_{\pm,n} \leqslant \lambda_n \int_I |\alpha|\,\mathcal{P}(d\alpha) \leqslant \lambda_n$, we may assume that $\nu_{\pm,n} \overset{*}{\rightharpoonup} \nu_\pm$ for some measures $\nu_\pm \in \mathcal{M}(\Omega)$. Our first result states that, similarly to the well-known case $\mathcal{P} = \delta_1$, the blow-up set is finite and that a "minimum mass" is necessary for blow-up to occur.

Theorem 2.1. *Let $\{v_n\}$ be a solution sequence to* (10) *with $\lambda_n \to \lambda_0$. Then, the following alternative holds.*

(i) *Compactness,* $\limsup_{n\to\infty} \|v_n\|_\infty < +\infty$: *There exist $v \in \mathcal{E}$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ in $\mathcal{E}$.*
(ii) *Concentration,* $\limsup_{n\to\infty} \|v_n\|_{L^\infty(M)} = +\infty$: *The sets $\mathcal{S}_\pm$ are finite and $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \neq \emptyset$. Moreover, we have $0 \leqslant s_\pm \in L^1(\Omega)$ such that*

$$\nu_\pm = s_\pm\,dx + \sum_{p\in\mathcal{S}_\pm} n_{\pm,p}\delta_p$$

with $n_{\pm,p} \geqslant 4\pi$ for all $p \in \mathcal{S}$.

Our main result is a finer description of the "blow-up masses" depending on α. To this end, it is convenient to consider the following measures defined on the product space $I \times \Omega$. For every fixed $\alpha \in I$ we define $\mu_\alpha^n \in \mathcal{M}(\Omega)$ by setting

$$\mu_\alpha^n(dx) = \lambda_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}\,dx.$$

We consider the following sequence of measures $\mu_n = \mu_n(d\alpha\, dx) \in \mathcal{M}(I \times \Omega)$ defined by

$$\mu_n = \mu_\alpha^n(dx)\,\mathcal{P}(d\alpha) = \lambda_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}\,\mathcal{P}(d\alpha)\,dx. \tag{12}$$

Since, in view of Fubini's theorem, for large values of n we have

$$\mu_n(I \times \Omega) = \iint\limits_{I \times \Omega} \lambda_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}\,\mathcal{P}(d\alpha)\,dx = \lambda_n \leqslant \lambda_0 + 1,$$

upon extracting a subsequence, we may assume that $\mu_n \overset{*}{\rightharpoonup} \mu$ for some Borel measure $\mu = \mu(d\alpha\, dx) \in \mathcal{M}(I \times \Omega)$. The following results hold.

Theorem 2.2. *Suppose that $\mathcal{P}$ is a Borel probability measure on $[-1, 1]$. Then:*

(i) *the limit measure μ has the form*

$$\mu(d\alpha\, dx) = \left[\sum_{p \in \mathcal{S}} m(\alpha, p)\delta_p(dx) + r(\alpha, x)\,dx\right]\mathcal{P}(d\alpha), \tag{13}$$

where $m(\cdot, p) \in L^\infty(I, \mathcal{P})$ for all $p \in \mathcal{S}$, δ_p denotes the Dirac mass on Ω centered at p and $r \in L^1(I \times \Omega)$;

(ii) *for every $p \in \mathcal{S}$ we have*

$$\sup_{\alpha \in I} m(\alpha, p) \leqslant \lambda_0 = \sum_{p \in \mathcal{S}} \int\limits_I m(\alpha, p)\,\mathcal{P}(d\alpha) + \iint\limits_{I \times \Omega} r(\alpha, x)\,\mathcal{P}(d\alpha)\,dx; \tag{14}$$

(iii) *for every fixed $p \in \mathcal{S}$, the following relation is satisfied by $m(\alpha, p)$:*

$$8\pi \int\limits_I m(\alpha, p)\,\mathcal{P}(d\alpha) = \left\{\int\limits_I \alpha m(\alpha, p)\,\mathcal{P}(d\alpha)\right\}^2; \tag{15}$$

(iv) *we have*

$$\int\limits_{I_\pm} |\alpha| m(\alpha, p)\,\mathcal{P}(d\alpha) = n_{\pm,p}, \qquad \int\limits_{I_\pm} |\alpha| r(\alpha, x)\,\mathcal{P}(d\alpha) = s_\pm(x),$$

with $n_{\pm,p}$ and $s_\pm(x)$ defined in Theorem 2.1. Furthermore $m(\alpha, p) \equiv 0$ for every $p \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$ and $\alpha \in I_\mp$.

Finally, we apply our blow-up analysis in order to derive an improved Trudinger–Moser inequality for the variational functional associated to (10).

Theorem 2.3. *Let $\mathcal{P}$ be a Borel probability measure on $I = [-1, 1]$. Then, $J_\lambda(v)$, $v \in \mathcal{E}$, defined by (8) is bounded below if*

$$\lambda \leqslant \frac{8\pi}{\max\{\int_{I_+} \alpha^2\,\mathcal{P}(d\alpha), \int_{I_-} \alpha^2\,\mathcal{P}(d\alpha)\}}. \tag{16}$$

An interpretation of (16) may be as follows. We recall the classical Trudinger–Moser inequality in the sharp form due to Fontana [10]:

$$\int_\Omega e^v \leqslant C_{TM} \exp\left\{\frac{1}{16\pi}\|\nabla v\|_2^2\right\}, \quad v \in \mathcal{E}, \tag{17}$$

where $C_{TM} > 0$ is a constant determined by Ω. It is not difficult to check that by rescaling (17) we obtain boundedness below of J_λ for all

$$\lambda \leqslant \frac{8\pi}{\int_{[-1,1]} \alpha^2 \mathcal{P}(d\alpha)},$$

see Lemma 6.1 below. Hence, (16) emphasizes the fact that "the positively supported part of $\mathcal{P}$ and the negatively supported part of $\mathcal{P}$ do not interact".

We now compare (16) with previously known results. In the special case $\mathcal{P} = \delta_1$, the Dirac measure on $[-1,1]$ concentrated at $\alpha = 1$, J_λ reduces to

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v.$$

Condition (16) yields boundedness below of $J_\lambda|_{\mathcal{P}=\delta_1}$ when $\lambda \leqslant 8\pi$. This condition is equivalent to (17). In the other case where $\mathcal{P}$ is given by (3), Eq. (9) is related to (4) with $n_+ = \tau$, $n_- = 1-\tau$. Then it holds that

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\left(\tau \log \int_\Omega e^v + (1-\tau)\log \int_\Omega e^{-v}\right).$$

This functional was derived by [14,22]. In this case, condition (16) yields boundedness below of $J_\lambda|_{\mathcal{P}=\tau\delta_1+(1-\tau)\delta_{-1}}$ when

$$\lambda \leqslant \frac{8\pi}{\max\{\tau, 1-\tau\}}. \tag{18}$$

The above is exactly the improved sharp Trudinger–Moser inequality recently derived in [26,20].

We conclude by some remarks on sharpness. It is not difficult to check that (16) is also a necessary condition if $\mathcal{P}$ is of the form

$$\mathcal{P} = \tau\delta_\alpha + (1-\tau)\delta_{-\beta}, \quad \alpha, \beta, \tau \in [0,1],$$

thus providing an extension of the optimal result in [20]. However, in general the sharpness of (16) may not be expected for every choice of $\mathcal{P}$, and the derivation of an inequality which is sharp for every choice of $\mathcal{P}$, if at all possible, seems to require an altogether different method. Some further remarks on sharpness are contained in Section 7.

3. Proof of Theorem 2.1

In order to prove Theorem 2.1, we need some lemmas. The first is a direct analogy of Corollary 4, p. 1234 in [2]. Let $D \subset \mathbb{R}^2$ be a bounded domain and for every $a \in \mathbb{R}$ let $a^+ = \max\{a, 0\}$ be the positive part of a. Recall that $I_+ = (0,1]$.

Lemma 3.1. *Suppose that $\{u_n\}$ is a solution sequence to*

$$-\Delta u_n = \int_{I_+} W_{\alpha,n} e^{\alpha u_n} \mathcal{P}(d\alpha) \quad \text{in } D,$$

where $\|W_{\alpha,n}\|_{L^p(D)} \leqslant C_1$, $p \in (1, \infty]$, $\|u_n^+\|_{L^1(D)} \leqslant C_2$. Suppose that for every n we have

$$\iint_{D\times I_+} |W_{\alpha,n}| e^{\alpha u_n} \mathcal{P}(d\alpha)\, dx \leqslant \varepsilon_0 < \frac{4\pi}{p'},$$

where $p' = p/(p-1)$ is the conjugate exponent to p. Then, $\{u_n^+\}$ is bounded in $L^\infty_{\mathrm{loc}}(D)$.

Proof. Without loss of generality, we may assume $D = B_R$. Split $u_n = w_{1,n} + w_{2,n}$, where $w_{1,n}$ satisfies

$$-\Delta w_{1,n} = \int_{I_+} W_{\alpha,n} e^{\alpha u_n} \mathcal{P}(d\alpha) \quad \text{in } D, \qquad w_{1,n} = 0 \quad \text{on } \partial D. \tag{19}$$

Then, $\Delta w_{2,n} = 0$ in Ω. By the mean value theorem for harmonic functions, we have

$$\|w_{2,n}^+\|_{L^\infty(B_{R/2})} \leqslant C\|w_{2,n}^+\|_{L^1(B_R)} \leqslant C\big(\|u_n^+\|_{L^1(B_R)} + \|w_{1,n}^+\|_{L^1(B_R)}\big) \leqslant C_3.$$

Now recall that, setting $\varphi_n = \int_{I_+} W_{\alpha,n} e^{\alpha u_n} \mathcal{P}(d\alpha)$, we have by assumption

$$\|\varphi_n\|_{L^1(D)} \leqslant \varepsilon_0 < 4\pi/p'.$$

In view of Theorem 1, p. 1226 in [2], we have

$$\int_D \exp\left\{\frac{4\pi(1-\eta)}{\|\varphi\|_1}|w_{1,n}|\right\} dx \leqslant \frac{\pi}{\eta}(\operatorname{diam}\Omega)^2, \quad \eta \in (0,1).$$

Let $\zeta_0 \in (0,1)$ be such that $\varepsilon_0 = 4\pi(1-\zeta_0)/p'$ and $\eta_0 \in (0, \zeta_0)$. Then, we have

$$\frac{4\pi(1-\eta_0)}{\|\varphi_n\|_1} \geqslant \frac{4\pi(1-\eta_0)}{\varepsilon_0} = \frac{1-\eta_0}{1-\zeta_0} p' > p'.$$

Putting $\delta = p'(\zeta_0 - \eta_0)/(1-\zeta_0)$, we have

$$\int_D e^{(p'+\delta)|w_{1,n}|}\, dx \leqslant \int_D \exp\left\{\frac{4\pi(1-\eta_0)}{\|\varphi\|_1}|w_{1,n}|\right\} dx \leqslant C_4.$$

Therefore, $\{e^{w_{1,n}}\}$ is bounded in $L^{p'+\delta}(\Omega)$ and consequently $\{e^{u_n}\}$ is bounded in $L^{p'+\delta}(B_{R/2})$ for some $\delta > 0$. In view of Fubini's theorem and of Hölder's inequality, it follows that:

$$\begin{aligned}
\int_{B_{R/2}} \left| \int_{I_+} W_{\alpha,n} e^{\alpha u_n} \, \mathcal{P}(d\alpha) \right|^r dx &\leqslant \int_{B_{R/2}} \mathcal{P}(I_+)^{r-1} \int_{I_+} \left| W_{\alpha,n} e^{\alpha u_n} \right|^r \mathcal{P}(d\alpha) \, dx \\
&\leqslant \iint_{I_+ \times B_{R/2}} \left| W_{\alpha,n} e^{\alpha u_n} \right|^r dx \, \mathcal{P}(d\alpha) \\
&\leqslant \iint_{I_+ \times B_{R/2}} |W_{\alpha,n}|^r e^{r|u_n|} \, dx \, \mathcal{P}(d\alpha) \\
&\leqslant \int_{I_+} \|W_{\alpha,n}\|^r_{L^p(\Omega)} \left(\int_{B_{R/2}} e^{(pr/(p-r))|u_n|} \right)^{(p-r)/p} \mathcal{P}(d\alpha) \\
&= \int_{I_+} \|W_{\alpha,n}\|^r_{L^p(\Omega)} \left\| e^{u_n} \right\|^r_{L^{p'+\delta}(B_{R/2})} \mathcal{P}(d\alpha),
\end{aligned}$$

where $r \in (1, p)$ is chosen to satisfy $pr/(p-r) = p' + \delta$. By elliptic estimates, we conclude that $w_{1,n}$ is bounded in $L^\infty(B_{R/4})$. Therefore, $\{u_n\}$ is bounded in $L^\infty(B_{R/4})$. □

Now we show the following result for equations defined on manifolds using some ideas from [19], Lemma 3.2, p. 188. Let (Ω, g) be a Riemannian surface. We consider solution sequences $\{u_n\}$ to the equation

$$-\Delta u_n = \int_{I_+} W_{\alpha,n} e^{\alpha u_n} \, \mathcal{P}(d\alpha) + f_n \quad \text{on } \Omega \tag{20}$$

and set

$$\sigma_n = \int_{I_+} |W_{\alpha,n}| e^{\alpha u_n} \, \mathcal{P}(d\alpha).$$

Lemma 3.2. *Suppose that u_n is a solution sequence to* (20)*, with $\|W_{\alpha,n}\|_p \leqslant C_5$, $\|f_n\|_\infty \leqslant C_6$, $\|u_n^+\|_1 \leqslant C_7$. Suppose that $\sigma_n \overset{*}{\rightharpoonup} \sigma$ and $\sigma(\{x_0\}) < 4\pi/p'$ for some $x_0 \in \Omega$. Then, there exists a neighborhood $U \subset \Omega$ of x_0 such that*

$$\limsup_{n\to\infty} \left\| u_n^+ \right\|_{L^\infty(U)} < +\infty.$$

Proof. We take a local isothermal chart (U, ψ) around x_0 such that $\psi(x_0) = 0$, $g = e^{\xi(X)}(dX_1^2 + dX_2^2)$. Then, $u_n(X) = u_n(\psi^{-1}(X))$ satisfies

$$-\Delta_X u_n = \left(\int_{I_+} W_{\alpha,n} e^{\alpha u_n} \, \mathcal{P}(d\alpha) + f_n \right) e^{\xi} \quad \text{in } D = \psi(U).$$

Let h_n be defined by

$$-\Delta_X h_n = f_n e^{\xi} \quad \text{in } D, \qquad h_n = 0 \quad \text{on } \partial D.$$

It follows that $\|h_n\|_{L^\infty(\Omega)} \leqslant C_8$, and $\tilde{u}_n = u_n - h_n$ satisfies

$$-\Delta_X \tilde{u}_n = e^{\xi} \int_{I_+} W_{\alpha,n} e^{\alpha h_n} e^{\alpha \tilde{u}_n} \, \mathcal{P}(d\alpha) \quad \text{in } D$$

with

$$\left\| e^{\xi} W_{\alpha,n} e^{\alpha h_n} \right\|_{L^p(D)} \leqslant e^{\|h_n\|_{L^\infty(D)} + \|\xi\|_{L^\infty(\Omega)}} \|W_{\alpha,n}\|_{L^p(\Omega)} \leqslant C_9,$$
$$\left\| \tilde{u}_n^+ \right\|_{L^1(D)} \leqslant \left\| u_n^+ \right\|_{L^1(\Omega)} + |D| \|h_n\|_{L^\infty(D)} \leqslant C_{10}.$$

We have

$$\int_D e^{\xi} \int_{I_+} |W_{\alpha,n}| e^{\alpha h_n} e^{\alpha \tilde{u}_n} \, \mathcal{P}(d\alpha) \, dX = \iint_{I_+ \times D} |W_{\alpha,n}| e^{\alpha u_n} e^{\xi} \, \mathcal{P}(d\alpha) \, dX$$
$$= \iint_{I_+ \times U} |W_{\alpha,n}| e^{\alpha u_n} \, \mathcal{P}(d\alpha) \, dx = \sigma_n(U).$$

From the assumptions, we derive that there exists $U' \subset U$ such that

$$\iint_{I_+ \times U'} |W_{\alpha,n}| e^{\alpha u_n} \, \mathcal{P}(d\alpha) \, dx \leqslant \varepsilon_0 < \frac{4\pi}{p'}.$$

Now the conclusion follows from Lemma 3.1. □

Now we return to the analysis of Eq. (10). We denote by $G = G(x, y)$ the Green's function associated to $-\Delta$ on Ω. Namely, G is defined by

$$\begin{cases} -\Delta_x G(x, y) = \delta_y - \dfrac{1}{\Omega} \quad \text{in } \Omega, \\ \displaystyle\int_{\Omega} G(x, y) \, dx = 0. \end{cases}$$

For every solution v_n to (10) we define a "positive part" $\tilde{u}_{+,n}$ and a "negative part" $\tilde{u}_{-,n}$ by setting $\tilde{u}_{\pm,n} = G \star \nu_{\pm,n}$, where $\nu_{\pm,n}$ is defined in (11). Then, $v_n = \tilde{u}_{+,n} - \tilde{u}_{-,n}$ and furthermore,

$$\begin{cases} -\Delta \tilde{u}_{\pm,n} = \lambda_n \displaystyle\int_{I_\pm} |\alpha| \left(\frac{e^{|\alpha|(\tilde{u}_{\pm,n} - \tilde{u}_{\mp,n})}}{\int_{\Omega} e^{|\alpha|(\tilde{u}_{\pm,n} - \tilde{u}_{\mp,n})}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha), \\ \displaystyle\int_{\Omega} \tilde{u}_{\pm,n} = 0. \end{cases} \tag{21}$$

Then, Theorem 2.1 is proven by the blow-up analysis to $\tilde{u}_{\pm,n}$.

Proof of Theorem 2.1. Let

$$\mathcal{S}_{\tilde{u}_+} = \left\{ p \in \Omega \;\middle|\; \nu_+\big(\{p\}\big) \geqslant 4\pi \right\}.$$

Since

$$\nu_{+,n}(\Omega) = \lambda_n \int_{I_+} \alpha\, \mathcal{P}(d\alpha) \to \nu_+(\Omega) = \lambda_0 \int_{I_+} \alpha\, \mathcal{P}(d\alpha) < +\infty,$$

it holds that $\sharp \mathcal{S}_{\tilde{u}_+} < +\infty$. Writing (21) in the form

$$-\Delta \tilde{u}_{+,n} = \int_{I_+} V_{\alpha,n}\, e^{\alpha \tilde{u}_{+,n}}\, \mathcal{P}(d\alpha) - \frac{\lambda_n}{|\Omega|} \int_{I_+} \alpha\, \mathcal{P}(d\alpha)$$

with

$$V_{\alpha,n} = \lambda_n \frac{\alpha e^{-\alpha \tilde{u}_{-,n}}}{\int_\Omega e^{\alpha v_n}},$$

first, we have $\int_\Omega e^{\alpha v_n} \geqslant |\Omega|$ by Jensen's inequality. Next, we have

$$\tilde{u}_{-,n} \geqslant -\lambda_n C_{11} \int_{I_-} |\alpha|\, \mathcal{P}(d\alpha) \geqslant -C_{12}$$

because $G(x, y)$ is bounded below, and consequently,

$$\|V_{\alpha,n}\|_{L^\infty(\Omega)} \leqslant C_{13}$$

uniformly for $\alpha \in I_+$. If $\mathcal{S}_{\tilde{u}_+} = \emptyset$, we have

$$\limsup_{n\to\infty} \big\| \tilde{u}^+_{+,n} \big\|_{L^\infty(\Omega)} < +\infty$$

by Lemma 3.2 with $p = +\infty$ and the compactness of Ω. Then, by elliptic estimates,

$$\limsup_{n\to+\infty} \big\| \tilde{u}^+_{+,n} \big\|_{W^{2,r}(\Omega)} < +\infty, \quad r \in [1, +\infty),$$

and therefore we may extract a subsequence $\{\tilde{u}_{+,n_k}\}$ such that $\tilde{u}_{+,n_k} \to \tilde{u}_+$, for some $\tilde{u}_+ \in \mathcal{E}$. Similarly, if $\mathcal{S}_{\tilde{u}_-} = \emptyset$, then there exists a subsequence $\tilde{u}_{-,n_l} \to \tilde{u}_-$ for some $\tilde{u}_- \in \mathcal{E}$, where

$$\mathcal{S}_{\tilde{u}_-} = \big\{ q \in \Omega \;\big|\; \nu_-\big(\{q\}\big) \geqslant 4\pi \big\}.$$

In the case of $\mathcal{S}_{\tilde{u}_+} \neq \emptyset$, we have

$$\limsup_{n\to+\infty} \big\| \tilde{u}^+_{+,n} \big\|_{L^\infty(\omega)} < +\infty$$

for every $\omega \Subset \Omega \setminus \mathcal{S}_{\tilde{u}_+}$, and therefore, there exists $s_+ \in L^\infty_{\mathrm{loc}}(\Omega \setminus \mathcal{S}_{\tilde{u}_+})$ such that $\nu_{+,n}|_\omega \to s_+$ in $L^p(\omega)$ for all $p \in [1, \infty)$. It follows that $\nu_+|_\omega = s_+\, dx$, while the singular part of ν_+ is supported on $\mathcal{S}_{\tilde{u}_+}$. Hence,

$$\nu_+ = s_+ + \sum_{p \in \mathcal{S}_{\tilde{u}_+}} n_{+,p} \delta_p$$

for some $n_{+,p} \geqslant 4\pi$, and similarly,

$$\nu_- = s_- + \sum_{q \in \mathcal{S}_{\tilde{u}_-}} n_{-,q}\delta_q,$$

where $n_{-,q} \geqslant 4\pi$. Finally, we claim

$$\mathcal{S}_{\tilde{u}_+} = \mathcal{S}_+, \qquad \mathcal{S}_{\tilde{u}_-} = \mathcal{S}_-. \tag{22}$$

To show the first equivalence, let $p_0 \notin \mathcal{S}_{\tilde{u}_+}$. Then, in view of Lemma 3.2 there exists a neighborhood $U \subset \Omega$ of p_0 such that

$$\limsup_{n\to\infty} \|\tilde{u}^+_{+,n}\|_{L^\infty(U)} < +\infty.$$

Recall that $v_n = \tilde{u}_{+,n} - \tilde{u}_{-,n} \leqslant \tilde{u}_{+,n} + C_{12}$. It follows that

$$\limsup_{n\to\infty} \|v_n^+\|_{L^\infty(U)} < +\infty$$

and consequently, $p_0 \notin \mathcal{S}_+$. We have thus $\mathcal{S}_+ \subset \mathcal{S}_{\tilde{u}_+}$. To show the reverse relation, we note that $\mathcal{S}_{\tilde{u}_+}$ coincides with the singular support of ν_+, and consequently the sequence of functions

$$\nu_{+,n} = \lambda_n \int_{I_+} \frac{\alpha e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)$$

is L^∞-unbounded near $p_0 \in \mathcal{S}_{\tilde{u}_+}$. We derive that, for every $r > 0$:

$$\begin{aligned} +\infty = \lim_{n\to\infty} \sup_{B(p_0,r)} \nu_{+,n} &= \lim_{n\to\infty} \sup_{x\in B(p_0,r)} \lambda_n \int_{I_+} \frac{\alpha e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\leqslant \lim_{n\to\infty} \sup_{B(p_0,r)} \lambda_n \frac{e^{v_n}}{|\Omega|}. \end{aligned}$$

In particular,

$$\lim_{n\to\infty} \sup_{B(p_0,r)} v_n = +\infty$$

and hence $p_0 \in \mathcal{S}_+$. The proof for $\mathcal{S}_-$ is analogous. □

4. Proof of Theorem 2.2, parts (i), (ii), (iv)

We begin with some lemmas. Let

$$\tilde{\mu}_{\pm,n}(dx) = \lambda_n \int_{I_\pm} \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)\, dx.$$

Since $\tilde{\mu}_{\pm,n}(\Omega) = \lambda_n \mathcal{P}(I_\pm) \leqslant \lambda_n$, upon extracting a subsequence we may assume $\tilde{\mu}_{\pm,n} \overset{*}{\rightharpoonup} \tilde{\mu}_\pm$ for some Borel measures $\tilde{\mu}_\pm = \tilde{\mu}_\pm(dx) \in \mathcal{M}(\Omega)$.

Lemma 4.1. *There exist $\tilde{s}_\pm \in L^1(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \mathcal{S}_\pm)$ and $\tilde{m}_\pm(p) \geqslant 4\pi$, $p \in \mathcal{S}_\pm$, such that*

$$\tilde{\mu}_\pm = \tilde{s}_\pm + \sum_{p \in \mathcal{S}_\pm} \tilde{m}_\pm(p)\delta_p. \tag{23}$$

Proof. By definition of $\mathcal{S}_\pm$, for every $\omega \Subset \Omega \setminus \mathcal{S}_\pm$ there exists $C_{14} = C_{14}(\omega)$ such that $\sup_\omega v_n \leqslant C_{14}$ for all $n \in \mathbb{N}$. It follows that, for any measurable $E \subset \omega$

$$\tilde{\mu}_{\pm,n}(E) = \lambda_n \int_E dx \int_{I_\pm} \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha) \leqslant \frac{\lambda_n e^{C_{14}}}{|\Omega|} |E|$$

by Jensen's inequality. Thus, the singular parts of $\tilde{\mu}_\pm$ are contained in $\mathcal{S}_\pm$ and therefore, we have (23) for some $\tilde{s}_\pm \in L^1(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \mathcal{S}_\pm)$ and for some $\tilde{m}_\pm(p) > 0$, $p \in \mathcal{S}_\pm$. Since $\tilde{\mu}_{\pm,n} \geqslant \nu_{\pm,n}$, where $\nu_{\pm,n}$ is the measure defined in (11), we conclude that $\tilde{m}(p) \geqslant n_{\pm,p} \geqslant 4\pi$. □

Recall from Section 2 that

$$\mu_n(d\alpha\, dx) = \lambda_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)\, dx$$

and that $\mu_n \overset{*}{\rightharpoonup} \mu$.

Lemma 4.2. *There exist $\zeta_p \in \mathcal{M}(I)$ and $r \in L^1(I \times \Omega)$, $r \geqslant 0$, such that*

$$\mu(d\alpha\, dx) = \sum_{p \in \mathcal{S}} \zeta_p(d\alpha)\, \delta_p(dx) + r(\alpha, x)\, \mathcal{P}(d\alpha)\, dx.$$

Proof. It suffices to show that the singular part of μ is supported on $I \times \mathcal{S}$. To see this, we take $\mathcal{A} \Subset I \times (\Omega \setminus \mathcal{S})$. Then there exists $C_{15} = C_{15}(\mathcal{A})$ such that $\|\alpha v_n\|_{L^\infty(\mathcal{A})} \leqslant C_{15}$. Hence, for large n we obtain

$$\lambda_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \leqslant (\lambda_0 + 1) \frac{e^{C_{15}}}{|\Omega|},$$

and therefore, μ_n does not concentrate on $\mathcal{A}$. □

Lemma 4.3. *For every $p \in \mathcal{S}$ and for every Borel set $\eta \subset I$, there holds that $\zeta_p(\eta) \leqslant \lambda_0 \mathcal{P}(\eta)$. In particular, ζ_p is absolutely continuous with respect to $\mathcal{P}$.*

Proof. Given $\eta \subset I$ and $\varepsilon > 0$, we have a compact set $\kappa \subset I$ and an open set $\omega \subset I$ such that $\kappa \subset \eta \subset \omega$ and $\mathcal{P}(\omega) \leqslant \mathcal{P}(\kappa) + \varepsilon$ because of the regularity properties of Borel measures. Let $\psi \in C(I)$ be such that $\psi \equiv 1$ on κ, $\operatorname{supp} \psi \subset \omega$, $0 \leqslant \psi \leqslant 1$, and for $\rho > 0$ sufficiently small let $\varphi \equiv 1$ on $B_\rho(p)$, $\operatorname{supp} \varphi \subset B(p, 2\rho)$, $0 \leqslant \varphi \leqslant 1$. Then,

$$\begin{aligned}
\iint_{I \times \Omega} \psi(\alpha)\varphi(x)\, \mu_n(d\alpha\, dx) &= \lambda_n \iint_{I \times \Omega} \varphi(x)\psi(\alpha) \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)\, dx \\
&\leqslant \lambda_n \iint_{I \times \Omega} \psi(\alpha) \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)\, dx = \lambda_n \int_I \psi(\alpha)\, \mathcal{P}(d\alpha) \\
&\leqslant \lambda_n \mathcal{P}(\omega) \leqslant \lambda_n \big(\mathcal{P}(\kappa) + \varepsilon\big) \leqslant \lambda_n \big(\mathcal{P}(\eta) + \varepsilon\big).
\end{aligned}$$

Taking limits, it follows that

$$\iint_{I\times\Omega} \psi(\alpha)\varphi(x)\,\mu(d\alpha\,dx) \leqslant \lambda_0\big(\mathcal{P}(\eta)+\varepsilon\big).$$

On the other hand, we have

$$\iint_{I\times\Omega} \psi(\alpha)\varphi(x)\,\mu(d\alpha\,dx) = \iint_{I\times\Omega} \psi(\alpha)\varphi(x)\Big[\sum_{q\in\mathcal{S}} \zeta_q\delta_q + r(\alpha,x)\mathcal{P}(d\alpha)\,dx\Big]$$
$$= \int_I \psi(\alpha)\zeta_p + \iint_{I\times\Omega} \psi(\alpha)\varphi(x) r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx \geqslant \zeta_p(\kappa).$$

Hence, we derive that

$$\zeta_p(\kappa) \leqslant \lambda_0\big(\mathcal{P}(\eta)+\varepsilon\big).$$

By Borel regularity of ζ_p, we obtain

$$\zeta_p(\eta) = \sup\big\{\zeta_p(\kappa)\ \big|\ \kappa \text{ compact},\ \kappa\subset\eta\big\} \leqslant \lambda_0\big(\mathcal{P}(\eta)+\varepsilon\big).$$

Finally, since $\varepsilon > 0$ is arbitrary, we conclude that

$$\zeta_p(\eta) \leqslant \lambda_0\mathcal{P}(\eta)$$

and the statement follows. □

Proof of Theorem 2.2. Proof of (i) and (ii). In view of Lemma 4.3, for every $p\in\mathcal{S}$ there exists $m(\alpha,p)\in L^1(I,\mathcal{P})$ such that $\zeta_p = m(\alpha,p)\,\mathcal{P}(d\alpha)$. Moreover, for every $\eta\subset I$ we have

$$\frac{1}{\mathcal{P}(\eta)}\int_\eta m(\alpha,p)\,\mathcal{P}(d\alpha) \leqslant \lambda_0.$$

Now, (13) and (14) follow from the Lebesgue differentiation theorem.

Proof of (iv). Let $\varphi\in C(\Omega)$, $\psi\in C(I)$, $0\leqslant\psi(\alpha)\leqslant 1$, $\psi\equiv 1$ on I_+, $\psi\equiv 0$ on $[-1,-\varepsilon]$, for some fixed $\varepsilon>0$. We have

$$\iint_{I\times\Omega} |\alpha|\varphi(x)\psi(\alpha)\,\mu_n(d\alpha\,dx)$$
$$= \int_\Omega \varphi(x)\,\nu_{+,n}(dx) + \lambda_n \int_{[-\varepsilon,0]} |\alpha|\psi(\alpha)\int_\Omega \frac{\varphi(x)e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}\,dx\,\mathcal{P}(d\alpha). \tag{24}$$

Taking limits on the left-hand side of (24) as $n\to\infty$, we have in view of (i)

$$\iint\limits_{I\times\Omega} |\alpha|\varphi(x)\psi(\alpha)\,\mu_n(d\alpha\,dx)$$
$$\to \sum_{p\in\mathcal{S}}\int\limits_I |\alpha|\psi(\alpha)m(\alpha,p)\,\mathcal{P}(d\alpha)\,\varphi(p) + \iint\limits_{I\times\Omega} |\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx.$$

Furthermore,

$$\int\limits_I |\alpha|\psi(\alpha)m(\alpha,p)\,\mathcal{P}(d\alpha) = \int\limits_{I_+} |\alpha|m(\alpha,p)\,\mathcal{P}(d\alpha) + \int\limits_{[-\varepsilon,0]} |\alpha|\psi(\alpha)m(\alpha,p)\,\mathcal{P}(d\alpha)$$

and

$$\iint\limits_{I\times\Omega} |\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx$$
$$= \iint\limits_{I_+\times\Omega} |\alpha|\varphi(x)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx + \iint\limits_{[-\varepsilon,0]\times\Omega} |\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx.$$

In view of (ii), we have

$$0 \leqslant \int\limits_{[-\varepsilon,0]} |\alpha|\psi(\alpha)m(\alpha,p)\,\mathcal{P}(d\alpha) \leqslant \varepsilon\mathcal{P}\big([-\varepsilon,0]\big)\lambda_0 \leqslant \varepsilon\lambda_0.$$

Moreover,

$$\left|\ \iint\limits_{[-\varepsilon,0]\times\Omega} |\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx\right| \leqslant \varepsilon\|\varphi\|_\infty \iint\limits_{I\times\Omega} r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx.$$

Similarly, taking limits on the right-hand side of (24), we have

$$\int\limits_\Omega \varphi\nu_{+,n} \to \sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) + \int\limits_\Omega s_+\varphi.$$

Furthermore, for large values of n,

$$\lambda_n \int\limits_{[-\varepsilon,0]} |\alpha|\psi(\alpha) \int\limits_\Omega \frac{\varphi(x)e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}\,dx\,\mathcal{P}(d\alpha) \leqslant \lambda_n\|\varphi\|_\infty\varepsilon \leqslant (\lambda_0+1)\|\varphi\|_\infty\varepsilon.$$

Therefore, we conclude from (24) and the estimates above that

$$\sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) + \int\limits_\Omega s_+\varphi + b_1\varepsilon\|\varphi\|_\infty$$
$$= \sum_{p\in\mathcal{S}}\int\limits_{I_+} |\alpha|m(\alpha,p)\,\mathcal{P}(d\alpha)\,\varphi(p) + \iint\limits_{I_+\times\Omega} \varphi(x)r(\alpha,x)|\alpha|\,\mathcal{P}(d\alpha)\,dx + b_2\varepsilon\|\varphi\|_\infty,$$

where b_1, b_2 are quantities which are uniformly bounded with respect to $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$, we obtain

$$\sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) + \int_\Omega s_+\varphi = \sum_{p\in\mathcal{S}} \int_{I_+} |\alpha| m(\alpha,p)\,\mathcal{P}(d\alpha)\,\varphi(p) + \iint_{I_+\times\Omega} |\alpha| r(\alpha,x)\varphi(x)\,\mathcal{P}(d\alpha)\,dx. \tag{25}$$

Now let $\varphi \in C(\Omega)$ be such that $\operatorname{supp}\varphi \subset \Omega \setminus \mathcal{S}$. We derive

$$\int_\Omega s_+\varphi = \int_\Omega \varphi(x)\left\{\int_{I_+} |\alpha| r(\alpha,x)\,\mathcal{P}(d\alpha)\right\} dx$$

and consequently

$$s_+ = \int_{I_+} |\alpha| r(\alpha,x)\,\mathcal{P}(d\alpha)$$

for a.e. $x \in \Omega$ since $\mathcal{S}$ is null set with respect to dx. Therefore (25) becomes

$$\sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) = \sum_{p\in\mathcal{S}} \int_{I_+} |\alpha| m(\alpha,p)\,\mathcal{P}(d\alpha)\,\varphi(p). \tag{26}$$

Now fix $p_0 \in \mathcal{S}_+$ and let $\varphi \in C(\Omega)$ be such that $\varphi(p_0) = 1$ and $\operatorname{supp}\varphi \subset B_\rho(p_0)$, with $B_\rho(p_0) \cap \mathcal{S} = \{p_0\}$. We conclude that

$$n_{+,p_0} = \int_{I_+} |\alpha| m(\alpha,p_0)\,\mathcal{P}(d\alpha)$$

for all $p_0 \in \mathcal{S}_+$.

Finally, for $p_0 \in \mathcal{S}_- \setminus S_+$, let $\varphi \in C(\Omega)$ as above. Then we get

$$0 = \int_{I_+} |\alpha| m(\alpha,p_0)\,\mathcal{P}(d\alpha)$$

from (26) and consequently we have $m(\alpha,p_0) \equiv 0$ for $\alpha \in I_+$ since $m(\cdot,p) \geqslant 0$ a.e. and $m(\cdot,p) \in L^1(I,\mathcal{P})$.

The proof of (iv) in the "I_- case" is analogous. □

5. Proof of Theorem 2.2, part (iii)

We first need a lemma.

Lemma 5.1. *Let $\Pi_n \in \mathcal{M}(I^2 \times \Omega^2)$ be the measure defined by*

$$\Pi_n\big(d\alpha\,d\alpha'\,dx\,dx'\big) = \mu_n(d\alpha\,dx)\mu_n\big(d\alpha'\,dx'\big),$$

where μ_n is the measure defined in (12)*. Then $\Pi_n \overset{*}{\rightharpoonup} \Pi$, where $\Pi \in \mathcal{M}(I^2 \times \Omega^2)$ is given by*

$$\Pi = \Bigg[\sum_{p,q\in\mathcal{S}} m(\alpha,p)m(\alpha',q)\delta_p(dx)\delta_q(dx') + \Bigg(\sum_{p\in\mathcal{S}} m(\alpha,p)\delta_p(dx)\Bigg) r(\alpha',x')\,dx' + \Bigg(\sum_{q\in\mathcal{S}} m(\alpha',q)\delta_q(dx')\Bigg) r(\alpha,x)\,dx + r(\alpha,x)r(\alpha',x')\,dx\,dx'\Bigg]\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha').$$

Proof. Let $\varphi = \varphi(\alpha,x)$, $\psi = \psi(\alpha',x') \in C(I \times \Omega)$. Then,

$$\iint\iint_{I^2\times\Omega^2} \varphi(\alpha,x)\psi(\alpha',x')\,\Pi_n(d\alpha\,d\alpha'\,dx\,dx') = \iint_{I\times\Omega} \varphi(\alpha,x)\,\mu_n(d\alpha\,dx) \iint_{I\times\Omega} \psi(\alpha',x')\,\mu_n(d\alpha'\,dx').$$

Therefore, as $n \to \infty$ we have

$$\begin{aligned}
&\iint\iint_{I^2\times\Omega^2} \varphi(\alpha,x)\psi(\alpha',x')\,\Pi_n(d\alpha\,d\alpha'\,dx\,dx') \\
&\to \int_I \Bigg[\sum_{p\in\mathcal{S}} m(\alpha,p)\varphi(\alpha,p) + \int_\Omega r(\alpha,x)\varphi(\alpha,x)\,dx\Bigg]\mathcal{P}(d\alpha) \\
&\quad \times \int_I \Bigg[\sum_{q\in\mathcal{S}} m(\alpha',q)\psi(\alpha',q) + \int_\Omega r(\alpha',x')\psi(\alpha',x')\,dx'\Bigg]\mathcal{P}(d\alpha') \\
&= \iint_{I^2} \Bigg[\sum_{p,q\in\mathcal{S}} m(\alpha,p)m(\alpha',q)\varphi(\alpha,p)\psi(\alpha',q) \\
&\quad + \sum_{p\in\mathcal{S}} m(\alpha,p)\varphi(\alpha,p) \int_\Omega r(\alpha',x')\psi(\alpha',x')\,dx' \\
&\quad + \sum_{q\in\mathcal{S}} m(\alpha',q)\psi(\alpha',q) \int_\Omega r(\alpha,x)\varphi(\alpha,x)\,dx \\
&\quad + \iint_{\Omega^2} r(\alpha,x)r(\alpha',x')\varphi(\alpha,x)\psi(\alpha',x')\,dx\,dx'\Bigg]\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha').
\end{aligned}$$

Since the linear combinations of functions of the type φ, ψ above are dense in $C(I^2 \times \Omega^2)$, the asserted representation of $\Pi = \Pi(d\alpha\,d\alpha'\,dx\,dx')$ follows. □

Given a solution $v \in \mathcal{E}$ to (9), for every $\alpha \in I$ we define

$$\mu_\alpha = \lambda \frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}}. \tag{27}$$

Let $u_\alpha \in \mathcal{E}$ be defined by

$$u_\alpha(x) = G \star \mu_\alpha(x) = \int_\Omega G\bigl(x, x'\bigr)\mu_\alpha\bigl(x'\bigr)\,dx',$$

where G denotes the Green's function (see Section 3). Then,

$$v = \int_I \alpha u_\alpha \,\mathcal{P}(d\alpha)$$

and $(u_\alpha)_{\alpha\in I}$ satisfies the "Liouville system":

$$-\Delta u_\alpha = \lambda\left(\frac{\exp\{\alpha \int_I \alpha' u_{\alpha'}\,\mathcal{P}(d\alpha')\}}{\int_\Omega \exp\{\alpha \int_I \alpha' u_{\alpha'}\,\mathcal{P}(d\alpha')\}} - \frac{1}{|\Omega|}\right), \qquad \int_\Omega u_\alpha = 0, \quad \alpha \in I. \tag{28}$$

In order to prove part (iv) in Theorem 2.2 we use the "symmetrization method" introduced in [25,18, 29]. Such a method in turn exploits the symmetry of the Green's function, namely

$$G\bigl(x, x'\bigr) = G\bigl(x', x\bigr), \quad \forall x, x' \in \Omega, \tag{29}$$

as well as a differentiation property of μ_α. More precisely, we use the fact that

$$\begin{aligned}\nabla\mu_\alpha &= \lambda\frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}}\nabla v = \alpha\mu_\alpha\nabla v = \alpha\mu_\alpha\nabla\int_I \alpha' u_{\alpha'}\,\mathcal{P}\bigl(d\alpha'\bigr)\\ &= \alpha\mu_\alpha\int_I \alpha'\nabla u_{\alpha'}\,\mathcal{P}\bigl(d\alpha'\bigr) = \alpha\mu_\alpha\int_I \alpha'(\nabla G)\star\mu_{\alpha'}\,\mathcal{P}\bigl(d\alpha'\bigr).\end{aligned} \tag{30}$$

Let χ be a C^1-vector field over Ω, and define

$$\rho_\chi : \Omega^2 \setminus \bigl\{\bigl(x, x'\bigr) \in \Omega^2 \bigm| x = x'\bigr\} \to \mathbb{R}$$

by

$$\rho_\chi\bigl(x, x'\bigr) = \frac{1}{2}\bigl[\chi(x)\cdot\nabla_x G\bigl(x, x'\bigr) + \chi\bigl(x'\bigr)\cdot\nabla_{x'} G\bigl(x, x'\bigr)\bigr]. \tag{31}$$

Since $|\nabla_x G(x, x')| = O(\operatorname{dist}(x, x')^{-1})$, $\rho_\chi(x, x')$ is a bounded function.

Lemma 5.2 *("Symmetrization"). Let v be a solution to* (9)*, and define μ_α by* (27)*. Then,*

$$\iint_{I\times\Omega} (\operatorname{div}\chi)\mu_\alpha\,\mathcal{P}(d\alpha)\,dx = -\iint_{I^2} \alpha\alpha'\,\mathcal{P}(d\alpha)\,\mathcal{P}\bigl(d\alpha'\bigr)\iint_{\Omega^2} \rho_\chi\bigl(x, x'\bigr)\mu_\alpha\mu_{\alpha'}\,dx\,dx'.$$

Proof. In view of (30), we have

$$-\int_I \mathcal{P}(d\alpha)\int_\Omega \mu_\alpha(\operatorname{div}\chi)\,dx = \iint_{I^2} \mathcal{P}(d\alpha)\,\mathcal{P}\big(d\alpha'\big)\iint_{\Omega^2} \alpha\alpha'\mu_\alpha(x)\mu_{\alpha'}\big(x'\big)\chi(x)\cdot\nabla_x G\big(x,x'\big)\,dx\,dx' =: A. \tag{32}$$

Then we "symmetrize" this A. That is, re-labeling x, x' and α, α', we derive from (29):

$$\begin{aligned} A &= \iint_{I^2} \alpha\alpha'\,\mathcal{P}(d\alpha)\,\mathcal{P}\big(d\alpha'\big)\iint_{\Omega^2} \mu_\alpha(x)\mu_{\alpha'}\big(x'\big)\chi(x)\cdot\nabla_x G\big(x,x'\big)\,dx\,dx' \\ &= \iint_{I^2} \alpha\alpha'\,\mathcal{P}(d\alpha)\,\mathcal{P}\big(d\alpha'\big)\iint_{\Omega^2} \mu_\alpha(x)\mu_{\alpha'}\big(x'\big)\chi\big(x'\big)\cdot\nabla_{x'} G\big(x',x\big)\,dx\,dx' \\ &= \iint_{I^2} \alpha\alpha'\,\mathcal{P}(d\alpha)\,\mathcal{P}\big(d\alpha'\big)\iint_{\Omega^2} \mu_\alpha(x)\mu_{\alpha'}\big(x'\big)\chi\big(x'\big)\cdot\nabla_{x'} G\big(x,x'\big)\,dx\,dx'. \end{aligned}$$

Addition of the first and the last terms yields:

$$A = \iint_{I^2} \alpha\alpha'\,\mathcal{P}(d\alpha)\,\mathcal{P}\big(d\alpha'\big)\iint_{\Omega^2} \rho_\chi\big(x,x'\big)\mu_\alpha(x)\mu_{\alpha'}\big(x'\big)\,dx\,dx'.$$

Thus, the proof is completed. □

Proof of Theorem 2.2(iv). Let $\{v_n\}$ be a solution sequence to (10) with $\lambda = \lambda_n \to \lambda_0$. For every $\alpha \in I$ and for every n, let

$$\mu_\alpha^n = \lambda_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}}.$$

In view of Lemma 5.2 we have, for any C^1-vector field χ:

$$\iint_{I\times\Omega} (\operatorname{div}\chi)\mu_\alpha^n\,\mathcal{P}(d\alpha)\,dx = -\iint_{I^2} \alpha\alpha'\,\mathcal{P}(d\alpha)\,\mathcal{P}\big(d\alpha'\big)\iint_{\Omega^2} \rho_\chi\big(x,x'\big)\mu_\alpha^n(x)\mu_{\alpha'}^n\big(x'\big)\,dx\,dx'.$$

Recalling the definitions of the measures $\mu_n = \mu_n(d\alpha\,dx)$ from (12) and $\Pi_n = \Pi_n(d\alpha\,d\alpha'\,dx\,dx')$ from Lemma 5.1, the above is equivalent to

$$\iint_{I\times\Omega} (\operatorname{div}\chi)\,\mu_n(d\alpha\,dx) = -\iint_{I^2}\iint_{\Omega^2} \alpha\alpha'\rho_\chi\big(x,x'\big)\,\Pi_n\big(d\alpha\,d\alpha'\,dx\,dx'\big). \tag{33}$$

If χ is such that ρ_χ is continuous on Ω^2, then taking limits in (33) and using Lemma 5.1, we obtain:

$$\sum_{p\in\mathcal{S}}\int_I (\operatorname{div}\chi)(p)m(\alpha,p)\,\mathcal{P}(d\alpha)+\iint_{I\times\Omega}(\operatorname{div}\chi)(x)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx$$

$$=\iint_{I^2}\Bigg[\sum_{p,q\in\mathcal{S}}m(\alpha,p)m(\alpha',q)\rho_\chi(p,q)+\sum_{p\in\mathcal{S}}m(\alpha,p)\int_\Omega r(\alpha',x')\rho_\chi(p,x')\,dx'$$

$$+\sum_{q\in\mathcal{S}}m(\alpha',q)\int_\Omega r(\alpha,x)\rho_\chi(x,q)\,dx$$

$$+\iint_{\Omega^2}r(\alpha,x)r(\alpha',x')\rho_\chi(x,x')\,dx\,dx'\Bigg]\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha'). \tag{34}$$

The continuity of ρ_χ is achieved by the modified second moment used in [18]. That is, we fix $p_0\in\mathcal{S}$ and take an isothermal coordinate chart (ψ,U) satisfying $\psi(p_0)=0$, $g(X)=e^{\xi}(dX_1^2+dX_2^2)$, and $\xi(0)=0$. Let $B(p_0,2r)\subset U$ and $B(p_0,2r)\cap\mathcal{S}=\{p_0\}$. We identify functions defined on $\psi(U)$ with their pullbacks to U. Then, the Green's function may be written in the following form:

$$G(X,X')=-\frac{1}{2\pi}\ln|X-X'|+\omega(X,X'),$$

$$\nabla_X G(X,X')=-\frac{1}{2\pi}\frac{X-X'}{|X-X'|^2}+\nabla_X\omega(X,X'),$$

$$\nabla_{X'} G(X,X')=\frac{1}{2\pi}\frac{X-X'}{|X-X'|^2}+\nabla_{X'}\omega(X,X'),$$

with ω satisfying

$$\|\omega\|_{L^\infty(B(p_0,2r)^2)}+\|\nabla_X\omega\|_{L^\infty(B(p_0,2r)^2)}+\|\nabla_{X'}\omega\|_{L^\infty(B(p_0,2r)^2)}=O(1)$$

as $r\to 0$. Let $\varphi\in C(\Omega)$ be a cut-off function such that $\varphi\equiv 1$ in $B(p_0,r)$ and $\varphi\equiv 0$ in $\Omega\setminus B(p_0,2r)$. We choose $\chi(X)=2X\varphi$. With this choice of χ we may write:

$$\rho_\chi(X,X')=\left(-\frac{1}{2\pi}+\eta\right)\varphi,$$

where $\eta(X,X')$ is a continuous function on Ω^2. Moreover, we have

$$\operatorname{div}\chi(X)=|g|^{-1/2}\partial_{X_j}\left(|g|^{1/2}(\chi)^j\right)=4+O(X).$$

Consequently, we may expand each term in (34), as $r\downarrow 0$:

$$\sum_{p\in\mathcal{S}}\int_I(\operatorname{div}\chi)(p)m(\alpha,p_0)\,\mathcal{P}(d\alpha)\to 4\int_I m(\alpha,p_0)\,\mathcal{P}(d\alpha);$$

$$\left|\iint_{I\times\Omega}(\operatorname{div}\chi)(x)r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx\right|$$

$$\leqslant\left(4+o(1)\right)\iint_{I\times B(p_0,2r)}r(\alpha,x)\,\mathcal{P}(d\alpha)\,dx=o(1);$$

$$\begin{aligned}&\iint\limits_{I^2}\sum_{p,q\in\mathcal{S}} m(\alpha,p)m(\alpha',q)\rho_\chi(p,q)\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha')\\&\quad\to -\frac{1}{2\pi}\iint\limits_{I^2} m(\alpha,p_0)m(\alpha',p_0)\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha');\end{aligned}$$

$$\begin{aligned}&\iint\limits_{I^2}\sum_{p\in\mathcal{S}} m(\alpha,p)\int\limits_\Omega r(\alpha',x')\rho_\chi(p,x')\,dx'\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha')\\&\quad=\iint\limits_{I^2} m(\alpha,p_0)\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha')\int\limits_{B(p_0,2r)} r(\alpha',x')\left(-\frac{1}{2\pi}+O(1)\right)dx'\\&\quad=o(1).\end{aligned}$$

Similarly,

$$\iint\limits_{I^2}\sum_{q\in\mathcal{S}} m(\alpha',q)\int\limits_\Omega r(\alpha,x)\rho_\chi(x,q)\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha')=o(1),$$

$$\begin{aligned}&\iint\limits_{I^2}\iint\limits_{\Omega^2} r(\alpha,x)r(\alpha',x')\rho_\chi(x,x')\,dx\,dx'\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha')\\&\quad=\iint\limits_{I^2}\iint\limits_{B(p_0,2r)^2} r(\alpha,x)r(\alpha',x')\left(-\frac{1}{2\pi}+O(1)\right)dx\,dx'\,\mathcal{P}(d\alpha)\,\mathcal{P}(d\alpha')\\&\quad=o(1).\end{aligned}$$

Now the asserted identity (iv) follows, and Theorem 2.2 is completely established. □

6. Proof of Theorem 2.3

The basic ideas of the proof of Theorem 2.3 are the following. Let

$$\Lambda=\left\{\lambda\in[0,+\infty)\;\middle|\;\inf_{v\in\mathcal{E}} J_\lambda(v)>-\infty\right\}$$

and

$$\bar\lambda=\frac{8\pi}{\max\{\int_{I_+}\alpha^2\,\mathcal{P}(d\alpha),\int_{I_-}\alpha^2\,\mathcal{P}(d\alpha)\}}. \tag{35}$$

In order to prove Theorem 2.3 we show $[0,\bar\lambda]\subset\Lambda$. Setting

$$\lambda^0=\sup\Lambda,$$

the proof is reduced to showing that

$$\lambda^0\geqslant\bar\lambda \tag{36}$$

and

$$\inf_{\mathcal{E}} J_{\lambda^0} > -\infty \quad \text{if } \lambda^0 = \bar{\lambda}. \tag{37}$$

To get (36) we show the existence of a blow-up sequence of solutions v_n to (10) with $\lambda_n \to \lambda^0$ by an argument attributed to Ding (see [12] or [20]). Then, the lower bound (36) for λ^0 follows from the mass identity (15). Next, we show (37) by the following splitting argument for J_λ. We take $\lambda_n \uparrow \lambda^0$. We have the boundedness below and coercivity of J_{λ_n} for all n. Let $v_n \in \mathcal{E}$ satisfy $J_{\lambda_n}(v_n) = \inf_{\mathcal{E}} J_{\lambda_n}$. Then v_n is a solution to Eq. (9) with $\lambda = \lambda_n$. Recall from Section 3, Eq. (21), that

$$\tilde{u}_{\pm,n}(x) = \lambda_n \iint_{I_\pm \times \Omega} G(x, x') \frac{|\alpha| e^{\alpha v_n(x')}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)\, dx'$$

and that $v_n = \tilde{u}_{+,n} - \tilde{u}_{-,n}$ and $\tilde{u}_{\pm,n} \geqslant -C_{12}$ for n. We may estimate:

$$J_{\lambda_n}(v_n) \geqslant J_{+,\lambda_n}(\tilde{u}_{+,n}) + J_{-,\lambda_n}(\tilde{u}_{-,n}) - \int_\Omega \nabla\tilde{u}_{+,n} \cdot \nabla\tilde{u}_{-,n} - C_{16}, \tag{38}$$

where we have set

$$J_{\pm,\lambda}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda \int_{I_\pm} \log\left(\int_\Omega e^{\alpha v}\right) \mathcal{P}(d\alpha) \tag{39}$$

for all $v \in \mathcal{E}$. By rescaling the standard Moser–Trudinger inequality (17), the functionals $J_{\pm,\lambda}$ are both bounded below if λ satisfies (16), see Lemma 6.1 below. Therefore, the main issue in proving (37) is to control the cross-term $\int_\Omega \nabla\tilde{u}_{+,n} \cdot \nabla\tilde{u}_{-,n}$. Integrating by parts, we have

$$\int_\Omega \nabla\tilde{u}_{+,n} \cdot \nabla\tilde{u}_{-,n} = \lambda_n \iint_{I_+ \times \Omega} \tilde{u}_{-,n} \frac{\alpha e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)\, dx.$$

Hence, we are reduced to showing that $\tilde{u}_{-,n}$ and v_n cannot be both unbounded above at a given point $p \in \Omega$. That is, we have to show that "two-sided blow-up" does not occur when $\lambda^0 = \lim_{n\to\infty} \lambda_n$ satisfies (16). This property will follow from Theorem 2.2.

We now proceed towards the detailed proof of Theorem 2.3. We begin by rescaling the Moser–Trudinger inequality (17).

Lemma 6.1. *The functional J_λ is bounded below if*

$$\lambda \leqslant \frac{8\pi}{\int_{[-1,1]} \alpha^2 \mathcal{P}(d\alpha)}. \tag{40}$$

Proof. From (17), it follows that

$$\int_\Omega e^{\alpha v} \leqslant C_{TM} \exp\left\{\frac{\alpha^2}{16\pi} \|\nabla v\|_2^2\right\} \tag{41}$$

for every $\alpha \in I$, and therefore,

$$\frac{1}{2}\|\nabla v\|_2^2 \geqslant \frac{8\pi}{\alpha^2}\log\int_\Omega e^{\alpha v} - \frac{8\pi}{\alpha^2}\log C_{TM}.$$

It follows that

$$\begin{aligned} J_\lambda(v) &= \int_I \left(\frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\int_\Omega e^{\alpha v}\right)\mathcal{P}(d\alpha) \\ &= \int_I \left\{\frac{\lambda\alpha^2}{8\pi}\left(\frac{1}{2}\|\nabla v\|_2^2 - \frac{8\pi}{\alpha^2}\log\int_\Omega e^{\alpha v}\right) + \frac{1}{2}\left(1-\frac{\lambda\alpha^2}{8\pi}\right)\|\nabla v\|_2^2\right\}\mathcal{P}(d\alpha) \\ &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{8\pi}\int_I \alpha^2\,\mathcal{P}(d\alpha)\right)\|\nabla v\|_2^2 - \lambda\log C_{TM} \end{aligned}$$

and hence the conclusion. □

Next, we derive an estimate for $\sup_{\alpha\in I} m(\alpha, p)$, using the mass identity (15).

Lemma 6.2. *Let $\{v_n\}$ be a solution sequence for* (10) *and let $p \in \mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ be a blow-up point. Then,*

$$\sup_{\alpha\in I} m(\alpha, p) \geqslant \bar{\lambda},$$

where we recall that $\bar{\lambda}$ is defined in (35)*. Moreover,*

$$\sup_{\alpha\in I} m(\alpha, p) > \bar{\lambda}$$

for all $p \in \mathcal{S}_+ \cap \mathcal{S}_-$.

Proof. Since $p \in \mathcal{S}$ is fixed, throughout this proof we put $m(\alpha, p) = m_\alpha$. Since $m_\alpha \geqslant 0$, we have

$$\begin{aligned} \left|\int_I \alpha m_\alpha\,\mathcal{P}(d\alpha)\right| &= \left|\int_{I_+} \alpha m_\alpha\,\mathcal{P}(d\alpha) - \int_{I_-} |\alpha| m_\alpha\,\mathcal{P}(d\alpha)\right| \\ &\leqslant \max\left\{\int_{I_+} |\alpha| m_\alpha\,\mathcal{P}(d\alpha),\ \int_{I_-} |\alpha| m_\alpha\,\mathcal{P}(d\alpha)\right\}. \end{aligned} \tag{42}$$

By Hölder's inequality, we have

$$\begin{aligned} \left(\int_{I_\pm} \alpha m_\alpha\,\mathcal{P}(d\alpha)\right)^2 &\leqslant \int_{I_\pm} \alpha^2\,\mathcal{P}(d\alpha)\int_{I_\pm} m_\alpha^2\,\mathcal{P}(d\alpha) \leqslant \sup_{I_\pm} m_\alpha \int_{I_\pm} \alpha^2\,\mathcal{P}(d\alpha)\int_{I_\pm} m_\alpha\,\mathcal{P}(d\alpha) \\ &\leqslant \sup_I m_\alpha \cdot \int_I m_\alpha\,\mathcal{P}(d\alpha) \cdot \int_{I_\pm} \alpha^2\,\mathcal{P}(d\alpha). \end{aligned} \tag{43}$$

From (42)–(43) we derive:

$$\left(\int_I \alpha m_\alpha\,\mathcal{P}(d\alpha)\right)^2 \leqslant \sup_{\alpha\in I} m_\alpha \cdot \int_I m_\alpha\,\mathcal{P}(d\alpha)\cdot \max\left\{\int_{I_+}\alpha^2\,\mathcal{P}(d\alpha), \int_{I_-}\alpha^2\,\mathcal{P}(d\alpha)\right\}.$$

Inserting this into the mass identity (15), we obtain

$$8\pi \int_I m_\alpha\,\mathcal{P}(d\alpha) \leqslant \sup_{\alpha\in I} m_\alpha \cdot \int_I m_\alpha\,\mathcal{P}(d\alpha)\cdot \max\left\{\int_{I_+}\alpha^2\,\mathcal{P}(d\alpha), \int_{I_-}\alpha^2\,\mathcal{P}(d\alpha)\right\}$$

and hence the first asserted estimate follows.

Now we suppose $p \in \mathcal{S}_+ \cap \mathcal{S}_-$ and we recall from Theorem 2.2(iv) that $n_{\pm,p} = \int_{I_\pm} |\alpha| m_\alpha\,\mathcal{P}(d\alpha)$, where $n_{\pm,p} \geqslant 4\pi$ are the masses defined in Theorem 2.1. Thus, the mass identity (15) may be written in the form

$$8\pi \int_I m_\alpha\,\mathcal{P}(d\alpha) = (n_{+,p} - n_{-,p})^2.$$

The strict inequality

$$|n_{+,p} - n_{-,p}| < \max\{n_{+,p}, n_{-,p}\}$$

is obvious. The same argument as above yields, keeping the strict inequality:

$$\begin{aligned} 8\pi \int_I m_\alpha\,\mathcal{P}(d\alpha) &< \max\left\{n_{+,p}^2, n_{-,p}^2\right\} \\ &= \max\left\{\left(\int_{I_+}|\alpha| m_\alpha\,\mathcal{P}(d\alpha)\right)^2, \left(\int_{I_+}|\alpha| m_\alpha\,\mathcal{P}(d\alpha)\right)^2\right\} \\ &\leqslant \max\left\{\int_{I_+}\alpha^2\,\mathcal{P}(d\alpha), \int_{I_-}\alpha^2\,\mathcal{P}(d\alpha)\right\}\cdot \int_I m_\alpha\,\mathcal{P}(d\alpha)\cdot \sup_{\alpha\in I} m_\alpha. \end{aligned}$$

We conclude that

$$\sup_{\alpha\in I} m_\alpha > \frac{8\pi}{\max\{\int_{I_+}\alpha^2\,\mathcal{P}(d\alpha), \int_{I_-}\alpha^2\,\mathcal{P}(d\alpha)\}} = \bar{\lambda}$$

for all $p \in \mathcal{S}_+ \cap \mathcal{S}_-$, as desired. □

In order to prove (36) we need the following.

Proposition 6.3. *There exist a sequence $\lambda_n \to \lambda^0$ and a solution sequence $\{v_n\} \subset \mathcal{E}$ to* (10) *such that $\|v_n\| \to +\infty$.*

We observe that

$$\inf_{\mathcal{E}} J_{t\lambda^0} > -\infty, \quad \text{for all } t \in (0,1),$$
$$\inf_{\mathcal{E}} J_{t\lambda^0} = -\infty, \quad \text{for all } t > 1.$$

Following ideas in [12,20], for every $\varepsilon \in (0,1)$ we introduce a "modified functional":

$$I_\varepsilon(v) = J_{(1-\varepsilon)\lambda^0}(v) - F\left(\frac{1}{2}\|v\|^2\right) = \frac{1}{2}\|v\|^2 - (1-\varepsilon)\lambda^0\mathcal{G}(v) - F\left(\frac{1}{2}\|v\|^2\right),$$

where

$$\mathcal{G}(v) = \int_I \left(\ln \int_\Omega e^{\alpha v}\,dx\right) d\mathcal{P}$$

and F is a suitable smooth function to be defined below. We shall prove that

$$\inf_{\mathcal{E}} I_0 = -\infty, \tag{44}$$
$$\inf_{\mathcal{E}} I_\varepsilon = I_\varepsilon(v_\varepsilon) > -\infty \quad \text{for some } v_\varepsilon \in \mathcal{E}. \tag{45}$$

The function F is defined using the following lemma from [12]:

Lemma 6.4. *(See [12, Lemma 4.4].) For any two sequences of non-negative real numbers $\{a_n\}$ and $\{b_n\}$ satisfying*

$$\lim_{n\to\infty} a_n = +\infty, \qquad \lim_{n\to\infty} \frac{b_n}{a_n} \leqslant 0$$

there exists a smooth concave function $F : [0,+\infty) \to \mathbb{R}$ such that $0 < F'(t) < 1$, $F'(t) \to 0$ as $t \to +\infty$ and $b_{n_k} - F(a_{n_k}) \to -\infty$ as $k \to \infty$ for some subsequence k.

Though it is not mentioned in [12, Lemma 4.4] that $F(t)$ is concave, it is clear from the proof.

We shall apply Lemma 6.4 with $a_n = \|v_n\|^2/2$ and $b_n = J_{\lambda^0}(v_n)$ for some suitable sequence v_n, as defined in the following.

Lemma 6.5. *There exists a sequence $\{v_n\} \subset \mathcal{E}$ such that*:

(i) $\lim_{n\to\infty} \|v_n\| = +\infty$,
(ii) $\lim_{n\to\infty} J_{\lambda^0}(v_n)/\|v_n\|^2 \leqslant 0$.

Proof. The proof is a consequence of the definition of λ^0, and of the general form of J. We first note that for every $0 < \delta < 1$ and for every $C > 0$ there exists $v \in \mathcal{E}$ such that

$$J_{\lambda^0}(v) < \frac{\delta}{2}\|v\|^2 - C.$$

Indeed, if not, there exist $\bar{\delta} \in (0,1)$ and $\bar{C} > 0$ such that

$$J_{\lambda^0}(v) \geqslant \frac{\bar{\delta}}{2}\|v\|^2 - \bar{C} \quad \forall v \in \mathcal{E}.$$

The above is equivalent to

$$\frac{1}{2}(1-\bar{\delta})\|v\|^2 - \lambda^0 \mathcal{G}(v) \geqslant -\bar{C} \quad \forall v \in \mathcal{E},$$

that is,

$$J_{\lambda^0/(1-\bar{\delta})}(v) \geqslant -\frac{\bar{C}}{1-\bar{\delta}} \quad \forall v \in \mathcal{E}.$$

Since $\lambda^0/(1-\bar{\delta}) > \lambda^0$, this contradicts the definition of λ^0. Now let $v_n \in \mathcal{E}$ satisfy

$$J_{\lambda^0}(v_n) < \frac{1}{2n}\|v_n\|^2 - n.$$

Note in particular that we have (ii).

Next we claim (i). Again, this is a consequence of the definition of λ^0. We fix $t \in (0,1)$ and denote

$$C(t) := \inf_{\mathcal{E}} J_{t\lambda^0} > -\infty.$$

We have

$$J_{\lambda^0}(v_n) = \frac{1}{t} J_{t\lambda^0}(v_n) + \frac{1}{2}\left(1 - \frac{1}{t}\right)\|v_n\|^2 \geqslant \frac{1}{t} C(t) - \frac{1-t}{2t}\|v_n\|^2.$$

Recalling the definition of v_n, it follows from the above that

$$\frac{1}{2n}\|v_n\|^2 - n > \frac{1}{t} C(t) - \frac{1-t}{2t}\|v_n\|^2.$$

That is,

$$\frac{1}{2}\left(\frac{1-t}{t} + \frac{1}{n}\right)\|v_n\|^2 > n + \frac{1}{t} C(t),$$

and the unboundedness of $\|v_n\|$ follows. □

At this point we set $a_n = \|v_n\|^2/2$, $b_n = J_{\lambda^0}(v_n)$, where v_n is the sequence defined in Lemma 6.5, and correspondingly we fix a function F, as given in Lemma 6.4. Here we recall that our F is concave and $t - F(t)$ is monotone non-decreasing. Therefore I_ε is weakly lower semi-continuous in $\mathcal{E}$. Now we prove the asserted properties (44)–(45) of I_ε.

Lemma 6.6. *The functional I_ε satisfies* (44)–(45)*.*

Proof. Property (44) follows readily from the definition of F. Indeed, we have $I_0(v_{n_k}) = b_{n_k} - F(a_{n_k}) \to -\infty$, where $\{n_k\}_k$ is the subsequence defined in Lemma 6.4. In order to prove (45), we fix $\sigma \in (0, \varepsilon)$. We note that in view of the properties of F there exists $C > 0$ such that $F(t) \leqslant \sigma t + C$ for all $t \geqslant 0$. Then,

$$I_\varepsilon(v) \geqslant \frac{1}{2}\|v\|^2 - (1-\varepsilon)\lambda^0 \mathcal{G}(v) - \frac{\sigma}{2}\|v\|^2 - C = (1-\sigma) J_{(1-\varepsilon)\lambda^0/(1-\sigma)}(v) - C.$$

Since $(1-\varepsilon)\lambda^0/(1-\sigma) < \lambda^0$, it follows that I_ε is coercive and bounded below. Therefore we get a minimizer since I_ε is weakly lower semi-continuous. □

Proof of Proposition 6.3. Let $\varepsilon_n \to 0$ and let $v_n \in \mathcal{E}$ be a minimizer of I_{ε_n}. We note that v_n satisfies Eq. (9) with $\lambda = \lambda_n$, where

$$\lambda_n = \frac{1-\varepsilon_n}{1-e_n}\lambda^0,$$

and

$$e_n = F'\left(\frac{1}{2}\|v_n\|^2\right).$$

We claim that

$$\|v_n\| \to +\infty. \tag{46}$$

Indeed, if not, there exists $v_\infty \in \mathcal{E}$ such that $v_n \rightharpoonup v_\infty$ weakly in $\mathcal{E}$, strongly in L^p for all $p \geqslant 1$ and a.e. Since $I_\varepsilon(v)$ is monotone non-decreasing in ε for fixed $v \in \mathcal{E}$ and I_0 is weakly lower semi-continuous in $\mathcal{E}$, it holds that

$$\liminf_{n\to\infty} I_{\varepsilon_n}(v_n) \geqslant \liminf_{n\to\infty} I_0(v_n) \geqslant I_0(v_\infty) > -\infty.$$

Since I_0 is unbounded below, there exists $v \in \mathcal{E}$ such that $I_0(v) < I_0(v_\infty)$. Set $\sigma = I_0(v_\infty) - I_0(v)$. Then for some large n, it follows that

$$I_{\varepsilon_n}(v) = I_0(v) + \varepsilon_n\lambda^0\mathcal{G}(v) < I_0(v_\infty) - \frac{\sigma}{2} \leqslant I_{\varepsilon_n}(v_n).$$

This contradicts the minimizing property of v_n, and therefore (46) is established. On the other hand, if (46) holds, then $e_n \to 0$ and $\lambda_n \to \lambda^0$. □

Proof of Theorem 2.3. As outlined in the beginning of this section, we divide the proof into showing two steps (36) and (37). The existence of a sequence of solutions obtained in Proposition 6.3 guarantees (36). Indeed from the property $\|v_n\| \to \infty$, the solution sequence $\{v_n\}$ cannot be compact in $\mathcal{E}$. Therefore the blow-up set $\mathcal{S}$ for this sequence is not empty in view of Theorem 2.1. Let $p \in \mathcal{S}$. We have

$$\lambda^0 \geqslant \sup_{\alpha\in I} m(\alpha, p) \geqslant \bar{\lambda}$$

from (14) and Lemma 6.2. Therefore we have

$$\lambda^0 \geqslant \bar{\lambda} = \frac{8\pi}{\max\{\int_{I_+}\alpha^2\,d\mathcal{P}, \int_{I_-}\alpha^2\,d\mathcal{P}\}},$$

and (36) is established.

In order to prove (37), we note that $J_{t\lambda^0}$ is coercive on $\mathcal{E}$ if $t \in (0,1)$. Indeed, we choose $\varepsilon > 0$ such that $t/(1-\varepsilon) < 1$. Then, it holds that

$$J_{t\lambda^0}(v)=\frac{1}{2}\|v\|_{\mathcal{E}}^2-t\lambda^0\mathcal{G}(v)=\frac{\varepsilon}{2}\|v\|_{\mathcal{E}}^2+(1-\varepsilon)\left[\frac{1}{2}\|v\|_{\mathcal{E}}^2-\frac{t}{1-\varepsilon}\lambda^0\mathcal{G}(v)\right]$$
$$=\frac{\varepsilon}{2}\|v\|_{\mathcal{E}}^2+(1-\varepsilon)J_{t\lambda^0/(1-\varepsilon)}(v)\geqslant\frac{\varepsilon}{2}\|v\|_{\mathcal{E}}^2+(1-\varepsilon)\inf_{\mathcal{E}}J_{t\lambda^0/(1-\varepsilon)}$$

and hence $J_{t\lambda^0}$ is coercive. Therefore, given $\lambda_n\uparrow\lambda^0$, we obtain $\underline{v}_n\in\mathcal{E}$ such that

$$J_{\lambda_n}(\underline{v}_n)=\inf_{\mathcal{E}}J_{\lambda_n}$$

by standard arguments. This $\{\underline{v}_n\}$ is a solution sequence for (10). Let $\nu_{\pm,n}$ be the measures defined in (11) with $v=\underline{v}_n$ and denote by $\tilde{u}_{\pm,n}$ the "positive" and the "negative" parts of $\underline{v}_n$, namely $\tilde{u}_{\pm,n}=G\star\nu_{\pm,n}$. We have

$$J_{\lambda_n}(\underline{v}_n)=\frac{1}{2}\int_\Omega|\nabla\tilde{u}_{+,n}|^2+\frac{1}{2}\int_\Omega|\nabla\tilde{u}_{-,n}|^2-\int_\Omega\nabla\tilde{u}_{+,n}\cdot\nabla\tilde{u}_{-,n}$$
$$-\lambda_n\int_{I_+}\log\left(\int_\Omega e^{\alpha(\tilde{u}_{+,n}-\tilde{u}_{-,n})}\right)\mathcal{P}(d\alpha)$$
$$-\lambda_n\int_{I_-}\log\left(\int_\Omega e^{|\alpha|(\tilde{u}_{-,n}-\tilde{u}_{+,n})}\right)\mathcal{P}(d\alpha).$$

Since $G(x,x')$ is bounded below, it follows that

$$J_{\lambda_n}(\underline{v}_n)\geqslant\frac{1}{2}\int_\Omega|\nabla\tilde{u}_{+,n}|^2+\frac{1}{2}\int_\Omega|\nabla\tilde{u}_{-,n}|^2-\int_\Omega\nabla\tilde{u}_{+,n}\cdot\nabla\tilde{u}_{-,n}$$
$$-\lambda_n\int_{I_+}\log\left(\int_\Omega e^{\alpha(\tilde{u}_{+,n}+C_{12})}\right)\mathcal{P}(d\alpha)$$
$$-\lambda_n\int_{I_-}\log\left(\int_\Omega e^{|\alpha|(\tilde{u}_{-,n}+C_{12})}\right)\mathcal{P}(d\alpha)$$
$$\geqslant J_{+,\lambda_n}(\tilde{u}_{+,n})+J_{-,\lambda_n}(\tilde{u}_{-,n})-\int_\Omega\nabla\tilde{u}_{+,n}\cdot\nabla\tilde{u}_{-,n}-C_{16}$$

with $J_{\pm,\lambda}(v)$ defined by (39). Since we are assuming $\lambda^0=\bar{\lambda}$ and $\lambda_n\uparrow\lambda^0$, we have $\lambda_n\leqslant\frac{8\pi}{\int_{I_\pm}\alpha^2\mathcal{P}(d\alpha)}$. Therefore,

$$J_{\pm,\lambda_n}(v)\geqslant-C_{17},\quad v\in\mathcal{E},$$

similarly to Lemma 6.1. The proof is thus reduced to

$$\left|\int_\Omega\nabla\tilde{u}_{+,n}\cdot\nabla\tilde{u}_{-,n}\right|\leqslant C_{18}. \tag{47}$$

In view of Lemma 6.2, $\mathcal{S}_+ \cap \mathcal{S}_- = \emptyset$ since $\lambda^0 = \bar{\lambda}$. Now, we take $r > 0$ such that $\bigcup_{p\in\mathcal{S}_+} B(p,r) \cap \mathcal{S}_- = \emptyset$. We have $\|\nu_{\pm,n}\|_1 \leqslant \lambda_0 + 1$ and $\|\tilde{u}_{\pm,n}\|_{W^{1,q}(\Omega)} \leqslant C_{19}$ by the L^1-estimate, see [3], and also $\{\nu_{\pm,n}\}$ and $\{\tilde{u}_{\pm,n}\}$ are locally uniformly bounded in $\Omega \setminus \mathcal{S}_\pm$ by (22). Writing

$$\int_\Omega \nabla\tilde{u}_{+,n} \cdot \nabla\tilde{u}_{-,n} = \int_\Omega \tilde{u}_{-,n}\nu_{+,n} = \int_{\bigcup_{p\in\mathcal{S}_+} B(p,r)} \tilde{u}_{-,n}\nu_{+,n} + \int_{\Omega\setminus\bigcup_{p\in\mathcal{S}_+} B(p,r)} \tilde{u}_{-,n}\nu_{+,n},$$

we obtain (47) and the proof of (37) is complete. Hence, Theorem 2.3 is completely established. □

7. Remarks on sharpness

As already mentioned, Theorem 2.3 is optimal when $\mathcal{P} = \tau\delta_\alpha + (1-\tau)\delta_{-\beta}$, $\tau, \alpha, \beta \in [0,1]$. In general, however, we cannot expect Theorem 2.3 to be sharp for every $\mathcal{P}$, in view of the following result which is derived using some dual inequalities from [26,27]. Such a result leads us to conjecture that condition (48) below should be optimal for every choice of $\mathcal{P}$.

Theorem 7.1 *(Discrete case). If $\mathcal{P}(d\alpha)$ is a finite sum of delta functions, then $J_\lambda(v)$ defined by* (8) *for $v \in \mathcal{E}$ is bounded below if*

$$\lambda \leqslant \inf\left\{ \frac{8\pi\mathcal{P}(K_\pm)}{(\int_{K_\pm} \alpha\,\mathcal{P}(d\alpha))^2} \;\middle|\; K_\pm \subset I_\pm \cap \operatorname{supp}\mathcal{P} \right\} \tag{48}$$

when $\mathcal{P} \neq \delta_0$ and for all $\lambda > 0$ if $\mathcal{P} = \delta_0$, where $I_- = [-1, 0)$ and $I_+ = (0, 1]$.

Proof. We rewrite

$$\mathcal{P} = m_0\delta_0 + \sum_{\alpha_i \neq 0} m_i\delta_{\alpha_i}. \tag{49}$$

The assertion is obvious if $m_0 = 1$. For the moment, we take the case $m_0 = 0$ because we can use

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda(1-m_0)\int_{[-1,1]\setminus\{0\}} \left(\log\int_\Omega e^{\alpha v}\right)\frac{\mathcal{P}(d\alpha)}{1-m_0} - \lambda m_0 \log|\Omega| \tag{50}$$

for the other case. Thus we assume

$$m_0 = 0, \qquad -1 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_L < 0 < \alpha_{L+1} \leqslant \cdots \leqslant \alpha_N \leqslant 1, \tag{51}$$

$$m_i > 0, \quad 1 \leqslant i \leqslant N, \quad \sum_{i=1}^N m_i = 1 \tag{52}$$

in (49).

Let

$$\mathcal{J}(w) = \frac{1}{2}\sum_{i,j\in\mathcal{B}} a_{ij}\int_\Omega \nabla w_i \cdot \nabla w_j - \sum_{i\in\mathcal{B}} M_i \log\int_\Omega \exp\left(\sum_{j\in\mathcal{B}} a_{ij}w_j\right)$$

be given, where $\mathcal{B} = \{1, \ldots, N\}$, $w = (w_i)$, and $w_i \in \mathcal{E}$. We assume that $a_{ij} = a_{ji}$ for $i, j \in \mathcal{B}$, $\mathcal{B}$ is a disjoint union of $\mathcal{B}_\ell$ for $\ell = 1, \ldots, k$, and $a_{ij} \geqslant 0$ for $i, j \in \mathcal{B}_\ell$, $\ell = 1, \ldots, k$. We assume, furthermore, $a_{ij} \leqslant 0$ for $i \in \mathcal{B}_\ell$, $j \in \mathcal{B}_m$, $\ell \neq m$, $1 \leqslant \ell, m \leqslant k$. For this functional, the following facts are known. If (a_{ij}) is positive definite, then $\mathcal{J}$ is bounded below if and only if (i) $A_\mathcal{K} \geqslant 0$ for $\emptyset \neq \mathcal{K} \subset \mathcal{B}_\ell$, where $\ell = 1, \ldots, k$ and

$$A_\mathcal{K} = 8\pi \sum_{i \in \mathcal{K}} M_i - \sum_{i,j \in \mathcal{K}} a_{ij} M_i M_j,$$

and (ii) in case $A_\mathcal{K} = 0$ it holds that $a_{ii} + A_{\mathcal{K} \setminus \{i\}} > 0$ for each $i \in \mathcal{K}$. Furthermore, the "if" part of the above assertion is valid even when (a_{ij}) is only non-negative definite. These results are proven for $\Omega = S^2$ in [26] but are also valid in the general case of Ω in view of the facts shown in the subsequent article [27] concerning the case $a_{ij} \geqslant 0$ for every i and j.

Given (49) with (51)–(52), we see that $\mathcal{B} = \{1, \ldots, N\}$ is a disjoint union of $\mathcal{B}_1 = \{1, \ldots, L\}$ and $\mathcal{B}_2 = \{L+1, \ldots, N\}$, and $A = (a_{ij})$, $a_{ij} = \alpha_i \alpha_j$ satisfies the above requirement with $k = 2$. Putting

$$w_i = v/(\alpha_i N), \qquad M_i = \lambda m_i,$$

furthermore, we have $\mathcal{J}(w) = J_\lambda(v)$. Then the above defined control functional $A_\mathcal{K}$, $\mathcal{K} \subset \mathcal{A} \cap I_\pm$, takes the form

$$A_\mathcal{K} = 8\pi\lambda \sum_{\alpha_i \in \mathcal{K}} m_i - \sum_{\alpha_i, \alpha_j \in \mathcal{K}} a_{ij} \lambda^2 m_i m_j = 8\pi\lambda \mathcal{P}(\mathcal{K}) - \lambda^2 \left(\int_\mathcal{K} \alpha \, \mathcal{P}(d\alpha) \right)^2,$$

and, therefore, it holds that $A_\mathcal{K} \geqslant 0$ by (48). The requirement $a_{ii} + A_{\mathcal{K} \setminus \{i\}} > 0$, $i \in \mathcal{K}$, for the residual case $A_\mathcal{K} = 0$ is always cleared because of $a_{ii} = \alpha_i^2 > 0$. Inequality (48) thus guarantees all the requirements of [26,27], and hence $J_\lambda(v)$, $v \in \mathcal{E}$, is bounded below.

Even in case $m_0 \neq 0, 1$, we can apply the above result, using (50). Thus $J_\lambda(v)$, $v \in \mathcal{E}$, is bounded below if

$$(1 - m_0)\lambda \leqslant \inf \left\{ \frac{8\pi \frac{\mathcal{P}(K_\pm)}{1 - m_0}}{(\int_{K_\pm} \alpha \frac{\mathcal{P}(d\alpha)}{1 - m_0})^2} \;\middle|\; K_\pm \subset I_\pm \cap \operatorname{supp} \mathcal{P} \right\}.$$

This inequality is equivalent to (48) and the proof is complete. □

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