Sharp Hölder exponent for some elliptic equations in two variables

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Let Ω be a bounded open subset of $\mathbb{R}^2$. We consider weak solutions $u \in H^1_{\mathrm{loc}}(\Omega)$ to the equation:

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \text{in } \Omega. \tag{1}$$

Here A is a 2×2 matrix whose entries are bounded, real measurable functions defined on Ω. Furthermore, we assume that A satisfies the following conditions:

$$\text{ellipticity :} \qquad \lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \tag{2}$$

$$\text{symmetry :} \qquad A(x) = A(x)^* \tag{3}$$

$$\text{unit determinant :} \qquad \det A(x) = 1 \tag{4}$$

for all $x \in \Omega$, for all $\xi \in \mathbb{R}^2$ and for some $0 < \lambda \le \Lambda$. Condition (4) is relevant in the context of quasiconformal mappings, see [2].

It is well-known [1, 3, 4] that assumption (2) implies the Hölder continuity of solutions to (1). More precisely, there exists $0 < \alpha < 1$ such that for every compact subset $K \Subset \Omega$ there holds

$$\sup_{x,y\in K, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < +\infty. \tag{5}$$

Here and in what follows, for every measurable function f we denote by $\sup f$ the essential upper bound of f. In [5] it is shown that the optimal value for α is $L^{-1/2}$, where $L = \Lambda/\lambda$ is the ellipticity constant. In [5] it is also shown that the optimal value of α increases if A is of the form $A = a(x)I$ for some bounded measurable function a.

These results motivate the following question:

Question: *What is the optimal value of* α *in* (5) *under assumptions* (2)–(3)–(4)*?*

In answer to the question above, in a recent note [6] we obtain a sharp integral estimate for α. More precisely, we establish the following

Theorem 1 ([6]). *Let* A *satisfy* (2), (3) *and* (4) *in* Ω *and let* $u \in H^1_{\mathrm{loc}}(\Omega)$ *satisfy* (1). *Then the least upper bound for the admissible values of the Hölder exponent for* u *is given by*

$$\bar{\alpha} = 2\pi \left(\sup_{x_0 \in \Omega} \inf_{0<r_0<d(x_0)} \sup_{0<r<r_0} \int_{|\xi|=1} \langle A(x_0 + r\xi)\xi, \xi \rangle \right)^{-1}. \tag{6}$$

Here $d(x_0) = \mathrm{dist}(x_0, \partial\Omega)$. We note that under assumption (4), we may choose $\lambda = \Lambda^{-1}$ in (2) and therefore the ellipticity constant takes the value $L = \Lambda^2$. Hence, the estimate obtained in [5] yields in this case $\alpha = L^{-1/2} = \Lambda^{-1}$. On the other hand, recalling that $\Lambda = \sup_{x\in\Omega} \sup_{|\xi|=1} \langle A(x)\xi, \xi \rangle$, it is clear that $\bar{\alpha} \geq \Lambda^{-1}$.

Theorem 1 is *sharp*, in the sense of the following

Example. Let $\Omega = B$ the unit ball in $\mathbb{R}^2$, let $\theta = \arg x$ and let

$$A(x) = \frac{1}{k(\theta)} I + \left(k(\theta) - \frac{1}{k(\theta)} \right) \frac{x \otimes x}{|x|^2} \qquad \text{in } B \setminus \{0\}, \tag{7}$$

where $k : \mathbb{R} \to \mathbb{R}^+$ is smooth 2π-periodic function bounded from above and below away from 0. Equivalently, setting $K(\theta) = \mathrm{diag}(k(\theta), 1/k(\theta))$, we may write

$$\begin{aligned} A(x) =& J(\theta) K(\theta) J^*(\theta) \\ =& \begin{pmatrix} k(\theta)\cos^2\theta + \frac{1}{k(\theta)}\sin^2\theta & \left(k(\theta) - \frac{1}{k(\theta)}\right)\sin\theta\cos\theta \\ \left(k(\theta) - \frac{1}{k(\theta)}\right)\sin\theta\cos\theta & k(\theta)\sin^2\theta + \frac{1}{k(\theta)}\cos^2\theta \end{pmatrix}, \end{aligned}$$

where

$$J(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{8}$$

Clearly, $\det A(x) \equiv 1$. By a suitable choice of k, we may obtain that

$$\bar{\alpha} = 2\pi \left(\int_{|\xi|=1} \langle A(r\xi)\xi, \xi \rangle \right)^{-1} = 2\pi \left(\int_0^{2\pi} k \right)^{-1}.$$

On the other hand the function $u \in H^1(B)$ defined by

$$u(x) = |x|^{\bar{\alpha}} \cos\left(\bar{\alpha}\int_0^{\arg x} k\right) \tag{9}$$

satisfies equation (1) with A given by (7). It is readily verified that the Hölder exponent of u is exactly $\bar{\alpha}$.

Sketch of the proof of Theorem 1. The point of the proof is to show that

$$\sup_{0<r<d(x_0)} r^{-2\alpha} \int_{|x-x_0|<r} \langle A\nabla u, \nabla u\rangle < +\infty, \tag{10}$$

for every $x_0 \in \Omega$ and for every $0 < \alpha < \bar{\alpha}$. Once estimate (10) is established, Theorem 1 follows by the well-known regularity results of Morrey [3]. In order to derive (10), we exploit some ideas in [5]. We set

$$g_{x_0}(r) = \int_{|x-x_0|<r} \langle A\nabla u, \nabla u\rangle.$$

Then, by (1) and the divergence theorem:

$$g_{x_0}(r) = \int_{S_r} (u-\mu)\langle A\nabla u, n\rangle = \int_{S_r} (u-\mu)\langle P\overline{\nabla} u, e_1\rangle,$$

where $x = x_0 + \rho e^{i\theta}$, $P = J^*AJ$, J is the rotation matrix defined in (8), $\overline{\nabla} u = (\partial_\rho u, \rho^{-1}\partial_\theta u)$, S_r is the circle of radius r centered at x_0, n is the outward normal to S_r and μ is any constant. By Hölder's inequality,

$$g_{x_0}(r) \le \left(\int_{S_r} p_{11}(u-\mu)^2\right)^{1/2} \left(\int_{S_r} \frac{\langle P\overline{\nabla} u, e_1\rangle^2}{p_{11}}\right)^{1/2}.$$

At this point, we observe that any 2×2 symmetric matrix B such that $b_{11} \neq 0$ satisfies the following identity:

$$\langle B\xi, \xi\rangle = \frac{\langle B\xi, e_1\rangle^2}{b_{11}} + \frac{\det B}{b_{11}}\langle \xi, e_2\rangle^2, \tag{11}$$

for any $\xi \in \mathbb{R}^2$. Let $C_P = C_P(x_0, r) > 0$ be the best constant in the

weighted Wirtinger inequality :

$$\int_0^{2\pi} p_{11}(x_0 + re^{i\theta}) w^2(\theta)\, \mathrm{d}\theta \le C_P \int_0^{2\pi} \frac{\det P}{p_{11}}(x_0 + re^{i\theta}) w'^2(\theta)\, \mathrm{d}\theta, \tag{12}$$

where $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ is 2π-periodic function such that

$$\int_0^{2\pi} p_{11}(x_0 + re^{i\theta}) w(\theta)\, \mathrm{d}\theta = 0.$$

(For ease of future reference, we do not use the assumption $\det P \equiv 1$ in the next few estimates). Then, by inequality (12) with

$$w(\theta) = u(x_0 + re^{i\theta}) - \mu, \qquad \mu = \frac{1}{2\pi} \int_0^{2\pi} p_{11}(x_0 + re^{i\theta}) u(x_0 + re^{i\theta})\, \mathrm{d}\theta,$$

we derive

$$g_{x_0}(r) \le C_P^{1/2} \left(\int_{S_r} \frac{\det P}{p_{11}} (x_0 + re^{i\theta})(\partial_\theta u)^2 \right)^{1/2} \left(\int_{S_r} \frac{\langle P\overline{\nabla} u, e_1\rangle^2}{p_{11}} \right)^{1/2}.$$

Recalling that $\partial_\theta u / r = (\overline{\nabla} u)_{22}$ we obtain, in view of the elementary inequality $\sqrt{ab} \le (a+b)/2$ and the identity (11) with $B = P$ and $\xi = \overline{\nabla} u$:

$$\begin{aligned}
g_{x_0}(r) \le & C_P^{1/2} r \left(\int_{S_r} \frac{\det P}{p_{11}} \left(\frac{\partial_\theta u}{\rho} \right)^2 \right)^{1/2} \left(\int_{S_r} \frac{\langle P\overline{\nabla} u, e_1\rangle^2}{p_{11}} \right)^{1/2} \\
\le & C_P^{1/2} r \left(\int_{S_r} \frac{\det P}{p_{11}} \left(\overline{\nabla} u\right)_{22}^2 \right)^{1/2} \left(\int_{S_r} \frac{\langle P\overline{\nabla} u, e_1\rangle^2}{p_{11}} \right)^{1/2} \\
\le & \frac{C_P^{1/2} r}{2} \int_{S_r} \left(\frac{\det P}{p_{11}} \left(\overline{\nabla} u\right)_{22}^2 + \frac{\langle P\overline{\nabla} u, e_1\rangle^2}{p_{11}} \right) \\
= & \frac{C_P^{1/2} r}{2} \int_{S_r} \langle P\overline{\nabla} u, \overline{\nabla} u\rangle = \frac{C_P^{1/2} r}{2} \int_{S_r} \langle A\nabla u, \nabla u\rangle.
\end{aligned}$$

Recalling the definition of g_{x_0}, we have that the above inequality is equivalent to:

$$g_{x_0}(r) \le \frac{C_P^{1/2}(x_0, r) r}{2} g'_{x_0}(r),$$

for almost every $0 < r < d(x_0)$. In turn, the above inequality implies that $\ln(r^{-2/\gamma} g_{x_0}(r))$ is non-decreasing, for every $\gamma \ge C_P^{-1/2}(x_0, r)$. At this point it is clear that (10) holds, with

$$\bar{\alpha}^{-1} = \sup_{x_0 \in \Omega} \inf_{0 < r_0 < d(x_0)} \sup_{0 < r < r_0} C_P(x_0, r)^{-1/2}.$$

In order to conclude the proof, we note that when $\det P \equiv 1$, the sharp constant in the generalized Wirtinger inequality (12) is given by

$$C_P(x_0, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} p_{11}(x_0 + re^{i\theta})\, \mathrm{d}\theta \right)^2 = \left(\frac{1}{2\pi} \int_{|\xi|=1} \langle A(x_0 + r\xi)\xi, \xi\rangle \right)^2.$$

This fact may be seen by a change of variables. □

We note that

Remark 1. *The functions of the form* (9) *may be of interest in the context of quasiconformal mappings.*

Indeed, let $\kappa : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic smooth function such that $\kappa \geq 1$. For $z \in \mathbb{C} \setminus \{0\}$ we define the mapping

$$f(z) = |z|^{1/\bar{\kappa}} \exp\left\{ \frac{i}{\bar{\kappa}} \int_0^{\arg z} \kappa \right\},$$

where $\bar{\kappa} = (2\pi)^{-1} \int_0^{2\pi} \kappa$. A computation shows that f satisfies the bounded distorsion *equality*

$$|Df(z)|^2 = \kappa(\arg z) J_f(z), \tag{13}$$

for all $z \in \mathbb{C} \setminus \{0\}$.

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References

[1] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Acc. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

[2] T. Iwaniec and C. Sbordone, Quasiharmonic fields, Ann. I.H.P. **18** No. 5 (2001), 519–572.

[3] C.B. Morrey, On the solutions of quasi-linar elliptic partial differential equations, Trans. Am. Math. Soc. **43** No. 1 (1938), 126–166.

[4] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. **80** No. 5 (1958), 931–954.

[5] L.C. Piccinini and S. Spagnolo, On the Hölder continuity of solutions of second order elliptic equations in two variables, Ann. Scuola Norm. Sup. Pisa **26** No. 2 (1972), 391–402.

[6] T. Ricciardi, A sharp Hölder estimate for elliptic equations in two variables, Proc. Roy. Soc. Edinburgh A, to appear. Preprint available on arXiv:math.AP/0406495.