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Some nonlinear elliptic problems from Maxwell-Chern-Simons vortex theory

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Introduction

The analysis of vortex solutions for self-dual Maxwell-Chern-Simons models may be generally reduced to the analysis of systems of two nonlinear elliptic equations, defined on 2-dimensional Riemannian manifolds, see [8, 11, 21]. In turn, these systems are equivalent to a scalar elliptic equation of the fourth order. In this note, we shall review our results in [18, 15, 16, 17] concerning the existence and asymptotics of solutions to such elliptic problems. We consider the case when the underlying manifold is compact.

More precisely, we begin by outlining our joint results with Tarantello on the self-dual $U(1)$ Maxwell-Chern-Simons model introduced in [12]. Motivated by the work of Chae and Nam [5] concerning the self-dual $CP(1)$ Maxwell-Chern-Simons model introduced in [7], we construct a general elliptic system which includes the $U(1)$ system and the $CP(1)$ system as special cases. We then outline the asymptotic analysis carried out in [16], which provides a unified proof of the asymptotics derived in [18] for the $U(1)$ system and in [5] for the $CP(1)$ system. Finally, we outline our proof in [17] of multiplicity of solutions for the general system. This result in particular implies multiplicity for the $CP(1)$ system, improving the existence result in [5].

1 The $U(1)$ system

We denote by M a compact Riemannian 2-manifold and we fix $n > 0$ points $p_1, \ldots, p_n \in M$. The system for vortex solutions for the $U(1)$ model introduced

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in [12] is given by:

$$\Delta\widetilde{u} = 2q^2 e^{\widetilde{u}} - 2\mu N + 4\pi \sum_{j=1}^{n} \delta_{p_j} \qquad \text{on } M$$

$$\Delta N = (\mu^2 + 2q^2 e^{\widetilde{u}})N - q^2(\mu + \frac{2q^2}{\mu})e^{\widetilde{u}} \qquad \text{on } M,$$

where $(\widetilde{u}, N)$ is the unknown pair of functions and $q, \mu > 0$ are constants. Setting $\lambda = 2q^2/\mu$, $\varepsilon = 1/\mu$, $v := \mu/q^2\, N$, the above system takes the form:

$$-\Delta\widetilde{u} = \varepsilon^{-1}\lambda(v - e^{\widetilde{u}}) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \qquad \text{on } M \tag{1}$$

$$-\Delta v = \varepsilon^{-1}\{\lambda e^{\widetilde{u}}(1 - v) - \varepsilon^{-1}(v - e^{\widetilde{u}})\} \qquad \text{on } M. \tag{2}$$

The following results were obtained in [18]:

Theorem 1.1 ([18]). *There exists $\kappa_* \in (0, \frac{1}{2}\sqrt{\frac{|M|}{\pi n}})$ such that if ε, λ satisfy $\lambda > \kappa_*^{-1}$ and $0 < \varepsilon < \frac{|M|}{2\pi n}(\lambda - \kappa_*^{-1})$, then there exist at least two solutions for system* (1)–(2).

It is of both mathematical and physical interest to consider the asymptotic behavior of solutions to (1)–(2) as $\varepsilon \to 0$ with λ fixed. By Theorem 1.1, such a limit is meaningful. We have:

Theorem 1.2 ([15]). *Let $(\widetilde{u}, v)$ be a sequence of solutions to* (1)–(2) *with λ fixed and $\varepsilon \to 0$. Then there exists a solution $\widetilde{u}_0$ for the equation:*

$$-\Delta\widetilde{u}_0 = \lambda^2 e^{\widetilde{u}_0}(1 - e^{\widetilde{u}_0}) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \qquad \text{on } M \tag{3}$$

such that $(e^{\widetilde{u}}, v) \to (e^{\widetilde{u}_0}, e^{\widetilde{u}_0})$ in $C^k(M) \times C^k(M)$, for any $k \geq 0$.

We note that $e^{\widetilde{u}}$ and $e^{\widetilde{u}_0}$ are smooth on M. Weaker versions of Theorem 1.2 were obtained in [18] and [4]. We note that (3), known as the Chern-Simons equation, has been widely investigated, see [3, 6, 14, 20] and references therein.

We refer to [12, 8, 18] for the derivation of (1)–(2) from the physics model. We shall only make a few considerations concerning the physical origin of (1)–(2), in order to motivate the above results. The physically relevant quantity in (1)–(2) is given by $e^{\widetilde{u}}$, and it represents a density. We note that $e^{\widetilde{u}}$ is smooth on M, and it vanishes exactly at p_j, $j = 1, \ldots, n$ (the "vortex points"). Solutions to (1)–(2) correspond to vortex-type solutions for the Euler-Lagrange equations for a lagrangian $\mathcal{L}_{\varepsilon,\lambda}$ of the form:

$$\mathcal{L}_{\varepsilon,\lambda} = \frac{\varepsilon}{\lambda}\mathcal{L}_{\text{Maxwell}} + \frac{1}{\lambda}\mathcal{L}_{\text{Chern-Simons}} + \text{kinetic terms} + V_{\varepsilon,\lambda}.$$

Such vortex solutions are time-independent, and they are obtained by a reduction mainly due to Bogomol'nyi and Taubes (see [11, 12]), which exploits the self-dual structure of $\mathcal{L}_{\varepsilon,\lambda}$. The potential $V_{\varepsilon,\lambda}$ admits two vacuum states, namely $(e^{\widetilde{u}}, v) = (1, 1)$ and the "degenerate" state $(e^{\widetilde{u}}, v) = (0, 0)$. This multiplicity of vacuum states explains the multiplicity of solutions as in Theorem 1.1 below. We denote by $\mathcal{L}_{0,\lambda}$ the lagrangian obtained by setting $\varepsilon = 0$ in $\mathcal{L}_{\varepsilon,\lambda}$. $\mathcal{L}_{0,\lambda}$ corresponds to the self-dual Chern-Simons model introduced in [10, 9]. Solutions to (3) correspond to vortex solutions for $\mathcal{L}_{0,\lambda}$. Thus, Theorem 1.2 provides a rigorous proof of the fact that $\mathcal{L}_{\varepsilon,\lambda}$ "tends" to $\mathcal{L}_{0,\lambda}$ as $\varepsilon \to 0$. We note that in the limit $\varepsilon \to 0$ with λ fixed, the Maxwell action $\mathcal{L}_{\text{Maxwell}}$ in $\mathcal{L}_{\varepsilon,\lambda}$ drops out of the lagrangian. Since $\mathcal{L}_{\text{Maxwell}}$ is of higher order with respect to $\mathcal{L}_{\text{Chern-Simons}}$, the resulting system (1)–(2) is of the singular perturbation type, and the estimates are somewhat delicate.

Sketch of the proof of Theorem 1.1. The proof is variational. Setting $\widetilde{u} = \sigma + u$, where σ is the Green function uniquely defined by

$$-\Delta\sigma = 4\pi\left(\frac{n}{|M|} - \sum_{j=1}^{n}\delta_{p_j}\right)$$

$$\int_M \sigma = 0,$$

it is clear that (1)–(2) is equivalent to:

$$-\Delta u = \varepsilon^{-1}\lambda(v - e^{\sigma+u}) - \frac{4\pi n}{|M|} \qquad \text{on } M \tag{4}$$

$$-\Delta v = \varepsilon^{-1}\{\lambda e^{\sigma+u}(1-v) - \varepsilon^{-1}(v - e^{\sigma+u})\} \qquad \text{on } M. \tag{5}$$

Solving (4) for v and inserting into (5), we find that u satisfies:

$$\varepsilon^2\Delta^2 u - \Delta u = -\varepsilon\lambda e^{\sigma+u}|\nabla(\sigma+u)|^2 + 2\varepsilon\lambda\Delta e^{\sigma+u} + \lambda^2 e^{\sigma+u}(1 - e^{\sigma+u}) - \frac{4\pi n}{|M|}. \tag{6}$$

Equation (6) has a variational structure. Indeed, solutions to (6) correspond to critical points in $H^2(M)$ for the functional:

$$I(u) = \frac{\varepsilon^2}{2}\int|\Delta u|^2 + \frac{1}{2}\int|\nabla u|^2 + \varepsilon\lambda\int e^{\sigma+u}|\nabla(\sigma+u)|^2 + \frac{\lambda^2}{2}\int(e^{\sigma+u})^2 + \frac{4\pi n}{|M|}\int u.$$

The proof of Theorem 1.1 consists in finding a local minimum and a "mountain pass" for I. The local minimum is obtained by exploiting a natural integral constraint for (6). Indeed, setting $u = w + c$ with $\int w = 0$, $c \in \mathbb{R}$ and integrating (6), we have that w is constrained to satisfy $w \in \mathcal{A}$, where

$$\mathcal{A} = \left\{ w \in H^2(M) \Big/ \begin{array}{c} \int w = 0 \\ (\int e^{\sigma+w} - \frac{\varepsilon}{\lambda}\int e^{\sigma+w}|\nabla(\sigma+w)|)^2 - \frac{16\pi n}{\lambda^2}\int e^{2(\sigma+w)} \geq 0 \end{array} \right\}$$

and c is constrained to take one of the values defined by:

$$e^{c_\pm(w)} = \left(2\int e^{2(\sigma+w)}\right)^{-1}\left\{\int e^{\sigma+w} - \varepsilon\lambda^{-1}\int e^{\sigma+w}|\nabla(\sigma+w)|^2 \pm\sqrt{\left(\int e^{\sigma+w} - \varepsilon\lambda^{-1}\int e^{\sigma+w}|\nabla(\sigma+w)|\right)^2 - \frac{16\pi n}{\lambda^2}\int e^{2(\sigma+w)}}\right\}.$$

We verify that the functional J_+ defined on $\mathcal{A}$ by $J_+(w) = I(w + c_+(w))$ is bounded below and coercive on $\mathcal{A}$, and for the values of ε, λ as in Theorem 1.1 its minimum yields a local minimum for I. Since $I(c) \to -\infty$ as $c \to -\infty$, I has a mountain pass geometry. Since I satisfies the Palais-Smale condition, the existence of a second critical point follows by the "mountain pass lemma" of Ambrosetti and Rabinowitz [1]. $\square$

For an outline of the proof of Theorem 1.2, see the more general case in Section 4.

2 A general system

In view of the results described in Section 1, the following question is natural:

Question 2.1. *What are the main features of system* (1)–(2), *which allow existence and asymptotics as in Theorem 1.1 and Theorem 1.2?*

A further motivation to answer Question 2.1 was provided by the analysis by Chae and Nam [5] of the vortex solutions for the $CP(1)$ Maxwell-Chern-Simons model introduced in [7]. The elliptic system for $CP(1)$ vortices is given by:

$$\Delta\widetilde{u} = 2q\left(-N + S - \frac{1-e^{\widetilde{u}}}{1+e^{\widetilde{u}}}\right) + 4\pi\sum_{j=1}^{n}\delta_{p_j} \qquad \text{on } M \tag{7}$$

$$\Delta N = -\kappa^2 q^2\left(-N + S - \frac{1-e^{\widetilde{u}}}{1+e^{\widetilde{u}}}\right) + q\frac{4e^{\widetilde{u}}}{(1+e^{\widetilde{u}})^2}N \qquad \text{on } M. \tag{8}$$

In [5] the authors obtain an asymptotic behavior of solutions analogous to the one described in Theorem 1.2. They also prove the existence of a solution by the super-subsolution method. However, multiplicity of solutions is not investigated. Thus, we were motivated to answer Question 2.1 in the following more specific form:

Question 2.2. *Does there exist a general system including* (1)–(2) *and* (7)–(8) *as special cases, whose solutions satisfy existence and asymptotic properties analogous to the ones described in Theorem 1.1 and Theorem 1.2?*

In [16, 17] we answer Question 2.2 in the affirmative. More precisely, we construct the following system:

$$-\Delta\widetilde{u} = \varepsilon^{-1}\lambda(v - f(e^{\widetilde{u}})) - 4\pi\sum_{j=1}^{n}\delta_{p_j} \qquad \text{on } M \tag{9}$$

$$-\Delta v = \varepsilon^{-1}\left[\lambda f'(e^{\widetilde{u}})e^{\widetilde{u}}(s - v) - \varepsilon^{-1}(v - f(e^{\widetilde{u}}))\right] \qquad \text{on } M. \tag{10}$$

We note that (1)–(2) and (7)–(8) are special cases of (9)–(10). Indeed, system (1)–(2) corresponds to (9)–(10) with $f(t) = t$ and $s = 1$. On the other hand, setting $v = N - S$, $s = -S$, $\lambda = 2/\kappa$, $\varepsilon = 1/(\kappa q)$, system (7)–(8) reduces to (9)–(10) with $f(t) = (t-1)/(t+1)$. We make the following

Assumptions on f:

($f0$) $f : [0, +\infty)$ is smooth and $f'(t) > 0$ for all $t \in [0, +\infty)$

($f1$) $f(0) < s < \sup_{t>0} f(t)$

($f2$) f, f', f'' have at most polynomial growth

($f3$) f satisfies one of the following conditions:

(a) $f''(t)t + f'(t) \geq 0$ and $\sup_{t>0} |f(t)|/[f'(t)t] < +\infty$

(b) $\sup_{t>0} f'(t)t(|\log t| + |f(t)|) < +\infty$.

We show:

Theorem 2.1 ([17]). *Suppose f satisfies assumptions ($f0$), ($f1$), ($f2$) and ($f3$). Then there exists $\lambda_0 > 0$ with the property that for every $\lambda \geq \lambda_0$ there exists $\varepsilon_\lambda > 0$ such that system* (9)–(10) *admits at least* two *solutions for all* $0 < \varepsilon < \varepsilon_\lambda$.

We note that assumption ($f3$)–(a) allows $f(t) = t^\alpha$, for every $\alpha > 0$, and therefore it includes the $U(1)$ case $f(t) = t$. On the other hand, assumption ($f3$)–(b) is satisfied by the $CP(1)$ case $f(t) = (t-1)/(t+1)$. It follows that the existence result stated in Theorem 2.1 includes indeed the $U(1)$ system and the $CP(1)$ system as special cases, as well as all power growths for f. Concerning the asymptotic behavior of solutions, we have:

Theorem 2.2 ([16]). *Let $(\widetilde{u}, v)$ be solutions to* (9)–(10), *with $\varepsilon \to 0$. There exists a solution $\widetilde{u}_0$ to*

$$-\Delta\widetilde{u}_0 = f'(e^{\widetilde{u}_0})e^{\widetilde{u}_0}(s - f(e^{\widetilde{u}_0})) - 4\pi\sum_{j=1}^{n}\delta_{p_j} \qquad \text{on } M, \tag{11}$$

such that a subsequence, still denoted $(\widetilde{u}, v)$, satisfies:

$$(e^{\widetilde{u}}, v) \to \left(e^{\widetilde{u}_0}, f(e^{\widetilde{u}_0})\right) \qquad \text{in } C^k(M) \times C^k(M),\ \forall k \geq 0.$$

Similarly as the $U(1)$ system (1)–(2), system (9)–(10) admits a *variational formulation.* Indeed, by analogous arguments as in Section 1, system (9)–(10) is equivalent to the following fourth order equation:

$$\begin{aligned}\varepsilon^2\Delta^2 u - \Delta u = &-\varepsilon\lambda[f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2 \\ &+ 2\varepsilon\lambda\Delta f(e^{\sigma+u}) + \lambda^2 f'(e^{\sigma+u})e^{\sigma+u}(s - f(e^{\sigma+u})) - \frac{4\pi n}{|M|} \qquad \text{on } M.\end{aligned} \tag{12}$$

In turn, solutions to (12) correspond to critical points for the functional

$$\begin{aligned}I_\varepsilon(u) = &\frac{\varepsilon^2}{2}\int(\Delta u)^2 + \frac{1}{2}\int|\nabla u|^2 \\ &+ \varepsilon\lambda\int f'(e^{\sigma+u})e^{\sigma+u}|\nabla(\sigma+u)|^2 + \frac{\lambda^2}{2}\int(f(e^{\sigma+u}) - s)^2 + \frac{4\pi n}{|M|}\int u,\end{aligned}$$

defined on the Sobolev space $H^2(M)$ (we choose to emphasize the dependence on ε only, since λ will be fixed).

In the remaining part of this note, we outline the proofs of Theorem 2.1 and Theorem 2.2.

3 Outline of the proof of Theorem 2.1

As in the proof of Theorem 1.1, we obtain the two solutions as a local minimum and a mountain pass for I_ε. However, due to the general form of f, it does not seem possible to adapt the method based on integral constraints described in Section 1 to obtain a local minimum. Instead, we adapt some ideas in [20]. Such an adaptation is not trivial, since the problem considered in [20] is of the second order, while (12) is of the fourth order, and thus the standard maximum principles do not apply. The key point is that (12) is a "good" perturbation of (11), and therefore a kind of "asymptotic maximum principle property" holds for small values of ε. Indeed, we may factor the higher order differential operator in (12) as follows:

$$\varepsilon^2\Delta - \Delta = (-\varepsilon^2\Delta + 1)(-\Delta). \tag{13}$$

The following lemma shows that the operator $-\varepsilon^2\Delta+1$ is a "good perturbation" of the identity:

Lemma 3.1. *Let $G_\varepsilon = G_\varepsilon(x, y)$ be the Green function defined by*

$$(-\varepsilon^2\Delta_x + 1)G_\varepsilon(x, y) = \delta_y \qquad \text{on } M.$$

Then

(i) $G_\varepsilon > 0$ on $M \times M$ and for every fixed $y \in M$ we have $G_\varepsilon \rightharpoonup \delta_y$ as $\varepsilon \to 0$, weakly in the sense of measures;

(ii) $\|G_\varepsilon * h\|_{H^k} \leq \|h\|_{H^k}$ for all $1 \leq q \leq +\infty$;

(iii) If $\Delta h \in L^q$ for some $1 < q < +\infty$, then $\|G_\varepsilon * h - h\|_q \leq \varepsilon^2\|\Delta h\|_q$.

Using Lemma 3.1, it is not difficult to construct a subsolution $\underline{u}_\varepsilon$ for (12) such that $\underline{u}_\varepsilon \to \underline{u}_0$ in H^2 and C^1, where $\underline{u}_0$ is a subsolution for (11). We recall that $\underline{u}_\varepsilon$ is a subsolution for (12) if it satisfies (12) with $\leq$. We define:

$$\mathcal{A}_\varepsilon = \{u \in H^2(M)/\ u \geq \underline{u}_\varepsilon \text{ on } M\}.$$

Then there exists a minimizer u_ε such that:

$$I_\varepsilon(u_\varepsilon) = \inf_{\mathcal{A}_\varepsilon} I_\varepsilon.$$

The main point now is to prove that

Claim: For ε sufficiently small there holds:

$$u_\varepsilon > \underline{u}_\varepsilon \qquad \text{on } M. \tag{14}$$

Proof of (14). We note that $I'_\varepsilon(u_\varepsilon) \geq 0$, i.e., u_ε is a supersolution for (12). However, since (12) is of the fourth order, we cannot derive (14) from the standard maximum principles. Nevertheless, we can prove the "asymptotic maximum principle property" (14) by first establishing some a priori estimates:

Lemma 3.2. *There exists a solution $u_0 \in H^1$ for (11) such that $u_\varepsilon \to u_0$ strongly in H^1. Furthermore,* *(i)* $\lim_{\varepsilon\to 0}\varepsilon\|\Delta u_\varepsilon\|_2 = 0$

(ii) $\lim_{\varepsilon\to 0}\varepsilon\int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2 = 0.$

Exploiting again the factorization (13), we can write the equation for u_ε in the form:

$$-\Delta u_\varepsilon + u_\varepsilon \geq G_\varepsilon * F_\varepsilon + u_\varepsilon,$$

with

$$F_\varepsilon = \varepsilon\lambda\, a(u_\varepsilon) + \lambda^2 f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}(s - f(e^{\sigma+u_\varepsilon})) - \frac{4\pi n}{|M|},$$

where

$$a(u) := -[f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2 + 2\Delta f(e^{\sigma+u}).$$

By the maximum principle, $u_\varepsilon \geq w_\varepsilon$, where w_ε is defined by

$$(-\Delta + 1)w_\varepsilon = G_\varepsilon * F_\varepsilon + u_\varepsilon.$$

The estimates is Lemma 3.2 imply that

$$\|w_\varepsilon - u_0\|_\infty \to 0,$$

where u_0 satisfies

$$-\Delta u_0 = \lambda^2 f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}(s - f(e^{\sigma+u_\varepsilon})) - \frac{4\pi n}{|M|}.$$

By the Hopf maximum principle, $u_0 > \underline{u}_0$ on M. It follows that for ε small the strict inequality (14) is satisfied.

Similarly as in the $U(1)$ case, it is readily checked that $I_\varepsilon(c) \to -\infty$ as $c \to -\infty$. Condition $(f3)$ ensures the Palais-Smale condition for I_ε. Hence, the proof of Theorem 2.1 follows again by the Ambrosetti-Rabinowitz mountain pass theorem [1].

4 Outline of the proof of Theorem 2.2

The main part of the proof of Theorem 2.2 is to obtain a priori estimates in H^k for all $k \geq 0$ for u and v, independent of $\varepsilon \to 0$. More precisely, we show:

Lemma 4.1. *For every $k \geq 0$ there exists $C_k \geq 0$ independent of ε such that*

$$\|u\|_{H^k} + \|v\|_{H^k} \leq C_k.$$

In order to establish Lemma 4.1 it is convenient to introduce a third variable $w = \varepsilon^{-1}(v - f(e^{\sigma+u}))$. Then (u, v, w) satisfies:

$$-\Delta u = w - \frac{4\pi n}{|M|} \tag{15}$$

$$-\varepsilon^2\Delta v + (1 + \varepsilon c(x,u))v = F_\varepsilon(x,u) \tag{16}$$

$$-\varepsilon^2\Delta w + (1 + \varepsilon c(x,u))w = G_\varepsilon(x,u,v,\nabla u), \tag{17}$$

$$
\begin{aligned}
c(x,u) &= f'(e^{\sigma+u})e^{\sigma+u} \\
F_\varepsilon(x,u) &= f(e^{\sigma+u}) + \varepsilon f'(e^{\sigma+u})e^{\sigma+u} \\
G_\varepsilon(x,u,v,\nabla u) &= f'(e^{\sigma+u})e^{\sigma+u}(s-v) \\
&\quad + \varepsilon(f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u}))e^{\sigma+u}|\nabla(\sigma+u)|^2.
\end{aligned}
$$

The proof of Lemma 4.1 is obtained by an induction argument. The basis of the induction is given by

Claim: There exists a constant $C > 0$ independent of ε such that:

$$\|w\|_2 \le C. \tag{18}$$

The proof of (18) is a consequence of some L^∞ estimates obtained by maximum principle:

$$
\begin{aligned}
&f(0) \le f(e^{\widetilde{u}}) \le s \\
&f(0) \le v \le s,
\end{aligned}
$$

together with the following identity:

$$\int |\nabla v|^2 + \int w^2 = \int (s-v)\left(f''(e^{\widetilde{u}})e^{\widetilde{u}} + f'(e^{\widetilde{u}})\right) e^{\widetilde{u}}|\nabla \widetilde{u}|^2.$$

Once (18) is established, we can iteratively obtain all the H^k estimates:

Claim: Suppose there exists $C_k > 0$ such that $\|w\|_{H^k} \le C$. Then there exists $C_{k+1} > 0$ such that $\|w\|_{H^{k+1}} \le C$.

The proof is mainly a consequence of Lemma 3.1–(ii). If $\|w\|_{H^k} \le C$, then:

$\|u\|_{H^{k+2}} \le C$	by (15) and elliptic estimates
$\|v\|_{H^{k+2}} \le C$	by (16) and Lemma 3.1–(ii)
$\|w\|_{H^{k+1}} \le C$	by (17) and Lemma 3.1–(ii).

Thus, Lemma 4.1 is established. Now the proof of Theorem 2.2 follows by taking limits in (9)–(10).

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