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PARTIAL DIFFERENTIAL EQUATIONS

Note on Lorentz Spaces and Differentiability of Weak Solutions to Elliptic Equations

by

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Summary. We show that weak solutions to the elliptic equations $(a_{ij}(x)u_{x_i})_{x_j} = (f_i)_{x_i}$ are differentiable almost everywhere when f_i belongs to the Lorentz space $L^{n,1}_{\mathrm{loc}}(\Omega)$. It is known that these solutions are continuous, but not necessarily Hölder continuous. Hence this proves the conjecture of B. Bojarski saying that differentiability a.e. of weak solutions is independent of their Hölder continuity.

1. Introduction. In 1987 Yu. G. Reshetnyak [6] considered a general nonlinear elliptic equation, and he proved that its weak solutions are differentiable almost everywhere. This result was a consequence of a theorem of Serrin [7] asserting the Hölder continuity of weak solutions. An analogous theorem in the case of the linear equation $(a_{ij}(x)u_{x_i})_{x_j} = 0$ was proved independently by B. Bojarski [2]. Instead of Hölder continuity, he used a weaker result on the local boundedness of solutions. Recently Hajłasz and Strzelecki [5] simplified the proof of Reshetnyak adopting the Bojarski method. The authors of [2] and [5] stress that their idea of the proof does not require the Hölder continuity of solutions. Moreover, Bojarski conjectured that the differentiability almost everywhere of weak solutions should be independent of their Hölder continuity (cf. [2, p. 5]). We are going to give an answer to this conjecture. We construct a large class of elliptic equations, which weak solutions are known to be continuous, differentiable a.e., but, in general, *not* Hölder continuous.

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In this note we consider weak solutions to the equation

$$\begin{cases} (a_{ij}(x)u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } \Omega, \\ u \in H^1_{\mathrm{loc}}(\Omega), \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ ($n \geqslant 3$) is an open set. The functions $a_{ij}(x)$ $(i, j = 1, \ldots, n)$ are bounded, measurable, and satisfy the uniform ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geqslant |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in \Omega.$$

Here summation over repeated indices is understood. Our result reads as follows.

THEOREM 1. *Assume that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution to the equation* (1) *with $f_i \in L^{n,1}_{\mathrm{loc}}(\Omega)$. Then u is differentiable almost everywhere in Ω.*

Some comments on Theorem 1 are in order.

Given $\mathcal{D} \subseteq \mathbb{R}^n$ the Lorentz space $L^{p,q}(\mathcal{D})$ with $0 < p, q \leqslant \infty$ consists of all measurable functions g for which the quantity

$$\|g\|_{p,q,\mathcal{D}} \equiv \begin{cases} \left(\int_0^{+\infty} (g^*(t)t^{1/p})^q t^{-1} dt\right)^{1/q} & \text{when } 0 < q < \infty, \\ \sup_{0<t<\infty}\{t^{1/p}g^*(t)\} & \text{when } q = \infty \end{cases}$$

is finite. Here $g^*(t) \equiv \sup\{s > 0 : |\{x \in \mathcal{D} : |g(x)| > s\}| > t\}$ denotes the decreasing rearrangement of a measurable function g (for $A \subseteq \mathbb{R}^n$, $|A|$ stands for its n-dimensional Lebesgue measure). Using the definition above, it is possible to prove that $L^{p,p}(\mathcal{D})$ coincides with the Lebesgue space $L^p(\mathcal{D})$, and $\|\cdot\|_{p,p,\mathcal{D}} = \|\cdot\|_{p,\mathcal{D}}$. Furthermore, the following inclusions hold

$$L^r(\mathcal{D}) \subset L^{p,q}(\mathcal{D}) \subset L^p(\mathcal{D}) \subset L^{p,r}(\mathcal{D}) \subset L^{p,\infty}(\mathcal{D}) \subset L^q(\mathcal{D})$$

whenever $0 < q < p < r \leqslant \infty$ and $\mathcal{D}$ is a bounded set. Other properties of the Lorentz spaces can be found in [1]. Let χ_A be the characteristic function of a set A. In this paper, we use the local version of the Lorentz spaces $L^{p,q}_{\mathrm{loc}}(\Omega)$ consisting of all measurable functions g in Ω such that $g\chi_A \in L^{p,q}(\Omega)$ for each compact set $A \subset \Omega$.

As it was said above, the Lorentz space $L^{n,1}_{\mathrm{loc}}(\Omega)$ used in Theorem 1 has the property of being between $L^n_{\mathrm{loc}}(\Omega)$ and all $L^p_{\mathrm{loc}}(\Omega)$ spaces with $p > n$. It is well known that for $f_i \in L^p_{\mathrm{loc}}(\Omega)$ with $p > n$ weak solutions to (1) are Hölder continuous (cf. e.g. [8]) and differentiable a.e. (cf. [6, 2, 5]). On the other hand, there is an example constructed in [4, p. 263], where a solution to the problem

$$\begin{cases} \Delta u = \sum_{i=1}^n (f_i)_{x_i} & \text{on } B(R) \\ u = 0 & \text{on } \partial B(R) \end{cases}$$

is continuous, but not Hölder continuous for some $f_i \in L^{n,1}(B(R))$ such that $f_i \notin L^{n+\varepsilon}(B(R))$ for each $\varepsilon > 0$. Now Theorem 1 asserts that u is always differentiable a.e.

We formulate Theorem 1 in the simplest case (1). Divergence form of equations with lower order terms can be treated in a similar way. One can also replace the right hand side of the equation (1) by $g \in L^{n/2,1}_{\mathrm{loc}}(\Omega)$, and the conclusion remains true. Since there are no essential novelties, we leave the details to the interested reader.

In this note, the ball centered in x_0, with radius R is denoted by $B(x_0, R)$ or simply by $B(R)$. We say also that a vector field f belongs to $L^{p,q}(\mathcal{D})$ if $|f|$ does.

2. Auxiliary results. We are going to list some auxiliary results needed to prove Theorem 1. We begin by the Stepanoff differentiability criterion [11]. The statement presented here is taken from [9, Ch. VIII, Thm 3].

THEOREM 2. *Let $u : \mathcal{D} \to \mathbb{R}$ be an arbitrary measurable function. Define*

$$E = \{a \in \mathcal{D} : \limsup_{x \to a}(|u(x) - u(a)|/|x - a|) < +\infty\},$$

then E is Lebesgue measurable and u is differentiable a.e. in E.

The next two theorems include estimates of weak solutions to the elliptic equation (1). The first of them is classical and states that weak solutions to (1) with $f_i \equiv 0$ are locally bounded (in fact, they are Hölder continuous) (cf. e.g. [8, Thm 5.1]).

THEOREM 3. *Assume that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution of the equation $(a_{ij}(x)u_{x_i})_{x_j} = 0$. Then for each $R > 0$ such that $B(2R) \subset \Omega$ there is a positive $C > 0$ independent of u such that*

$$\|u\|_{\infty,B(R)} \leqslant C\|u\|_{2,B(2R)}.$$

The second estimate we shall need was proved by V. Ferone [4]. It was used to get continuity of weak solutions to (1) for the Dirichlet problem, in the so-called limit case.

THEOREM 4 [4]. *Assume that $u \in H^1_0(\Omega)$ is a weak solution to the equation (1). Suppose that $f_i \in L^{n,1}(\Omega)$. Then the estimate*

$$\|u\|_{\infty,\Omega} \leqslant C\|u\|_{n,1,\Omega}$$

holds, where C is independent of u.

We conclude this section with some considerations concerning a maximal operator defined on the Lorentz spaces. Let us define for $f \in L^{p,q}_{\mathrm{loc}}(\mathcal{D})$ the operator

$$(Mf)(x) \equiv \limsup_{h\to 0} \frac{1}{|B(x,h)|^{1/p}} \|f\|_{p,q,B(x,h)},$$

for all $x \in \mathcal{D}$.

THEOREM 5. *Let $1 \leqslant q \leqslant p \leqslant \infty$. Suppose that $f \in L^{p,q}(\mathcal{D})$. Then $(Mf)(x)$ is finite almost everywhere, and there is a constant $C > 0$ such that*

$$|\{x \in \mathcal{D} : (Mf)^p(x) > \lambda\}| \leqslant C \frac{\|f\|^p_{p,q,\mathcal{D}}}{\lambda} \tag{2}$$

for all $\lambda > 0$.

This theorem was formulated and proved by Chung *et al.* [3]. They generalized an idea of Stein [10], where a particular case of Theorem 5 was considered. Let us stress here that the assumption $q \leqslant p$ is crucial in the proof of Theorem 5. We refer the reader to [3] for examples when (2) fails for $q > p$.

Remark. Proceeding analogously as in the case of the classical maximal operator (cf. [9, Ch. I]) and using Theorem 5 one can show that, in fact,

$$(Mf)(x) = \lim_{h\to 0} \frac{1}{|B(x,h)|^{1/p}} \|f\|_{p,q,B(x,h)} = \left(\frac{p}{q}\right)^{p/q} |f(x)|$$

for a.e. $x \in \mathcal{D}$. However, we shall not need this result in the remainder of this note.

3. Proof of Theorem 1. The differentiability of u follows from Theorem 2. To see this, fix $x_0 \in \Omega$ and $h > 0$ such that $B(x_0, 2h) \subset \Omega$. For $X \in B(2)$, let us define the difference quotient

$$v_h(X) = \frac{u(x_0 + hX) - u(x_0)}{h}.$$

This is well-defined function belonging to $H^1(B(2))$. By the change of variables $x = x_0 + hX$ and the definition of weak solutions of (1) one can prove that $v_h(X)$ solves the equation

$$(a^h_{ij}(X) v_{X_i})_{X_j} = (f^h_i(X))_{X_i} \quad \text{in } B(2),$$

where $a^h_{ij}(X) = a_{ij}(x_0 + hX)$ and $f^h_i(X) = f_i(x_0 + hX)$.

We decompose $v_h = v' + v''$, where the functions v' and v'' are defined as solutions to the problems

$$\begin{cases} (a^h_{ij}(X) v'_{X_i})_{X_j} = (f^h_i)_{X_i} & \text{in } B(2), \\ v' \in H^1_0(B(2)), \end{cases} \tag{3}$$

$$(4) \qquad \begin{cases} (a_{ij}^h(X)v''_{X_i})_{X_j} = 0 & \text{in } B(2), \\ v'' - v_h \in H_0^1(B(2)). \end{cases}$$

Applying Theorems 4 and 3 respectively to v' and v'' we obtain

$$(5) \qquad \|v'\|_{\infty,B(2)} \leqslant C\|f_i^h\|_{n,1,B(2)} \quad \text{and} \quad \|v''\|_{\infty,B(1)} \leqslant C\|v''\|_{2,B(2)}$$

with some constants C independent of x_0. Moreover, using (5) we get

$$(6) \quad \|v''\|_{2,B(2)} = \|v_h - v'\|_{2,B(2)} \leqslant \|v_h\|_{2,B(2)} + \|v'\|_{2,B(2)} \\ \leqslant \|v_h\|_{2,B(2)} + |B(2)|^{1/2}\|v'\|_{\infty,B(2)} \leqslant \|v_h\|_{2,B(2)} + C\|f^h\|_{n,1,B(2)}.$$

Combining (5) and (6) we obtain the following estimate of the function v_h.

$$(7) \quad \|v_h\|_{\infty,B(1)} \leqslant \|v'\|_{\infty,B(1)} + \|v''\|_{\infty,B(1)} \leqslant C(\|f^h\|_{n,1,B(2)} + \|v_h\|_{2,B(2)}).$$

The proof will be completed by showing that the right hand side of (7) remains bounded as $h \to 0$ for almost all $x_0 \in \Omega$.

To handle the first term, we observe that for $p, q \geqslant 1$

$$\|f^h\|_{p,q,B(1)}^p = \frac{\omega_n}{|B(x_0,h)|}\|f\|_{p,q,B(x_0,h)}^p$$

by a simple change of variables. Here ω_n denotes the volume of the unit n-dimensional ball. Theorem 5 now yields

$$\limsup_{h\to 0}\|f^h\|_{n,1,B(1)} = \omega_n^{1/n}(Mf)(x_0) < \infty$$

for a.e. $x_0 \in \Omega$.

To estimate the second term in (7), we use theorem of Calderón and Zygmund [9, Ch. VIII, Thm 1], which states that for $u \in H^1(B(2))$, $h \to 0$, and almost all $x_0 \in \Omega$, the following function of $X \in B(2)$

$$\frac{u(x_0 + hX) - u(x_0)}{h} - \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x_0) \cdot X_i$$

tends to zero in $L^2(B(2))$. Consequently, $\limsup_{h\to 0}\|v^h\|_{2,B(2)} < \infty$.

Now the proof of Theorem 1 is complete.

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