Rend. Acc. Sc. fis. mat. Napoli
Vol. LXIII, (1996) pp. 7-16

CONTINUITY PROPERTIES FOR LINEAR ELLIPTIC EQUATIONS WITH LOWER-ORDER TERMS

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Presentata dal Socio Angelo Alvino

Adunanza del 13 gennaio 1996

Riassunto. Sono date, in termini di spazi di Lorentz, condizioni che assicurino limitatezza e continuitá delle soluzioni del problema (1.1).

Abstract. Boundedness and continuity conditions for solutions to problem (1.1) are given in the framework of Lorentz spaces.

1. Introduction

We are concerned with the regularity of solutions to the Dirichlet problem

$$(1.1)\qquad \begin{cases} Lu = -(a_{ij}u_{x_i} + b_j u)_{x_j} + d_i u_{x_i} + cu = (f_i)_{x_i} + g & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

where a_{ij}, b_i, d_i, c, f_i and g are measurable functions defined in a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and ellipticity condition

$$(1.2)\qquad a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \ \ a.e. \text{ in } \Omega, \qquad \forall \xi \in \mathbb{R}^n$$

holds, where ν is a positive constant. In the framework of Lebesgue spaces the regularity theory for problem (1.1) has been widely studied, and can be found in classical references such as [St], [LU]. One of the main results is boundedness and local Hölder-continuity of u when the following conditions are satisfied for some $\epsilon > 0$:

$$(i) \qquad a_{ij} \in L^{\infty}(\Omega), \qquad d \in L^{n}(\Omega);$$

$$(ii) \qquad b \in L^{n+\epsilon}(\Omega), \qquad c \in L^{n/2+\epsilon}(\Omega);$$

$$(iii) \qquad f \in L^{n+\epsilon}(\Omega), \qquad g \in L^{n/2+\epsilon}(\Omega),$$

where $b = (\sum_i b_i^2)^{1/2}$, $d = (\sum_i d_i^2)^{1/2}$, $f = (\sum_i f_i^2)^{1/2}$. If we allow $\epsilon = 0$, solutions may be discontinuous (a counterexample may be found in [LU], Chapter 1), so in terms of Lebesgue spaces conditions (ii) and (iii) may not be weakened without losing continuity.

For some special cases of (1.1), by symmetrization techniques, Alvino [Al] and Ferone [Fe] have shown boundedness and continuity of u when the coefficients and the data satisfy weaker conditions than (ii) and (iii), formulated in terms of the Lorentz spaces. Using different methods, results of the same kind have been obtained in [FF] for minimizers of integral functionals and for solutions of nonlinear equations.

Symmetrization has been extended to linear equations with all lower-order terms by Ferone and Posteraro [FP] in the case when all coefficients are bounded, and by Alvino, P.L. Lions and Trombetti [ALT] when the negative part of $-(b_j)_{x_j} + c$ is bounded, but their results yield estimates for u only when existence holds for a so-called "symmetrized problem" associated to (1.1).

Our purpose is to show how by some simple arguments one can obtain continuity results for solutions of (1.1), under assumptions which are weaker than (ii) and (iii). Such results are contained in Section 2.

In Section 3 we extend our results to the case when the matrix $[a_{ij}]$ satisfies a degenerate ellipticity condition. We refer to the papers by Murthy and Stampacchia [MS] and Trudinger [Tr] for the basic theory of degenerate elliptic equations, and to Alvino and Trombetti [AT] and Betta [Be] for sharp estimates obtained via symmetrization.

2. Main results

First of all, let us briefly define the Lorentz spaces (see e.g. [BR] for more details).

For any measurable function φ defined in Ω, let us denote by

$$\varphi^*(s) = \sup\{t > 0 : |\{x \in \Omega : |\varphi(x)| > t\}| > s\}$$

the decreasing rearrangement of φ, and by

$$\bar{\varphi}(t) = \frac{1}{t}\int_0^t \varphi^*(s)ds$$

the average function of φ.

Then φ belongs to the Lorentz space $L^{p,q}(\Omega)$, $1 < p < \infty$, $1 \leq q \leq \infty$, if and only if

$$\| \varphi \|_{p,q} = \begin{cases} \left(\int_0^{|\Omega|} (\bar{\varphi}(t)t^{1/p})^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \leq q < \infty \\ \sup\limits_{t>0} \bar{\varphi}(t)t^{1/p} & \text{if } q = \infty \end{cases}$$

is finite.

The Lorentz spaces interpolate the Lebesgue spaces, and the following embeddings hold:

$$\text{(2.1)} \qquad \begin{cases} L^p(\Omega) \subset L^{r,q}(\Omega) \subset L^r(\Omega) & \text{if } 1 < r < p, \quad 1 \leq q < r; \\ L^r(\Omega) \subset L^{r,q}(\Omega) \subset L^t(\Omega) & \text{if } t < r, \quad r < q \leq \infty . \end{cases}$$

We prove the following theorems:

Theorem 2.1. *Let $u \in H_0^1(\Omega)$ be a solution of problem (1.1) under assumptions (1.2), $(i), (ii)$ and*

$(iii)'$ $\qquad f \in L^{n,1}(\Omega), \qquad g \in L^{n/2,1}(\Omega).$

Then u is continuous in Ω.

Remark 2.1. We restrict ourselves to the case of homogeneous boundary data for simplicity, but actually the proof of Theorem 2.1 yields continuity for local solutions of the equation

$$Lu = (f_i)_{x_i} + g \qquad \text{in } \Omega$$

when hypotheses $(i), (ii)$ and $(iii)'$ are satisfied.

Theorem 2.2. *Let $u \in H_0^1(\Omega)$ be a solution of problem (1.1) under assumptions (1.2), $(i), (iii)'$ and*

$$(ii)' \qquad b \in L^{n,1}(\Omega), \qquad c \in L^{n/2,1}(\Omega).$$

Moreover, suppose that u is unique. Then u is bounded and continuous in Ω.

Remark 2.2. Embedding properties (2.1) show that conditions $(ii)'$ and $(iii)'$ are indeed weaker than the classical conditions (ii) and (iii), respectively.

Remark 2.3. As counterexamples in [Fe] show, local Hölder-continuity may not be expected under the hypotheses of Theorems 2.1 and 2.2.

For the proofs, we shall need the following two lemmas. The first one is essentially contained in [Al], when $f = 0$ and in [Fe], when $g = 0$.

Lemma 2.1. *Suppose $u \in H_0^1(\Omega)$ is a solution of problem (1.1) where $b = c = 0$ and conditions (1.2), (i) and $(iii)'$ are satisfied. Then u is bounded and continuous in Ω and there exists $C_1 = C_1(n, \|d\|_n)$ such that*

$$\|u\|_\infty \leq C_1(\|f\|_{n,1} + \|g\|_{n/2,1}).$$

Lemma 2.2. *Suppose $u \in H_0^1(\Omega)$ is the unique solution of problem (1.1) under assumptions (1.2), $(i), (ii)'$ and $(iii)'$. If $\|b\|_{n,1}$ and $\|c\|_{n/2,1}$ are sufficiently small, then u is bounded and continuous in Ω.*

PROOF. We consider the operator

$$L_o v = -(a_{ij} v_{x_i})_{x_j} + d_i v_{x_i},$$

which by maximum principle is bicontinuous from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Let us denote by G_o its inverse, and by $C_b(\Omega)$ the Banach space of bounded and continuous functions, equipped with the sup-norm. The operator

$$Tv = G_o[(b_i v + f_i)_{x_i} - cv + g]. \tag{2.2}$$

is a well-defined mapping of C_b into itself, by Lemma 2.1 . Clearly, the continuity of the unique solution of (1.1) is equivalent to the existence of a fixed point for T.

But it is easily seen that T is a contraction if

$$\|b\|_{n,1} + \|c\|_{n/2,1} < \frac{1}{C_1},$$

as by Lemma 2.1 the following estimates hold:

$$\|Tu - Tv\|_\infty = \|G_o[(b_j(u-v))_{x_j} - c(u-v)]\|_\infty \leq$$

$$\leq C_1(\|b(u-v)\|_{n,1} + \|c(u-v)\|_{n/2,1}) \leq$$

$$\leq C_1(\|b\|_{n,1} + \|c\|_{n/2,1})\|u-v\|_\infty. \blacksquare$$

PROOF OF THEOREM 2.1. We fix $x_0 \in \Omega$ and a ball $B \subset \Omega$ centered in x_0 such that uniqueness holds in B and

$$\|b\chi_B\|_{n,1} + \|c\chi_B\|_{n/2,1} < \frac{1}{C_1}, \tag{2.3}$$

where χ_B is the characteristic function of B. In B we can write $u = v + w$, where v is the unique solution of

$$\begin{cases} Lv = 0 & \text{in } B \\ v - u \in H_0^1(B), \end{cases}$$

and w is the unique solution of

$$\begin{cases} Lw = (f_i)_{x_i} + g & \text{in } B \\ w \in H_0^1(B). \end{cases}$$

Then, since v is locally Hölder-continuous in B by classical theory and w is bounded and continuous in B by Lemma 2.2, theorem follows. ■

PROOF OF THEOREM 2.2. We use a partition of unity argument. Let $\{B_k\}_{k=1}^l$ be an open covering of Ω by balls, such that $\|b\chi_k\|_{n,1}$ and $\|c\chi_k\|_{n/2,1}$ are small enough in the sense of Lemma 2.2, where χ_k is the characteristic function of $\Omega_k = B_k \cap \Omega$, and such that uniqueness holds in Ω_k for every k. The functions

$$\zeta_k(x) = \frac{\chi_k(x)}{\sum_{k=1}^l \chi_k(x)}, \qquad x \in \Omega$$

satisfy $0 \le \zeta_k(x) \le 1$ and $\sum_k \zeta_k(x) = 1$. Let

$$\begin{cases} Lu_k = (f_i\zeta_k)_{x_i} + g\zeta_k & \text{in } \Omega \\ u_k \in H_0^1(\Omega). \end{cases}$$

Then

$$v = \sum_k u_k$$

satisfies (1.1), so by uniqueness

$$u = v\,.$$

By Lemma 2.2 each u_k is bounded and continuous in Ω, hence u is bounded and continuous in Ω and the statement is proved. ∎

3. The degenerate case

Let us substitute ellipticity condition (1.2) by the following :

$$a_{ij}(x)\xi_i\xi_j \ge \nu(x)|\xi|^2 \qquad a.e. \text{ in } \Omega,\ \forall \xi \in \mathbb{R}^n, \tag{3.1}$$

where $\nu \ge 0$ is a nonnegative measurable function defined in Ω. Then the natural generalization of (1.1) is the following degenerate Dirichlet problem:

$$\begin{cases} Lu = (f_i)_{x_i} + g & \text{in } \Omega \\ u \in H_0^1(\nu), \end{cases} \tag{3.2}$$

where $H_0^1(\nu) = H_0^1(\Omega, \nu)$ is the completion of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{H^1(\nu)} = \left(\int_\Omega u^2 \nu dx\right)^{1/2} + \left(\int_\Omega |Du|^2 \nu dx\right)^{1/2}.$$

It is well-known (see [MS] and [Tr]) that the solution u is Hölder-continuous under assumptions (3.1) and the following:

$$\nu \in L^s(\Omega), \quad \nu^{-1} \in L^t(\Omega), \quad t > n, \quad \frac{1}{s} + \frac{1}{t} \le \frac{2}{n}; \tag{3.3}$$

$$a_{ij}\nu^{-1} \in L^\infty(\Omega), \qquad d\nu^{-1/2} \in L^r(\Omega); \tag{j}$$

(jj) $\qquad b\nu^{-1/2} \in L^{r+\epsilon}(\Omega), \qquad c \in L^{r/2+\epsilon}(\Omega);$

(jjj) $\qquad f\nu^{-1/2} \in L^{r+\epsilon}(\Omega), \qquad g \in L^{r/2+\epsilon}(\Omega),$

where $r^{-1} = n^{-1} - (2t)^{-1}$ and $\epsilon > 0$.

By essentially the same arguments as in Section 2, conditions (jj) and (jjj) can be weakened, without losing boundedness and continuity of the solution u. The analogues of Lemmas 2.1 and 2.2 for the degenerate case are stated here below.

Lemma 3.1. *Let u be a solution of the problem (3.2) where $b = c = 0$ and conditions (3.1), (3.3), (j) and*

$(jjj)'$ $\qquad f\nu^{-1/2} \in L^{r,2t/(2t-1)}(\Omega), \qquad g \in L^{r/2,t/(t-1)}(\Omega)$

are satisfied. Then u is continuous and bounded in Ω and the estimate

$$\|u\|_\infty \leq C_2(\|f\nu^{-1/2}\|_{r,2t/(2t-1)} + \|g\|_{r/2,t/(t-1)}) \tag{3.4}$$

holds, where C_2 depends on n, $\|\nu^{-1}\|_t$, $\|d\nu^{-1/2}\|_r$.

Proof. The proof of the a priori estimate (3.4) is contained in Theorem 3.1 in [Be]. For the continuity, we fix $x_0 \in \Omega$ and denote for $\rho > 0$, $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$. We take ρ such that $B_{8\rho}(x_0) \subset \Omega$, and we put $u = v + w$, where v is the solution of the problem:

$$\begin{cases} -\left(a_{ij}(x)v_{x_j}\right)_{x_i} + d_i v_{x_i} = (f_{x_i})_{x_j} + g & \text{in } B_{8\rho}(x_0) \\ v \in H_0^1\left(B_{8\rho}(x_0), \nu\right) \end{cases} \tag{3.5}$$

and w is the solution of the problem:

$$\begin{cases} -\left(a_{ij}(x)w_{x_j}\right)_{x_i} + d_i w_{x_i} = 0 & \text{in } B_{8\rho}(x_0) \\ w - u \in H_0^1\left(B_{8\rho}(x_0), \nu\right). \end{cases}$$

Denoting by $osc(w,\rho)$ the oscillation of $w(x)$ in $B_\rho(x_0)$, we have by Theorem 11.2 in [MS] that a constant $\eta < 1$ exists, such that

$$osc(w,\rho) \leq \eta\, osc(w, 4\rho). \tag{3.6}$$

Furthermore, as the estimate (3.4) may be applied to the solution v of (3.5), we have

$$osc(v,\rho) \le osc(v,4\rho) \le 2\|v\|_\infty \le F(\rho), \tag{3.7}$$

where

$$F(\rho) = 2\,C_2(\|f\nu^{-1/2}\chi_{B_{8\rho}(x_0)}\|_{r,2t/(2t-1)} + \|g\chi_{B_{8\rho}(x_0)}\|_{r/2,t/(t-1)})$$

is a function such that $F(\rho) \to 0$ as $\rho \to 0$. Then from (3.6) and (3.7) we get

$$\begin{aligned} osc(u,\rho) \le& \eta\, osc(w,4\rho) + F(\rho) \le \\ &\le \eta\, osc(u,4\rho) + (\eta+1)F(\rho) \end{aligned}$$

and continuity of u follows. ■

Lemma 3.2. *Suppose u is the unique solution of problem (3.2) under assumptions (3.1), (3.3), (j), (jjj)′ and*

$$(jj)' \qquad b\nu^{-1/2} \in L^{r,2t/(2t-1)}(\Omega), \qquad c \in L^{r/2,t/(t-1)}(\Omega).$$

If $\|b\nu^{-1/2}\|_{r,2t/(2t-1)}$ and $\|c\|_{r/2,t/(t-1)}$ are sufficiently small, then u is bounded and continuous in Ω.

Sketch of Proof. Using the same arguments as in Lemma 2.2, we consider the operator

$$L_o u = -(a_{ij}u_{x_i})_{x_j} + d_i u_{x_i},$$

which by maximum principle is bicontinuous from $H^1_0(\nu)$ onto $H^{-1}(\nu^{-1})$. If we denote by G_o its inverse, and by T the operator as in (2.2), the proof follows as in Lemma 2.2. ■

We obtain the following theorems:

Theorem 3.1. *Suppose u is a solution of (3.2) under assumptions (3.1), (3.3), (j), (jj) and (jjj)′. Then u is continuous in Ω.*

Theorem 3.2. *Suppose u is a solution of (3.2) under assumptions (3.1), (3.3), (j), (jj)′ and (jjj)′. Moreover, suppose that u is unique. Then u is bounded and continuous in Ω.*

Remark 3.1. We observe that conditions $(jj)'$ and $(jjj)'$ are indeed weaker than conditions (jj) and (jjj), respectively, by the embedding properties (2.1).

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