# Vortices in the Maxwell-Chern-Simons Theory

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**Abstract**

Our aim is to prove rigorously that the Chern-Simons model of Hong, Kim, and Pac [13] and Jackiw and Weinberg [14] (the CS model) and the Abelian Higgs model of Ginzburg and Landau (the AH model, see [15]) are unified by the Maxwell-Chern-Simons theory introduced by Lee, Lee, and Min in [16] (MCS model). In [16] the authors give a formal argument that shows how to recover both the CS and AH models out of their theory by taking special limits for the values of the physical parameters involved. To make this argument rigorous, we consider the existence and multiplicity of periodic vortex solutions for the MCS model and analyze their asymptotic behavior as the physical parameters approach these limiting values. We show that, indeed, the given vortices approach (in a strong sense) vortices for the CS and AH models, respectively. For this purpose, we are led to analyze a system of two elliptic PDEs with exponential nonlinearities on a flat torus. © 2000 John Wiley & Sons, Inc.

## Introduction

Our purpose is to study periodic multivortices (also known as "condensates") for a model introduced by Lee, Lee, and Min in [16]. Such a model provides a *self-dual* field theory inclusive of both the Maxwell and Chern-Simons terms and where self-duality is attained by the presence of a neutral scalar field.

In gauge theory, a self-dual structure is always very advantageous, because it permits the identification of a special class of (static) solutions (e.g., instantons, monopoles, vortices, etc.) by solving appropriate *first-order* equations, known as the "Bogomol′nyĭ equations" because of Bogomol′nyĭ's pioneer work in this direction (see [3]). From an analytical viewpoint, the Bogomol′nyĭ equations allow a reduction of the more complicated *second-order* equations of motion.

In the framework of superconductivity, a first self-dual situation was represented by the Abelian Higgs model (AH) of Ginzburg and Landau (see [15]) whose self-dual vortices are completely characterized by the work of Taubes [23] in a full space setting and Wang and Yang [25] in the periodic case. On the other hand, if we wish to investigate anyonic superconductivity (high critical temperature) we

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need to consider a "charged" vortex theory and include the Chern-Simons term into the model which, in principle, could spoil its self-dual properties.

In [16], self-duality was restored by virtue of a theory inclusive of a neutral scalar field (MCS model). However, a first attempt to obtain a self-dual Chern-Simons theory without the help of a neutral scalar field was considered by Hong, Kim, and Pac [13] and Jackiw and Weinberg [14], who proposed a model (CS model) whose electrodynamics were governed solely by the Chern-Simons term and characterized by the presence of a *sixth*-order potential in place of the usual quadratic potential of Ginzburg and Landau.

Even though several results are now available for vortices in the "pure" CS model [4, 18, 20, 21, 22, 24], it is still important to treat rigorously more complete models that include the Maxwell term as well. For this reason we consider the MCS model and show that it gives rise to a periodic vortex theory that in many respects is *equivalent* to the "pure" Chern-Simons theory relative to the CS model. Furthermore, in [16] (see also [10, pp. 113–115]) it is argued that, at least formally, their model includes both the CS model and the AH model as limiting cases. Thus in support of the formal arguments in [10, 16], we show rigorously that the periodic vortices of the MCS model converge (in a suitably "strong" sense) to those of the CS model and the AH model when taking the appropriate limits.

To be more precise, let us recall that the Lagrangean density for the MCS model is defined in the $(2+1)$–Minkowski space $\mathbb{R}^{1+2}$ in terms of the (real-valued) potential field $A = A_\alpha\,\mathrm{d}x^\alpha$, $\alpha = 0,1,2$, the complex-valued Higgs field $\phi$, and the real-valued neutral scalar field $N$ as follows:

$$\mathcal{L}(A,\phi,N) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} + \frac{\mu}{2q^2}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma - D_\alpha\phi(D^\alpha\phi)^* - \frac{1}{2q^2}\partial_\alpha N\partial^\alpha N - V(|\phi|,N) \tag{0.1}$$

with self-dual potential

$$V(|\phi|,N) = |\phi|^2\left(N - \frac{q^2}{\mu}\right)^2 + \frac{q^2}{2}\left(|\phi|^2 - \frac{\mu}{q^2}N\right)^2, \tag{0.2}$$

covariant derivative $D_A = D_\alpha\,\mathrm{d}x^\alpha$, $D_\alpha = \partial_\alpha + iA_\alpha$, $\alpha = 0,1,2$, Maxwell field $F_A = \mathrm{d}A = \frac{1}{2}F_{\alpha\beta}\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and where to lower or raise indices we use the metric tensor $\mathrm{diag}(-1,1,1)$ in the usual way. The constant $q>0$ denotes the electric charge. The Chern-Simons term in the theory is represented by the quantity $(\mu/2q^2)\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma$, where $\varepsilon^{\alpha\beta\gamma}$ is the totally skew-symmetric tensor fixed so that $\varepsilon^{012} = 1$ and $\mu > 0$ is the Chern-Simons mass scale.

Now if in $\mathcal{L}$ we let $\mu, q \to +\infty$ while keeping fixed the ratio $\mu/q^2$ (the CS limit), then, at least formally, both the Maxwell term for $A$ (i.e., $F_{\alpha\beta}F^{\alpha\beta}$) and the kinetic term for $N$ should drop out of the Lagrangean (0.1) while $N$ is forced to be evaluated at $(q^2/\mu)|\phi|^2$.

So, setting $\mu/q^2 = \kappa$ and inserting the identity $N = (q^2/\mu)|\phi|^2$ into the potential $V$ in (0.2), we get a sixth-order potential and a "limiting" model that corresponds exactly to the CS model, whose Lagrangean is given by

$$\mathcal{L}^{CS}(A,\phi) = \frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma - D_\alpha\phi(D^\alpha\phi)^* - \frac{1}{\kappa^2}|\phi|^2(|\phi|^2-1)^2\,. \tag{0.3}$$

Similarly, if we let $\mu \to 0$ while keeping $q$ fixed (the AH limit), then the Chern-Simons term drops out of the Lagrangean (0.1) while $N$ is forced to satisfy

$$\frac{\mu}{q^2}N = 1 \tag{0.4}$$

at $\mu = 0$. As above, by "formally" inserting this identity into the potential $V$ in (0.2), we obtain the familiar double-well potential of Ginzburg-Landau and a "limiting" Lagrangean that coincides with the following AH model:

$$\mathcal{L}^{\mathrm{AH}}(A,\phi) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} - D_\alpha\phi(D^\alpha\phi)^* - \frac{q^2}{2}(|\phi|^2-1)^2\,. \tag{0.5}$$

By now much is known about periodic vortices corresponding to both theories described by $\mathcal{L}^{CS}$ and $\mathcal{L}^{\mathrm{AH}}$; see [4, 22, 25].

For instance, we mention that while the vortex points, corresponding to the zeroes of $\phi$, *uniquely* characterize (up to gauge transformations) a periodic AH vortex, on the contrary, *multiple* periodic CS vortices with the same vortex points can coexist; see [22].

However, in the class of all possible CS vortex solutions that share the same set of vortex points, it is still possible to distinguish a "special" one that maximizes the magnitude $|\phi|$. Those have been identified by Caffarelli and Yang in [4] as the "maximal vortices" relative to a prescribed configuration of vortex points. Such a maximality notion has been extended by Chae and Kim in [6] to include the MCS vortex solutions for $\mathcal{L}$, and it proved useful in establishing some convergence results. In fact, in [6] the authors show that a maximal vortex for $\mathcal{L}$ converges (in a suitably strong sense) to the corresponding *maximal* vortex for $\mathcal{L}^{\mathrm{CS}}$ and to the *unique* vortex for $\mathcal{L}^{\mathrm{AH}}$ when taking the appropriate limits.

Our goal here is to complete these results and prove that the MCS (periodic) vortex theory for $\mathcal{L}$ is very much in line with the CS (periodic) vortex theory of $\mathcal{L}^{\mathrm{CS}}$. Namely, in analogy with $\mathcal{L}^{\mathrm{CS}}$, we establish the existence of *multiple* (periodic) vortices for each assigned set of vortex points. In addition, regardless of their maximality property, each of those vortices is shown to converge (in suitably strong norms) to CS vortices for $\mathcal{L}^{\mathrm{CS}}$ after taking the CS limit indicated above. On the contrary, concerning the AH limit, we show that only a particular one of those MCS vortices survives the passage to the limit as suggested above, while the other one diverges in a suitable sense.

## 1 Preliminaries and Statement of the Main Results

Periodic vortices (or condensates) relative to (0.1) are defined as the *static* solutions (i.e., independent of the $x^0$-variable) for the equations of motion corresponding to $\mathcal{L}$:

$$\begin{cases} \frac{1}{q^2}\partial_\beta F^{\alpha\beta} + \frac{\mu}{2q^2}\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma} = J^\alpha = -i(\phi^* D^\alpha\phi - (D^\alpha\phi)^*\phi) \\ D_\alpha D^\alpha\phi = \frac{\partial V}{\partial\phi^*} \\ \partial_\alpha\partial^\alpha N = \frac{\partial V}{\partial N} \end{cases} \tag{1.1}$$

where $J^\alpha$ is the relativistic current

subject to appropriate periodic boundary conditions. More precisely, in order to account for the invariance of (1.1) with respect to the gauge transformations,

$$\phi \to \phi e^{-i\omega}, \quad A_\alpha \to A_\alpha + \partial_\alpha\omega, \;\; \alpha = 0,1,2, \quad N \to N,$$

for every $\omega = \omega(x^0, x^1, x^2)$ smoothly defined in $\mathbb{R}^{1+2}$, the boundary conditions are specified as follows:

Given the periodic cell domain

$$\Omega = \left\{x \in \mathbb{R}^2 : x = s^1\mathbf{a}_1 + s^2\mathbf{a}_2,\, 0 < s^1, s^2 < 1\right\}$$

with $\mathbf{a}_1$ and $\mathbf{a}_2$ linearly independent vectors in $\mathbb{R}^2$, let

$$\Gamma_k = \left\{x \in \mathbb{R}^2 : \; x = s^k\mathbf{a}_k,\, 0 < s^k < 1\right\}, \quad k = 1,2,$$

so that

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \{\Gamma_1 + \mathbf{a}_2\} \cup \{\Gamma_2 + \mathbf{a}_1\}.$$

For $(A, \phi, N)$ a static (i.e., independent of the $x^0$-variable) solution of (1.1), we require that there exist smooth functions $\omega_k = \omega_k(x^1, x^2)$, $k = 1,2$, defined in a neighborhood of $\Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$ such that $(A, \phi, N)$ satisfies

$$\begin{cases} A_j(x + \mathbf{a}_k) = A_j(x) + \partial_j\omega_k(x), & j,k = 1,2, \\ A_0(x + \mathbf{a}_k) = A_0(x), & \\ \phi(x + \mathbf{a}_k) = e^{-i\omega_k(x)}\phi(x), & \\ N(x + \mathbf{a}_k) = N(x), & k = 1,2, \end{cases} \tag{1.2}$$

for all $x \in \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$, $k = 1,2$.

Since $\phi$ is required to be single-valued in $\Omega$, setting

$$\omega_k(s^1, s^2) = \omega_k(s^1\mathbf{a}_1, s^2\mathbf{a}_2), \quad k = 1,2,$$

by virtue of (1.2) we have the compatibility condition

$$\omega_1(0, 0^+) - \omega_1(0, 1^-) + \omega_2(1^-, 0) - \omega_2(0^+, 0) = 2\pi n, \tag{1.3}$$

satisfied for some $n \in \mathbb{Z}$. The integer $n$ is called the *vortex number* associated to the given periodic vortex, and it will permit us to distinguish between topologically distinct periodic vortices.

As a first consequence of (1.2) and (1.3), we find "quantized" values for both the magnetic flux $\Phi = \int_\Omega F_{12}$ and the electric charge $Q = \int_\Omega J^0$. In fact, we compute

$$\Phi = \int_\Omega F_{12} = \int_\Omega \partial_1 A_2 - \partial_2 A_1 = \int_{\Gamma_2} \partial_2 \omega_1 - \int_{\Gamma_1} \partial_1 \omega_2 = -2\pi n\,.$$

In addition, by the $\alpha = 0$ component of (1.1), we derive the Gauss law governing the system. For static solutions this law takes the form

$$-\frac{1}{q^2}\partial_j F^{0j} + \frac{\mu}{q^2} F_{12} = J^0 = -2A_0|\phi|^2 \tag{1.4}$$

and so

$$Q = \frac{\mu}{q^2}\Phi = -2\pi\frac{\mu}{q^2}n\,.$$

We can also use (1.4) to eliminate the $A_0$-component of $A$ from $\mathcal{L}$ and consequently compute the *energy* $E$ for a periodic vortex $(A,\phi,N)$ to find

$$\begin{aligned} E(A,\phi,N) = \int_\Omega \Bigg\{ & \frac{1}{2q^2}\sum_{j=1}^{2}(F_{j0} \pm \partial_j N)^2 + \left|\left(\frac{q^2}{\mu} - N \mp A_0\right)\phi\right|^2 \\ & + \frac{1}{2q^2}(\pm F_{12} - q^2|\phi|^2 + \mu N)^2 + |(D_1 \pm iD_2)\phi|^2 \Bigg\} dx \mp \Phi\,. \end{aligned}$$

In particular, if $(A,\phi,N)$ admits vortex number $n$, then

$$E \geq 2\pi|n|\,. \tag{1.5}$$

Thus, for fixed $n \in \mathbb{Z}$, in the class of periodic $n$-vortices, that is, vortices with vortex number $n$, we may identify the energy minimizers that attain equality in (1.5) as the solutions of the following (first-order) equations:

$$\begin{cases} \partial_j N \pm F_{j0} = 0 \\ (\frac{q^2}{\mu} - N \mp A_0)\phi = 0 \\ \pm F_{12} = q^2|\phi|^2 - \mu N \\ (D_1 \pm iD_2)\phi = 0\,, \end{cases} \tag{1.6}$$

together with the Gauss law (1.4). Since in the static case $F_{j0} = \partial_j A_0$, in order to obtain nontrivial solutions we reduce (1.4)–(1.6) to the following Bogomol′nyĭ equations:

$$\begin{cases} (D_1 \pm iD_2)\phi = 0 \\ \pm F_{12} = q^2|\phi|^2 - \mu N \\ \mp A_0 = N - \frac{q^2}{\mu} \\ -\Delta A_0 + \mu F_{12} = -2q^2 A_0|\phi|^2\,. \end{cases} \tag{1.7}$$

By direct computation one can rigorously show that solutions of (1.2)–(1.7) define periodic $n$-vortices with $\Phi = (q^2/\mu)Q = -2\pi n$ and $E = 2\pi|n|$.

To obtain solutions for (1.2)–(1.7), we notice first that the choice of the "upper" or "lower" sign in (1.7) reflects upon the sign of the vortex number for the corresponding solution. Indeed, if $(A,\phi,N)$ satisfies (1.7) with the "upper" sign (and, of course, the boundary condition (1.2)), then we may rewrite the first equation $D_1\phi + iD_2\phi = 0$ equivalently as follows:

$$2i\bar{\partial}\ln\phi = A_1 + iA_2 \tag{1.8}$$

and use the $\bar{\partial}$-Poincaré lemma to find that, up to a nonvanishing multiple factor, $\phi$ is holomorphic in $\Omega$, and so it admits a finite number of zeroes with integral multiplicity. In view of the periodicity conditions, we can arrange so that the degree of $\phi$ at zero in $\Omega$ is well-defined, and by (1.2) and (1.3) we obtain

$$\deg(\phi,\Omega,0) = n\,.$$

Hence, in this case, the vortex number is *nonnegative* since it coincides with the number of zeroes of $\phi$ in $\Omega$ counted according to their multiplicities.

On the other hand, solutions of (1.7) with the "lower" sign may be derived from solutions of (1.7) with the "upper" sign via the transformations

$$\phi \to \phi^*\,, \quad A \to -A\,, \quad N \to N\,,$$

and so, without loss of generality, we shall limit our attention to the Bogomol′nyĭ equations (1.7) with the upper sign.

For prescribed $n \ge 0$, in order to obtain a (self-dual) periodic $n$-vortex as a solution for

$$\begin{cases} (D_1 + iD_2)\phi = 0 \\ F_{12} = q^2|\phi|^2 - \mu N \\ -A_0 = N - \frac{q^2}{\mu} \\ -\Delta A_0 + \mu F_{12} = -2q^2 A_0|\phi|^2\,, \end{cases} \tag{1.9}$$

together with the boundary conditions (1.2), we follow Taubes [23]; in view of the above discussion, we prescribe the zeroes of $\phi$ in $\Omega$ and their multiplicities.

The case $n = 0$, where $\phi$ *never* vanishes in $\Omega$, is easily solved by the constant solution

$$(A,|\phi|^2,N) = \left(0,1,\frac{q^2}{\mu}\right).$$

For $n > 0$ we shall prove the following results:

THEOREM 1.1 *For $n \in \mathbb{N}$, any periodic $n$-vortex $(A,\phi,N)$ solution for* (1.2)–(1.9) *satisfies*

$$|\phi| < 1 \quad \textit{and} \quad 0 < N < \frac{q^2}{\mu} \quad \textit{in}\,\Omega\,.$$

*Remark.* Theorem 1.1 shows that the admissibility property introduced in [5, 6] is in fact a general property of periodic vortices for $\mathcal{L}$.

THEOREM 1.2 *Given $n \in \mathbb{N}$, the conditions*

$$q^2 > \frac{2\pi n}{|\Omega|} \quad \textit{and} \quad 0 < \mu < \frac{1}{2}\sqrt{\frac{|\Omega|}{\pi n}}\left(q^2 - \frac{2\pi n}{|\Omega|}\right)$$

*are necessary for the existence of a periodic n-vortex for $\mathcal{L}$, the solution for* (1.9).

*Given $p_1, \dots, p_s$ fixed points in $\Omega$ and $n_1, \dots, n_s \in \mathbb{N}$ such that $\sum_{j=1}^{s} n_j = n$, there exists a suitable constant $\kappa_* \in (0, \frac{1}{2}\sqrt{|\Omega|/(\pi n)})$ (depending on $p_j$ and $n_j$, $j = 1, \dots, s$) such that for every*

$$0 < \mu < \kappa_*\left(q^2 - \frac{2\pi n}{|\Omega|}\right)$$

*there exist at least* two *gauge distinct periodic n-vortices $(A, \phi, N)^{\pm}_{\mu}$, solutions for* (1.9)–(1.2)*, with the following properties*:

(i) *$\phi$ vanishes exactly in $p_1, \dots, p_s$ with multiplicities $n_1, \dots, n_s$.*
(ii) *The energy $E$, the magnetic flux $\Phi$, and the electric charge $Q$ satisfy*

$$E = 2\pi n, \qquad \Phi = \frac{q^2}{\mu} Q = -2\pi n.$$

Our next goal is to characterize the asymptotic behavior of the vortices as given in Theorem 1.2 in terms of the CS limit and the AH limit. Concerning the CS limit, note that as an immediate consequence of Theorem 1.2, we obtain the following:

COROLLARY 1.3 *For fixed $n \in \mathbb{N}$, $p_1, \dots, p_s \in \Omega$, $n_1, \dots, n_s \in \mathbb{N}$, with $\sum_{j=1}^{s} n_j = n$ and $0 < \kappa < \kappa_*$, there exists $\mu_\kappa > 0$ sufficiently large* (*depending on $\kappa$*) *such that problem* (1.9)–(1.2) *with $q^2 = \mu/\kappa$ and $\mu > \mu_\kappa$ admits two distinct solutions $(A, \phi, N)^{\pm}_{\mu}$ satisfying* (i) *and* (ii) *of Theorem* 1.2.

We show that along a sequence $\mu_j \to +\infty$, the periodic vortices $(A, \phi, N)^{\pm}_{\mu}$ converge to periodic vortices for $\mathcal{L}^{\mathrm{CS}}$ in the following sense:

THEOREM 1.4 *For fixed $n \in \mathbb{N}$, $p_1, \dots, p_s \in \Omega$, $n_1, \dots, n_s \in \mathbb{N} : \sum_{j=1}^{s} n_j = n$, and $0 < \kappa < \kappa_*$ ($\kappa_*$ as given in Theorem* 1.2)*, there exist two periodic n-vortices for $\mathcal{L}^{\mathrm{CS}}$, denoted $(\tilde{A}, \tilde{\phi})^{\pm}_{\kappa}$, satisfying the analogous properties* (i) *and* (ii) *in Theorem* 1.2 *with $\mu/q^2 = \kappa$ and such that along a sequence $\mu_j \to +\infty$ we have*

(i) *$(A, \phi, N)^{\pm}_{\mu_j} \to ((\tilde{A}, \tilde{\phi})^{\pm}_{\kappa}, \frac{1}{\kappa}|\tilde{\phi}|^2)$ as $j \to +\infty$, in $[L^2(\Omega)]^3 \times H^1(\Omega) \times L^p(\Omega)$ and $[L^2(\Omega)]^3 \times C^0(\Omega) \times L^p(\Omega)$, $\forall p \geq 1$.*
(ii) *As $\kappa \to 0^+$, $|\tilde{\phi}^+|^2 \to 1$ in $W^{1,q}(\Omega)$ $\forall q \in [1, 2)$, and uniformly on compact subsets of $\Omega \setminus \{p_1, \dots, p_s\}$, $\tilde{\phi}^- \to 0$ $C^k$-uniformly, $\forall k \geq 0$, provided $n = 1$.*

*Remark.* The existence of periodic $n$-vortices for $\mathcal{L}^{\mathrm{CS}}$ with property (ii) was established in [22]. Here we show that those vortices are in fact limits of the MCS periodic $n$-vortices constructed in Theorem 1.2. Although it is believed that (ii) should hold for $\tilde{\phi}^-$ without the restriction $n = 1$, so far only partial results have been obtained in this direction; see [7, 8, 9, 18].

For fixed $q$, recall that for $\mathcal{L}^{\mathrm{AH}}$ there exists a unique periodic $n$-vortex $(A,\phi)_0$ satisfying the corresponding properties (i) and (ii) in Theorem 1.2; see [25]. We show that in the AH limit only $(A,\phi,N)_\mu^+$ converges to $(A,\phi)_0$ as $\mu \to 0$, while $(A,\phi,N)_\mu^-$ diverges, in the following sense:

THEOREM 1.5 *Let $n \in \mathbb{N}$, $q^2 > 2\pi n/|\Omega|$, and $p_1,\dots,p_s \in \Omega$, $n_1,\dots,n_s \in \mathbb{N} : \sum_{j=1}^s n_j = n$. For $\mu > 0$ small, denote by $(A,\phi,N)_\mu^\pm$ the two periodic n-vortices of Theorem* 1.2 *and by $(A,\phi)_0$ the* unique *periodic n-vortex for $\mathcal{L}^{\mathrm{AH}}$ satisfying the analogous properties* (i) *and* (ii) *in Theorem* 1.2.

*As $\mu \to 0^+$ we have*

(i) $(A,\phi)_\mu^+ \to (A,\phi)_0$, $|\nabla N^+|^2 \to 0$, $C^k$*-uniformly* $\forall k \ge 0$, $\mu \fint N^+ \to q^2$.

(ii) $\phi^- \to 0$, $|\nabla A_0^-|^2 = |\nabla N^-|^2 \to 0$, $F_{12}^- \to -2\pi n/|\Omega|$, $C^k$*-uniformly* $\forall k \ge 0$, $\mu \fint N^- \to 2\pi n/|\Omega|$ *and* $\mu \fint A_0 \to q^2 - 2\pi n/|\Omega|$.

(iii) *For small values of $\mu$, $(A,\phi,N)_\mu^\pm$ are the* unique *solutions for* (1.2)–(1.9) *satisfying* (i) *and* (ii) *of Theorem* 1.2. *Furthermore, they form smooth curves parametrized by $\mu$ and belong to a connected solution set of* (1.2)–(1.9).

In other words, Theorem 1.5 states that, for fixed $q$, there exists a set of solutions of (1.2)–(1.9) along which the Higgs field connects $\phi_0$ with $\phi \equiv 0$ and the $A_0$ component of the gauge potential connects $A_0 \equiv 0$ with $\infty$.

Theorems 1.2 through 1.5 will be established by a further reduction of problem (1.2)–(1.9) to a system of elliptic equations subject to periodic boundary conditions. This approach has been introduced by Taubes in [23] for the AH model in $\mathbb{R}^2$, and it has been successfully adapted in several other self-dual Chern-Simons theories; see, for example, [4, 5, 6, 20, 21, 22, 24].

The key ingredient for Taubes's approach to apply is represented by the "self-duality" equation $(D_1 + iD_2)\phi = 0$ or its equivalent formulation (1.8). In fact, as already observed, (1.8) gives that (up to a nonvanishing multiple factor) $\phi$ is holomorphic and so, by its behavior near the zeroes $p_1,\dots,p_s$, $\phi$ can be written as

$$\phi(x) = |\phi(x)| \exp\left\{ i \sum_{j=1}^s n_j \operatorname{Arg}\left( \frac{x-p_j}{|x-p_j|} \right) \right\}, \quad x \in \Omega \setminus \{p_1,\dots,p_s\}. \tag{1.10}$$

Using (1.10), it is possible to extend equation (1.8) smoothly through the zeroes of $\phi$ and derive $A_1$ and $A_2$ only in terms of $\phi$ as follows:

$$A_1 + iA_2 = i(\partial_1 + i\partial_2)\ln\phi \quad \text{in } \Omega. \tag{1.11}$$

As a consequence of (1.10) and (1.11) we obtain, in the sense of distributions,

$$\Delta \ln|\phi|^2 = 2F_{12} + 4\pi \sum_{j=1}^s n_j \delta_{p_j}, \tag{1.12}$$

where $\delta_{p_j}$ denotes the Dirac measure with pole $p_j$, $j = 1, \dots, s$. Thus, by (1.9) for the unknowns $u = \ln|\phi|^2$ and $N$, we are reduced to solving

$$\begin{cases} \Delta u = 2q^2 e^u - 2\mu N + 4\pi \sum_{j=1}^{s} n_j \delta_{p_j} & \text{in } \Omega \\ \Delta N = (\mu^2 + 2q^2 e^u)N - q^2(\mu + \frac{2q^2}{\mu})e^u & \text{in } \Omega \\ u, N \text{ doubly periodic} & \text{on } \partial\Omega\,. \end{cases} \tag{1.13}$$

Clearly, for any $(u,N)$-solution of (1.13), we recover, up to a gauge transformation, the whole periodic $n$-vortex $(A,\phi,N)$ solution of (1.2)–(1.9) by setting

$$\begin{cases} \phi(x) = \exp\{\frac{1}{2}u + i\sum_{j=1}^{s} n_j \operatorname{Arg}\left(\frac{x-p_j}{|x-p_j|}\right)\} \\ A_1 + iA_2 = 2i\bar{\partial}\ln\phi \\ -A_0 = N - \frac{q^2}{\mu}\,. \end{cases} \tag{1.14}$$

Theorems 1.2 through 1.5 will be established by means of (1.13). Equivalently, in view of the periodicity required on $(u,N)$, we shall analyze (1.13) on the flat 2-torus, so with abuse of notation, we identify $\Omega$ with the flat torus obtained as the quotient of $\mathbb{R}^2$ with respect to the lattice generated by $\mathbf{a}_1$ and $\mathbf{a}_2$.

Also, it will be convenient to set

$$\frac{2q^2}{\mu} =: \lambda > 0 \quad \text{and} \quad N' = 2N\,. \tag{1.15}$$

With this change of variables, system (1.13) reduces to (omitting primes)

$$\begin{cases} \Delta u = \lambda\mu e^u - \mu N + 4\pi\sum_{j=1}^{s} n_j \delta_{p_j} \\ \Delta N = \mu(\mu + \lambda e^u)N - \lambda\mu(\mu+\lambda)e^u\,, \end{cases} \tag{1.16}$$

which sometimes it is convenient to consider as formed by the first equation in (1.16) and the linear combination

$$\Delta\left(u + \frac{N}{\mu}\right) = \lambda e^u(N - \lambda) + 4\pi\sum_{j=1}^{s} n_j \delta_{p_j}\,.$$

In this notation, the CS limit and the AH limit correspond to the analysis of solutions for (1.16) as $\mu \to +\infty$ with $\lambda$ constant and as $\mu \to 0$ with $\lambda\mu$ constant, respectively.

We devote the next sections to the study of (1.16).

## 2 A Priori Estimates and the Proof of Theorem 1.1

We collect in this section all the necessary estimates for solutions to (1.16). In the following, unless otherwise specified, all the integrals are taken over $\Omega$, identified with the flat 2-torus obtained as the quotient of $\mathbb{R}^2$ with respect to the lattice generated by $\mathbf{a}_1$ and $\mathbf{a}_2$. Furthermore, using standard notation, we let $\|\cdot\|_p$ denote the norm in $L^p(\Omega)$, $1 \le p \le \infty$, $\|\cdot\|$ denote the norm in $H^1(\Omega)$, and $⨍_\Omega = (1/|\Omega|)\int_\Omega$.

We distinguish between the singular and regular part of a solution $u$ of (1.16) by setting

$$u = u_0 + v\,,$$

with $u_0$ the *unique* solution on the compact manifold $\Omega$ for the problem

$$\begin{cases} \Delta u_0 = 4\pi \sum_{j=1}^{s} n_j \delta_{p_j} - \frac{4\pi n}{|\Omega|} \\ \int u_0 = 0\,; \end{cases} \tag{2.1}$$

see [2]. Thus, for the unknowns $(v, N)$ we are reduced to to the following elliptic system over $\Omega$:

$$\begin{cases} \Delta v = \lambda\mu e^{u_0+v} - \mu N + \frac{4\pi n}{|\Omega|} \\ \Delta N = \mu(\mu + \lambda e^{u_0+v})N - \lambda\mu(\mu+\lambda)e^{u_0+v}\,, \end{cases} \tag{2.2}$$

which, in particular, leads to

$$\Delta\left(v + \frac{N}{\mu}\right) = \lambda e^{u_0+v}(N-\lambda) + \frac{4\pi n}{|\Omega|}\,. \tag{2.3}$$

*Remark.* Notice that in (2.1) we have

$$u_0(x) = \ln|x-p_j|^{2n_j} + \gamma_j(x) \quad \text{as } x \to p_j \tag{2.4}$$

with $\gamma_j$ regular over $\Omega$, $j = 1, \dots, s$ (see [2]). Hence $e^{u_0}$ is smoothly defined over $\Omega$ and vanishes exactly at $p_j$ with multiplicity $2n_j$, $j = 1, \dots, s$. Moreover, $e^{u_0}|\nabla u_0|$ can be continuously extended over $\Omega$ and $e^{u_0}|\nabla u_0|^2 \in L^q(\Omega)$ for all $q \in [1,2)$.

We start with the following:

LEMMA 2.1 *Let* $(u, N)$ *satisfy* (1.16) *over* $\Omega$*; then* $0 < N < \lambda$ *and* $e^u < 1$ *pointwise in* $\Omega$.

PROOF: Since $N$ is smooth over $\Omega$, let $\underline{x} \in \Omega : N(x) = \min_\Omega N$ and so $\Delta N(\underline{x}) \geq 0$. Evaluating at $\underline{x}$ the right-hand side of the second equation in (1.16) we immediately find that $\min_\Omega N = N(\underline{x}) \geq 0$. Therefore, $N(\underline{x}) \geq 0$ and (by (1.16)) $\Delta N + hN \leq 0$ on $\Omega$, with $h(x) = -\mu(\mu + \lambda e^u) < 0$ smooth in $\Omega$. Thus, by the "strong" maximum principle we conclude that $N > 0$ in $\Omega$, since $N \equiv 0$ is not admissible as a solution for the second equation in (1.16).

Let $\overline{x} \in \Omega$ satisfy $N(\overline{x}) = \max_\Omega N > 0$. Then $\Delta N(\overline{x}) \leq 0$ and

$$N(\overline{x}) \leq \lambda \frac{(\mu+\lambda)e^{u(\overline{x})}}{\mu + \lambda e^{u(\overline{x})}}\,. \tag{2.5}$$

On the other hand, by (2.4), $u(x) \to -\infty$ as $x \to p_j$, $j = 1, \dots, s$, so $u$ attains its maximum value at a point $\overline{y} \in \Omega \setminus \{p_1, \dots, p_s\}$, where it satisfies

$$0 \geq \Delta u(\overline{y}) = \mu(\lambda e^{u(\overline{y})} - N(\overline{y}))\,, \quad \text{that is,} \quad \lambda e^{u(\overline{y})} \leq N(\overline{y})\,.$$

Consequently,

$$\lambda e^{u(\overline{x})} \leq \lambda e^{u(\overline{y})} \leq N(\overline{y}) \leq N(\overline{x}) \tag{2.6}$$

and we may insert (2.6) into (2.5) to derive

$$0 < N(\bar{x}) \le \frac{\mu+\lambda}{\mu+\lambda e^{u(\bar{x})}} N(\bar{x}),$$

which implies $e^{u(\bar{x})} \le 1$ and, in turn, by (2.5)

$$\max_{\Omega} N = N(\bar{x}) \le \lambda. \tag{2.7}$$

Thus

$$e^{u(\bar{y})} < \lambda^{-1} N(\bar{y}) \le \lambda^{-1} N(\bar{x}) \le 1.$$

On the other hand, by (1.16) and (2.7), on any regular subset $\Omega' \subset \Omega \setminus \{p_1, \dots, p_s\}$, $u$ satisfies

$$\Delta u = \mu(\lambda e^u - N) \ge \mu\lambda(e^u - 1), \quad \text{that is,} \quad \Delta u + h(x) u \ge 0 \ \text{ in } \Omega', \ \ u \le 0,$$

with $h(x) = -\mu\lambda(e^u - 1)/u$ smooth over $\Omega'$. Since we cannot take $u \equiv 0$ as a solution for (1.16), we conclude again by the "strong" maximum principle that $u < 0$ on every regular subset $\Omega' \subset \Omega \setminus \{p_1, \dots, p_s\}$. That is, by (2.1)

$$e^u < 1 \ \text{ in } \Omega, \quad \text{which by (2.5) also implies} \quad \max_{\Omega} N = N(\bar{x}) < \lambda,$$

and the conclusion follows. □

PROOF OF THEOREM 1.1: Recalling the transformations of (1.15), we obtain Theorem 1.1 as a direct consequence of Lemma 2.1. □

LEMMA 2.2 *Let $\lambda, \mu > 0$. The conditions*

$$\lambda\mu > \frac{4\pi n}{|\Omega|} \quad \textit{and} \quad \mu < \frac{1}{4}\sqrt{\frac{|\Omega|}{\pi n}}\left(\lambda\mu - \frac{4\pi n}{|\Omega|}\right)$$

*are necessary to the existence of solutions for* (1.16).

PROOF: Integrating (1.16) over $\Omega$, we get the constraints

$$\lambda\mu \int e^u - \mu \int N + 4\pi n = 0, \tag{2.8}$$

$$\lambda \int e^u (N - \lambda) + 4\pi n = 0. \tag{2.9}$$

Using the estimate $N < \lambda$ on $\Omega$ (Lemma 2.1), we immediately get the first of the necessary conditions

$$4\pi n \le \mu \int N - \lambda\mu \int e^u < \lambda\mu |\Omega|.$$

Decompose $N = S + \fint N$ so that $\int S = 0$ and use (2.8) and (2.9) to obtain a quadratic equation for $\fint e^u$ as follows:

$$\lambda^2 \left(\fint e^u\right)^2 - \lambda\left(\lambda - \frac{4\pi n}{\mu|\Omega|}\right) \fint e^u + \frac{4\pi n}{|\Omega|} + \lambda \fint e^u S = 0. \tag{2.10}$$

Thus, the discriminant corresponding to (2.10) must be nonnegative, and we get

$$\frac{1}{4}\left(\lambda - \frac{4\pi n}{\mu|\Omega|}\right)^2 \geq \frac{4\pi n}{|\Omega|} + \lambda \fint e^u S. \tag{2.11}$$

In order to determine the sign for the integral $\fint e^u S$, we multiply by $S$ the second equation in (1.16) and integrate over $\Omega$,

$$\begin{aligned} -\frac{1}{\mu}\|\nabla S\|_2^2 &= \mu\|S\|_2^2 + \lambda \int e^u NS - \lambda(\mu+\lambda)\int e^u S \\ &= \mu\|S\|_2^2 + \lambda\int e^u S^2 + \lambda \int e^u S\left(\fint N - \mu - \lambda\right). \end{aligned} \tag{2.12}$$

By Lemma 2.1, $\fint N < \lambda$, so in view of (2.12) we must necessarily have

$$\int e^u S > 0. \tag{2.13}$$

At this point, (2.11) and (2.13) imply

$$\lambda\mu - \frac{4\pi n}{|\Omega|} > 4\mu\sqrt{\frac{\pi n}{|\Omega|}}$$

and the desired conclusion follows immediately. □

Recall that by (1.15) the asymptotic limits involved in Theorems 1.4 and 1.5 correspond to studying the behavior of solutions for (1.16) in the case $\lambda$ fixed and $\mu \to +\infty$ and the case $\lambda\mu$ fixed and $\mu \to 0$.

For this purpose we point out the following estimates:

LEMMA 2.3 *There exists a constant $C > 0$ independent of $\lambda$ and $\mu$ such that any solution $(v, N)$ for (2.2) satisfies*

(i) $\|\nabla v\|_2 + \|\nabla N\|_2 \leq C\lambda\mu$,
(ii) $\|\nabla v\|_2 \leq C(1+\lambda^2)$, *and*
(iii) $|\fint v| \leq C(\|\nabla v\|_2^2 + \ln\lambda + 1)$.

PROOF: In the sequel, $C > 0$ will denote a general constant independent of $\lambda$ and $\mu$, and may vary from line to line. Write $v = w + \fint v$ and $N = S + \fint N$ with $\int S = 0 = \int w$.

Multiply the first equation of (2.2) by $w$ and integrate over $\Omega$; by means of Lemma 2.1 we have

$$\begin{aligned} \|\nabla v\|_2^2 = \|\nabla w\|_2^2 &= -\lambda\mu\int e^{u_0+v} w + \mu\int Nw \\ &\leq 2\lambda\mu\int|w| \leq C\lambda\mu\|\nabla w\|_2. \end{aligned} \tag{2.14}$$

Analogously, multiply the second equation in (2.2) by $N-\lambda$ and integrate over $\Omega$:

$$\begin{aligned}\int |\nabla N|^2 &= -\mu^2 \int (N-\lambda)^2 - \mu\lambda \int e^{u_0+v}(N-\lambda)^2 \\ &\quad + \lambda\mu^2 \int (\lambda - N) - \lambda\mu^2 \int e^{u_0+v}(\lambda - N) \\ &\leq |\Omega|(\lambda\mu)^2 .\end{aligned} \tag{2.15}$$

Thus, estimate (i) is an immediate consequence of (2.14) and (2.15).

In order to obtain (ii), multiply (2.3) by $w$ and integrate over $\Omega$ to obtain

$$\begin{aligned}\|\nabla v\|_2^2 = \|\nabla w\|_2^2 &= -\int \nabla \frac{N}{\mu} \cdot \nabla w + \lambda \int e^{u_0+v}(\lambda - N)w \\ &\leq \|\nabla \frac{N}{\mu}\|_2 \|\nabla w\|_2 + \lambda^2 \int |w| \\ &\leq C\left(\|\nabla \frac{N}{\mu}\|_2 + \lambda^2\right)\|\nabla w\|_2 .\end{aligned} \tag{2.16}$$

Therefore, we get (ii) as a consequence of (i) and (2.16).

Finally, to obtain (iii) recall the Moser-Trudinger inequality (see [2, 11]):

$$\int e^w \leq C \exp\left(\frac{1}{16\pi}\|\nabla w\|_2^2\right), \quad \forall w \in H^1(\Omega) : \int w = 0, \tag{2.17}$$

with $C > 0$ a suitable constant depending only on $\Omega$. Integrating (2.3) over $\Omega$ yields

$$\frac{4\pi n}{\lambda} = \int e^{u_0+v}(\lambda - N) \leq \lambda \max_{\Omega} e^{u_0} e^{⨍ v} \int e^w \leq \lambda C e^{\frac{1}{16\pi}\|\nabla w\|_2^2} e^{⨍ v},$$

that is,

$$⨍ v \geq -C(\ln\lambda + \|\nabla w\|_2^2 + 1).$$

On the other hand, by Lemma 2.1,

$$⨍ v \leq -⨍ u_0 \leq 0,$$

and we conclude estimate (iii). □

As an immediate consequence of Lemmas 2.1 and 2.3, we have the following:

COROLLARY 2.4 *Any solution* $(v,N)$ *of* (2.2) *satisfies*

$$\|v\| \leq C((\lambda\mu)^2 + \ln\lambda + 1), \qquad \|N\| \leq C\lambda\mu\left(1 + \frac{1}{\mu}\right),$$

*with* $C > 0$ *a suitable constant independent of* $\lambda$ *and* $\mu$.

## 3 A Multiplicity Result and the Proof of Theorem 1.2

The goal of this section is to prove an existence result for (2.2) that leads to the conclusion of Theorem 1.2. To this purpose we solve for $N$ the first equation in (2.2) and obtain

$$N = \frac{1}{\mu}\left(\lambda\mu e^{u_0+v} + \frac{4\pi n}{|\Omega|} - \Delta v\right),$$

which we may insert into the second equation of (2.2) and derive the following *biharmonic* equation for $v$:

$$\frac{\lambda}{\mu}\Delta e^{u_0+v} - \frac{1}{\mu^2}\Delta^2 v = (\mu + \lambda e^{u_0+v})\left(\lambda e^{u_0+v} + \frac{4\pi n|\Omega|^{-1} - \Delta v}{\mu}\right) - \lambda(\mu+\lambda)e^{u_0+v}.$$

Using the fact that $e^{u_0}\Delta u_0 = -(4\pi n/|\Omega|)e^{u_0}$ (in the sense of distributions) and performing straightforward calculations, we get

$$\frac{1}{\mu^2}\Delta^2 v - \frac{\lambda}{\mu}\left[\Delta e^{u_0+v} - \left(\frac{4\pi n}{|\Omega|} - \Delta v\right)e^{u_0+v}\right] - \Delta v = \lambda^2 e^{u_0+v}(1 - e^{u_0+v}) - \frac{4\pi n}{|\Omega|}. \tag{3.1}$$

In order to get an idea of the type of existence result we may expect, we integrate (3.1) over $\Omega$ to obtain the constraint

$$\int e^{2(u_0+v)} - \frac{1}{\lambda}\int\left(\lambda - \frac{4\pi n|\Omega|^{-1} - \Delta v}{\mu}\right)e^{u_0+v} + \frac{4\pi n}{\lambda^2} = 0. \tag{3.2}$$

Using the notation of Section 2, set $v = w + c$ with $\int w = 0$; from (3.2) we get a quadratic equation in $e^c$,

$$e^{2c}\int e^{2(u_0+w)} - e^c\frac{1}{\lambda}\int\left(\lambda - \frac{4\pi n|\Omega|^{-1} - \Delta v}{\mu}\right)e^{u_0+w} + \frac{4\pi n}{\lambda^2} = 0, \tag{3.3}$$

which yields $w$ to satisfy the necessary condition

$$\int\left(\lambda - \frac{4\pi n|\Omega|^{-1} - \Delta v}{\mu}\right)e^{u_0+w} \ge 4\sqrt{\pi n}\left(\int e^{2(u_0+w)}\right)^{1/2} \tag{3.4}$$

and

$$e^c = \left[2\lambda\int e^{2(u_0+w)}\right]^{-1}\left[\int\left(\lambda - \frac{4\pi n|\Omega|^{-1} - \Delta v}{\mu}\right)e^{u_0+w} \pm\sqrt{\left(\int\left(\lambda - \frac{4\pi n|\Omega|^{-1} - \Delta v}{\mu}\right)e^{u_0+w}\right)^2 - 16\pi n\int e^{2(u_0+w)}}\,\right]. \tag{3.5}$$

We will show the following:

THEOREM 3.1 *There exists a constant* $\kappa_* > 0$ (*depending only on* $n$, $u_0$, *and* $\Omega$) *such that if* $\mu > 0$ *and* $\lambda > 0$ *satisfy*

$$\lambda - \frac{4\pi n}{\mu|\Omega|} > \frac{1}{\kappa_*},$$

*then problem* (3.1) *admits at least* two *distinct solutions* $v^+ = w^+ + c^+$ *and* $v^- = w^- + c^-$. *Furthermore,* $w^+$ *satisfies* (3.4) *with a* strict *inequality and* $c^+$ *satisfies* (3.5) *with the "plus" sign and the following estimates hold*:

(i) $\frac{1}{\mu}\|\Delta w^+\|_2 + \|\nabla w^+\|_2 \le C\lambda$ *and*

(ii) $\int e^{2(u_0+v^+)} \ge C\big((1-\frac{4\pi n}{\lambda\mu|\Omega|})^2 + \frac{1}{\lambda^2}\ln(1-\frac{4\pi n}{\lambda\mu|\Omega|})\big)$ *for a suitable constant* $C>0$ *independent of* $\lambda$ *and* $\mu$.

*If* $n=1$, *then* $w^-$ *also satisfies* (3.4) *with a strict inequality and* $c^-$ *satisfies* (3.5) *with the "minus" sign.*

*Remark.* Notice that a lower bound of type (ii) cannot be expected for every solution of (3.1). For instance, we show that for the second solution $v^-$, we have $e^{u_0+v^-} \to 0$ uniformly as $\mu \to 0$ and $\lambda\mu$ is fixed.

By scaling we may assume without loss of generality that $|\Omega| = 1$. We shall use the variational formulation of (3.1) to prove Theorem 3.1.

Consider the functional defined for $v \in H^2(\Omega)$,

$$\begin{aligned} I(v) = {}& \frac{1}{2\mu^2}\|\Delta v\|_2^2 + \frac{\lambda}{\mu}\int (4\pi n - \Delta v)e^{u_0+v} \\ & + \frac{1}{2}\|\nabla v\|_2^2 + \frac{\lambda^2}{2}\int (e^{u_0+v}-1)^2 + 4\pi n\int v\,. \end{aligned}$$

By the Sobolev embeddings and the Moser-Trudinger inequality (2.17), $I$ is well-defined and Fréchet-differentiable on $H^2(\Omega)$, and critical points of $I$ in $H^2(\Omega)$ define (weak) solutions for (3.1).

In order to derive solutions for (3.1) with the desired sign condition in (3.5) as stated by Theorem 3.1, we insert the constraints directly into the functional $I$ as follows: Consider the subspace $X_2$ of $H^2(\Omega)$ defined by

$$X_2 = \left\{ w \in H^2(\Omega) : \int w = 0 \right\}$$

and notice that on $X_2$ the norm $\|w\|_{X_2} = \|\Delta w\|_2$ is equivalent to the standard one induced by $H^2(\Omega)$. On $X_2$ we define the set

$$\mathcal{A} = \left\{ w \in X_2;\ 4\sqrt{\pi n}\left(\int e^{2(u_0+w)}\right)^{1/2} \le \int \left(\lambda - \frac{4\pi n - \Delta w}{\mu}\right)e^{u_0+w} \right\},$$

and for every $w \in \mathcal{A}$ we define $c_\pm(w) \in \mathbb{R}$ by

$$e^{c_\pm(w)} = \left(2\lambda \int e^{2(u_0+w)}\right)^{-1} \left[\int (\lambda - \frac{4\pi n - \Delta w}{\mu}) e^{u_0+w} \pm \sqrt{\left(\int \left(\lambda - \frac{4\pi n - \Delta w}{\mu}\right) e^{u_0+w}\right)^2 - 16\pi n \int e^{2(u_0+w)}}\right]. \tag{3.6}$$

Hence $\forall w \in \mathcal{A}$, we have that $e^{c_\pm(w)}$ satisfies (3.3) and

$$\begin{aligned} I(w + c_\pm(w)) &= \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{\lambda}{\mu} e^{c_\pm(w)} \int (4\pi n - \Delta w) e^{u_0+w} + \frac{1}{2}\|\nabla w\|_2^2 \\ &\quad + \frac{\lambda^2}{2} e^{2c_\pm(w)} \int e^{2(u_0+w)} - \lambda^2 e^{c_\pm(w)} \int e^{u_0+w} + \frac{\lambda^2}{2} + 4\pi n c_\pm(w) \\ &= \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{\lambda^2}{2} e^{2c_\pm(w)} \int e^{2(u_0+w)} + 4\pi n c_\pm(w) \\ &\quad + \frac{\lambda^2}{2} - 4\pi n . \end{aligned}$$

Thus, setting

$$J_\pm(w) = \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{\lambda^2}{2} e^{2c_\pm(w)} \int e^{2(u_0+w)} + 4\pi n c_\pm(w),$$

we have that the functionals $J_\pm$ are well-defined on $\mathcal{A}$ and

$$I(w + c_\pm(w)) = J_\pm(w) + \frac{\lambda^2}{2} - 4\pi n . \tag{3.7}$$

Furthermore, it is not difficult to check that for every $w \in \mathcal{A}$ satisfying the strict inequality in (3.4), namely,

$$\frac{4\sqrt{\pi n}}{\lambda} \left(e^{2(u_0+w)}\right)^{1/2} < \int \left(1 - \frac{4\pi n - \Delta w}{\lambda\mu}\right) e^{u_0+w}, \tag{3.8}$$

the functionals $J_\pm$ are Fréchet-differentiable in $w$ and, as expected, critical points of $J_\pm$ give rise to critical points for $I$ (hence to solutions of (3.1)), which satisfy the constraint (3.5) accordingly with the plus or minus sign.

More precisely, we have the following:

LEMMA 3.2 *If $w_0 \in \mathcal{A}$ satisfying* (3.8) *is a critical point for $J_\pm$, then $v = w_0 + c_\pm(w_0)$ defines a critical point for $I$, hence a solution for* (3.1).

PROOF: By the definition of $c_\pm(w)$, we have that necessarily

$$\langle I'(w + c_\pm(w)), c\rangle = 0, \quad \forall w \in \mathcal{A}, \forall c \in \mathbb{R} .$$

Hence, let $\varphi \in X_2$. By (3.8), for small $t \in \mathbb{R}$, we have $w_0 + t\varphi \in \mathcal{A}$ and $|c_\pm(w_0 + t\varphi) - c_\pm(w_0)| = O(t)$ as $t \to 0$. Consequently,

$$\begin{aligned} I(w_0&+t\varphi+c_\pm(w_0)) - I(w_0+c_\pm(w_0)) \\ &= I(w_0+t\varphi+c_\pm(w_0)) - I(w_0+t\varphi+c_\pm(w_0+t\varphi)) \\ &\quad + I(w_0+t\varphi+c_\pm(w_0+t\varphi)) - I(w_0+c_\pm(w_0)) \\ &= \langle I'(w_0+t\varphi+c_\pm(w_0+t\varphi)), c_\pm(w_0) - c_\pm(w_0+t\varphi)\rangle + o(t) \\ &\quad + J_\pm(w_0+t\varphi) - J_\pm(w_0) \\ &= J_\pm(w_0+t\varphi) - J_\pm(w_0) + o(t)\,. \end{aligned}$$

Thus,

$$\begin{aligned} \langle I'(w_0+c_\pm(w_0)),\varphi\rangle &= \lim_{t\to 0} \frac{I(w_0+t\varphi+c_\pm(w_0)) - I(w_0+c_\pm(w_0))}{t} \\ &= \lim_{t\to 0} \frac{J_\pm(w_0+t\varphi) - J_\pm(w_0)}{t} = \langle J'_\pm(w_0),\varphi\rangle = 0\,, \end{aligned}$$

and the conclusion follows. □

Thus, to derive Theorem 3.1 we will show that under the given assumptions there exist a minimum for $J_+$ and a minimum for $J_-$ (provided $n = 1$) satisfying (3.8). The minimum for $J_+$ will provide a local minimum for the functional $I$ in $H^2(\Omega)$. Therefore, for every $n \in \mathbb{N}$, a *second* critical point of $I$ will be obtained by a mountain-pass-type construction.

We start with the following:

LEMMA 3.3 *Let $\mu > 0$ and $\lambda > 0$ be such that*

$$\lambda - \frac{4\pi n}{\mu} > 4\sqrt{\pi n}\left(\frac{\|e^{u_0}\|_2}{\|e^{u_0}\|_1}\right). \tag{3.9}$$

*Then*:

(i) $w \equiv 0$ *satisfies the strict inequality* (3.8) (*in particular, $\mathcal{A}$ contains functions satisfying the strict inequality* (3.8)).

(ii) *The functionals $J_\pm$ are bounded below in $\mathcal{A}$ and attain their infimum at $w^\pm \in \mathcal{A}$ satisfying*

$$\frac{1}{\mu}\|\Delta w^\pm\|_2 + \|\nabla w^\pm\|_2 \le C(\lambda+1)$$

*with $C > 0$ a suitable constant independent of $\lambda$ and $\mu$.*

PROOF: Recalling that $\int |\nabla u_0|^2 e^{u_0} = 4\pi n \int e^{u_0}$, property (i) is readily checked from the definition of $\mathcal{A}$. To obtain (ii), notice that every $w \in \mathcal{A}$ in particular satisfies

$$\frac{\int e^{2(u_0+w)}}{(\int e^{u_0+w})^2} \le \frac{\lambda^2}{16\pi n}\,. \tag{3.10}$$

Indeed, in the sense of distributions we have that $e^{u_0}\Delta u_0 = -4\pi n e^{u_0}$ and consequently

$$(4\pi n - \Delta v)e^{u_0+w} = -\Delta(u_0+w)e^{u_0+w} = -\Delta e^{u_0+w} + |\nabla(u_0+w)|^2 e^{u_0+w}.$$

Thus

$$\int (4\pi n - \Delta v)e^{u_0+w} = \int |\nabla(u_0+v)|^2 e^{u_0+w} > 0 \tag{3.11}$$

and (3.10) follows for every $w \in \mathcal{A}$. In particular, for every $\tau \in (0,1]$ and $a = 1/(2-\tau)$, we have

$$\begin{aligned}\int e^{u_0+w} &= \int (e^{u_0+w})^{\tau a}(e^{u_0+w})^{2(1-a)}\\ &\le \left(\int e^{\tau(u_0+w)}\right)^a \left(\int e^{2(u_0+w)}\right)^{1-a}\\ &\le \left(\int e^{\tau(u_0+w)}\right)^a \left(\frac{\lambda^2}{16\pi n}\right)^{1-a} \left(\int e^{u_0+w}\right)^{2(1-a)},\end{aligned}$$

that is,

$$\int e^{u_0+w} \le \max_\Omega e^{u_0}\left(\frac{\lambda^2}{16\pi n}\right)^{\frac{1-\tau}{\tau}} \left(\int e^{\tau w}\right)^{\frac{1}{\tau}}, \quad \forall \tau \in (0,1]. \tag{3.12}$$

Furthermore, it easy to see that

$$\frac{4\pi n}{\lambda^2 \int e^{u_0+w}} \le e^{c_\pm(w)} \le \frac{\int e^{u_0+w}}{\int e^{2(u_0+w)}}. \tag{3.13}$$

Therefore, by (3.12) and (3.13) we have

$$\begin{aligned}J_\pm(w) &= \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{\lambda^2}{2}e^{2c_\pm(w)}\int e^{2(u_0+w)} + 4\pi n c_\pm(w)\\ &\ge \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{\lambda^2}{2}\frac{(\int e^{u_0+w})^2}{\int e^{2(u_0+w)}} - 4\pi n \ln\frac{\lambda^2\int e^{u_0+w}}{4\pi n}\\ &\ge \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{\lambda^2}{2} - 4\pi n\ln\frac{\lambda^2}{4\pi n}\\ &\quad - 4\pi n \ln\left[\max_\Omega e^{u_0}\left(\frac{\lambda^2}{16\pi n}\right)^{\frac{1-\tau}{\tau}}\left(\int e^{\tau w}\right)^{\frac{1}{\tau}}\right]\\ &\ge \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{4\pi n}{\tau}\ln\int e^{\tau w} - \frac{\lambda^2}{2} - \frac{4\pi n}{\tau}\ln\lambda^2 - C_{n,\tau}\end{aligned} \tag{3.14}$$

with $C_{n,\tau}$ a suitable constant depending on $n$ and $\tau$ only. At this point, we may use the Moser-Trudinger inequality (2.17) to obtain

$$J_\pm(w) \ge \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\left(1 - \frac{n\tau}{2}\right)\|\nabla w\|_2^2 - \frac{\lambda^2}{2} - \frac{4\pi n}{\tau}\ln\lambda^2 - C_{n,\tau}. \tag{3.15}$$

Thus, by choosing $0 < \tau < \min\{1, \frac{2}{n}\}$, we find that $J_\pm$ is bounded below and coercive in $\mathcal{A}$.

Since $J_\pm$ is (sequentially) lower semicontinuous with respect to the weak topology in $X_2$, we conclude that in the (sequentially) weakly closed set $\mathcal{A}$ it attains its infimum at some point $w^\pm \in \mathcal{A}$.

Finally, by choosing $\tau = 1/n$ in (3.15), we derive a suitable constant $C_n > 0$ such that

$$\frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{4}\|\nabla w\|_2^2 \le \frac{\lambda^2}{2} + 4\pi n^2 \ln \lambda^2 + J_\pm(w^\pm) + C_n .$$

Since

$$J_\pm(w^\pm) \le J_\pm(0) = -\frac{\lambda^2}{2} e^{2c_\pm(0)} \int e^{2u_0} + 4\pi n c_\pm(0) \le 4\pi n \ln \frac{\int e^{u_0}}{\int e^{2u_0}} ,$$

the desired estimate immediately follows for any $\lambda$ satisfying (3.9). □

In view of Lemma 3.2, to obtain Theorem 3.1 we need to ensure that, under those assumptions, $w_\pm$ satisfies the strict inequality (3.8). To this purpose, we analyze the behavior of $J_\pm$ on the set

$$\Gamma = \left\{ w \in X_2 : \int \left(1 - \frac{4\pi n - \Delta w}{\lambda\mu}\right) e^{u_0 + w} = \frac{4\sqrt{\pi n}}{\lambda} \left(\int e^{2(u_0+w)}\right)^{\frac{1}{2}} \right\},$$

and notice that $J_+$ and $J_-$ coincide on $\Gamma$.

We have the following:

LEMMA 3.4 *Let $\mu > 0$ and $\lambda > 0$ satisfy* (3.9). *There exists a suitable constant $C_n$ depending on n only such that*

$$\inf_\Gamma J_+ = \inf_\Gamma J_- \ge -4\pi n \ln \lambda - 4\pi n(n-1) \ln \left(\lambda - \frac{4\pi n}{\mu}\right) - C_n . \tag{3.16}$$

PROOF: Note that for $w \in \Gamma$, we have

$$\lambda^2 e^{2c_\pm(w)} \int e^{2(u_0+w)} = 4\pi n . \tag{3.17}$$

Consequently, for all $w \in \Gamma$ we have

$$\begin{aligned} J_\pm(w) &= \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - 2\pi n + 2\pi n \ln \frac{4\pi n}{\lambda^2 \int e^{2(u_0+w)}} \\ &= \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - 2\pi n \ln \int e^{2(u_0+w)} - 4\pi n \ln \lambda \\ &\quad + 2\pi n \ln(4\pi n) - 2\pi n . \end{aligned}$$

Let

$$C_0 = \inf_{w \in X_2 \setminus \{0\}} \frac{\|\Delta w\|_2^2}{\|\nabla w\|_2^2} . \tag{3.18}$$

By the Moser-Trudinger inequality (2.17) and (3.18), we derive

$$\int e^{2w} \le C\exp\left(\frac{1}{4\pi}\|\nabla w\|_2^2\right) \le C\exp\left(\frac{1}{4\pi n}\|\nabla w\|_2^2 + \frac{1}{4\pi n}(n-1)\|\nabla w\|_2^2\right)$$
$$\le C\exp\left(\frac{1}{4\pi n}\|\nabla w\|_2^2 + \frac{\mu^2}{4\pi C_0}\left(1-\frac{1}{n}\right)\frac{\|\Delta w\|_2^2}{\mu^2}\right).$$

Therefore, for a suitable constant $C_n > 0$ (depending on $n$ only), for every $w \in \Gamma$ we obtain

$$J_\pm(w) \ge \left[1 - \frac{\mu^2}{C_0}(n-1)\right]\frac{\|\Delta w\|_2^2}{2\mu^2} - 4\pi n \ln\lambda - C_n\,. \tag{3.19}$$

Since (3.9) implies the lower bound $\lambda - 4\pi n/\mu > 1$, (3.19) immediately gives (3.16) for $n = 1$, and for $n > 1$ when $0 < \mu^2 \le C_0/(n-1) =: \mu_n^2$. Thus we are left to prove (3.16) when $n > 1$ and $\mu \ge \mu_n$. To this purpose, we argue as in Lemma 3.3 with $\tau = 2/n$ and use (3.17) to derive, for $w \in \Gamma$,

$$\begin{aligned} J_\pm(w) &= \frac{1}{2\mu^2}\|\Delta w\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - 2\pi n + 4\pi n c_\pm(w) \\ &\ge \frac{1}{2\mu^2}\|\Delta w\|_2^2 - 4\pi n^2 \ln\lambda - C_n \\ &\ge -4\pi n\ln\lambda - 4\pi n(n-1)\ln\lambda - C_n\,, \end{aligned} \tag{3.20}$$

with suitable $C_n > 0$ (depending on $n$ only). Since we assume $\mu \ge \mu_n$, by (3.9) we get a suitable constant $\overline{\mu}_n > 0$ such that $\lambda\mu \ge \overline{\mu}_n$ and so

$$\ln\lambda \le \ln\left(\lambda - \frac{4\pi n}{\mu}\right) + C_n$$

for a suitable constant $C_n > 0$. As $n > 1$, at this point (3.16) immediately follows from (3.20). □

At this point, we can complete the description of the minimization problem concerning $J_+$ on $\mathcal{A}$ as follows:

PROPOSITION 3.5 *There exists a sufficiently small $\kappa_1 > 0$ such that, if $\lambda - 4\pi n/\mu > 1/\kappa_1$, then*

(i) *the minimum $w^+$ of $J_+$ satisfies the strict inequality* (3.8), *namely,*

$$\int\left(1 - \frac{4\pi n - \Delta w^+}{\lambda\mu}\right)e^{u_0 + w^+} > \frac{4\sqrt{\pi n}}{\lambda}\left(\int e^{2(u_0+w^+)}\right)^{1/2}$$

(*hence, it defines a critical point for $J_+$*).

(ii) *The function $v_* = w^+ + c_+(w^+)$ defines a* local minimum *for the functional $I$ on $H^2(\Omega)$.*

(iii) *The following estimates hold*:

(a) $\frac{1}{\mu}\|\Delta w^+\|_2 + \|\nabla w^+\|_2 \le C\lambda$

(b) $\int e^{2(u_0+v_*)} \ge C\big((1 - \frac{4\pi n}{\lambda\mu})^2 + \frac{1}{\lambda^2}\ln(1 - \frac{4\pi n}{\lambda\mu})\big)$

*for a suitable constant $C > 0$ independent of $\lambda$ and $\mu$.*

PROOF: We shall use Lemma 3.4 and compare the values of $J_+$ on $\Gamma$ with that of $J_+$ at $0 \in \mathcal{A} \setminus \Gamma$. To this purpose, recall that

$$e^{c_+(0)} = \frac{\int e^{u_0}}{2\int e^{2u_0}} \left[ \left(1 - \frac{4\pi n}{\lambda\mu}\right) + \sqrt{\left(1 - \frac{4\pi n}{\lambda\mu}\right)^2 - \frac{16\pi n}{\lambda^2} \frac{\int e^{2u_0}}{(\int e^{u_0})^2}} \right]$$

and so

$$\left(1 - \frac{4\pi n}{\lambda\mu}\right) \frac{\int e^{u_0}}{2\int e^{2u_0}} \le e^{c_+(0)} \le \left(1 - \frac{4\pi n}{\lambda\mu}\right) \frac{\int e^{u_0}}{\int e^{2u_0}}\,.$$

Thus,

$$\begin{aligned} J_+(0) &= -\frac{\lambda^2}{2} e^{2c_+(0)} \int e^{2u_0} + 4\pi n c_+(0) \\ &\le -\frac{\lambda^2}{2} \left(1 - \frac{4\pi n}{\lambda\mu}\right)^2 \frac{(\int e^{u_0})^2}{4\int e^{2u_0}} + 4\pi n \ln\left(1 - \frac{4\pi n}{\lambda\mu}\right), \end{aligned}$$

and in view of (3.16) we may assert that

$$\inf_{\Gamma} J_+ > J_+(0) \ge \inf_{\mathcal{A}} J_+ = J_+(w^+)\,,$$

provided there holds

$$\frac{1}{8} \frac{(\int e^{u_0})^2}{\int e^{2u_0}} \left(\lambda - \frac{4\pi n}{\mu}\right)^2 > 4\pi n^2 \ln\left(\lambda - \frac{4\pi n}{\mu}\right) + C_n\,, \tag{3.21}$$

with $C_n$ the constant in Lemma 3.4. Thus, we may find a suitably small constant $\kappa_1 > 0$ (depending on $n$ only) such that if $\lambda - 4\pi n/\mu > 1/\kappa_1$, then (3.21) holds and $\inf_\Gamma J_+ > J_+(w^+)$. Consequently, $w^+ \in \mathcal{A} \setminus \Gamma$ and (i) is established.

To obtain (ii), observe that, by (3.7), $I(w + c_+(w)) \ge I(w^+ + c_+(w^+))$, $\forall w \in \mathcal{A}$. In view of (i) we find $\varepsilon_0 > 0$ such that

$$\begin{gathered} \int \left(1 - \frac{4\pi n - \Delta w^+}{\lambda\mu}\right) e^{u_0 + w^+} - \frac{4\sqrt{\pi n}}{\lambda} \left(\int e^{2(u_0 + w^+)}\right)^{1/2} \ge \varepsilon_0\,, \\ c_+(w^+) - c_-(w^+) \ge \varepsilon_0\,. \end{gathered}$$

Thus, for $\delta_1 > 0$ sufficiently small, if $\|w - w^+\|_{H^2} < \delta_1$, then

$$\begin{gathered} \int \left(1 - \frac{4\pi n - \Delta w}{\lambda\mu}\right) e^{u_0 + w} - \frac{4\sqrt{\pi n}}{\lambda} \left(\int e^{2(u_0 + w)}\right)^{1/2} \ge \frac{\varepsilon_0}{2}, \\ c_+(w^+) - c_-(w) \ge \frac{\varepsilon_0}{2}, \quad |c_+(w) - c_+(w^+)| \le \frac{\varepsilon_0}{4}\,. \end{gathered}$$

In particular, this implies that if $v = w + c \in H^2(\Omega)$ satisfies $\|w - w^+\|_{H^2} \le \delta_1$ and $|c - c_+(w^+)| \le \varepsilon_0/4$, then $w \in \mathcal{A}$ and $c - c_-(w) > 0$. Consequently,

$$I(w + c) \ge \min_{c \ge c_-(w)} I(w + c) = I(w + c_+(w)) \ge I(w^+ + c_+(w^+))\,,$$

and the desired conclusion (ii) follows.

Finally, to obtain (iii) notice that (i) follows from Lemma 3.3 when $\lambda$ is sufficiently large as in the hypothesis. To establish (ii) we can argue as in Lemma 3.3 with $\tau = 2/n$ to derive a constant $\overline{C}_n > 0$ (depending on $n$ only) such that

$$\begin{aligned}
&\frac{\lambda^2}{2} e^{2c_+(w^+)} \int e^{2(u_0+w^+)} \\
&\quad \geq -4\pi n^2 \ln\lambda - \overline{C}_n - J_+(w^+) \\
&\quad \geq -4\pi n^2 \ln\lambda - \overline{C}_n - J_+(0) \\
&\quad \geq -4\pi n^2 \ln\lambda - \overline{C}_n + \frac{\lambda^2}{8}\left(1 - \frac{4\pi n}{\lambda\mu}\right)^2 \frac{\|e^{u_0}\|_1^2}{\|e^{u_0}\|_2^2} - 4\pi n \ln\left(1 - \frac{4\pi n}{\lambda\mu}\right).
\end{aligned}$$

Hence, by choosing $\kappa_1 > 0$ smaller if necessary, we may insure that, for $\lambda - 4\pi n/\mu > 1/\kappa_1$ we have

$$\frac{\lambda^2}{16}\left(1 - \frac{4\pi n}{\lambda\mu}\right)^2 \frac{\|e^{u_0}\|_1^2}{\|e^{u_0}\|_2^2} \geq 4\pi n^2 \ln\left(\lambda - \frac{4\pi n}{\mu}\right) + \overline{C}_n\,.$$

Consequently,

$$\int e^{2(u_0+v_*)} \geq \frac{1}{8}\left(1 - \frac{4\pi n}{\lambda\mu}\right)^2 \frac{\|e^{u_0}\|_1^2}{\|e^{u_0}\|_2^2} + \frac{8\pi n(n-1)}{\lambda^2} \ln\left(1 - \frac{4\pi n}{\lambda\mu}\right)$$

and (ii) easily follows. □

From the existence of a local minimum for $I$, we immediately derive a *second* critical point, because, as we shall see, $I$ exhibits a mountain pass structure. Towards this goal, we begin by showing that $I$ satisfies the Palais-Smale condition. Actually, $I$ satisfies the following stronger condition:

LEMMA 3.6 *Let* $\{v_j\} \subset H^2(\Omega)$ *satisfy* $\|I'(v_j)\|_{H^{-2}(\Omega)} \to 0$ *as* $j \to \infty$. *Then* $v_j$ *admits a strongly convergent subsequence in* $H^2(\Omega)$.

PROOF: Recall that, for all $v, \varphi \in H^2(\Omega)$, we have

$$\begin{aligned}
\langle I'(v), \varphi\rangle = {}& \frac{1}{\mu^2}\int \Delta v \Delta\varphi + \frac{\lambda}{\mu}\int (4\pi n - \Delta v) e^{u_0+v}\varphi - \frac{\lambda}{\mu}\int \Delta e^{u_0+v}\varphi \\
&+ \int \nabla v \cdot \nabla\varphi + \lambda^2 \int (e^{u_0+v} - 1) e^{u_0+v}\varphi + 4\pi n \int \varphi\,.
\end{aligned}$$

Therefore, applying the assumption at $\varphi = 1$, we derive

$$\begin{aligned}
o(1) &= \langle I'(v_j), 1\rangle \\
&= \frac{\lambda}{\mu}\int (4\pi n - \Delta v_j) e^{u_0+v_j} + \lambda^2 \int (e^{u_0+v_j} - 1) e^{u_0+v_j} + 4\pi n
\end{aligned} \tag{3.22}$$

as $j\to\infty$. Using the inequality $(e^x-1)e^x \ge -\frac{1}{4}\ \forall x\in\mathbb{R}$ and (3.11), we obtain the bounds

$$0 \le \int |\nabla(u_0+v_j)|^2 e^{u_0+v_j} = \int (4\pi n - \Delta v_j)e^{u_0+v_j} \le C_1, \tag{3.23}$$

$$\|e^{u_0+v_j}\|_2 \le C_2\,, \tag{3.24}$$

with $C_1$ and $C_2$ independent of $j$.

Writing $v_j = w_j + c_j$, with $\int w_j = 0$, we derive

$$\begin{aligned} o(1)\|\Delta w_j\|_2 &= \langle I'(v_j), w_j\rangle \\ &= \frac{1}{\mu^2}\|\Delta w_j\|_2^2 + \frac{\lambda}{\mu}\int (4\pi n - \Delta v_j)e^{u_0+v} w_j \\ &\quad - \frac{\lambda}{\mu}\int \Delta e^{u_0+v_j} w_j + \|\nabla w_j\|_2^2 + \lambda^2 \int (e^{u_0+v_j}-1)e^{u_0+v_j} w_j\,. \end{aligned} \tag{3.25}$$

By (3.11) and Sobolev embeddings, we have

$$\begin{aligned} \left|\int (4\pi n - \Delta v_j)e^{u_0+v_j} w_j\right| &= \left|\int (-\Delta e^{u_0+v_j} + |\nabla(u_0+v_j)|^2 e^{u_0+v_j}) w_j\right| \\ &\le \left|\int e^{u_0+v_j}\Delta w_j\right| + \|w_j\|_\infty \int |\nabla(u_0+v_j)|^2 e^{u_0+v_j} \\ &\le \|e^{u_0+v_j}\|_2 \|\Delta w_j\|_2 + \|w_j\|_\infty \int (4\pi n - \Delta v_j) e^{u_0+v_j} \\ &\le (C_2 + C_1 C_3)\|\Delta w_j\|_2\,, \end{aligned}$$

with $C_3$ the constant of the Sobolev embedding. Therefore, from (3.25) we conclude that

$$\|\Delta w_j\|_2 \le C\,, \tag{3.26}$$

which, together with the Moser-Trudinger inequality, (2.17) leads to

$$\int e^{u_0+w_j} \le C \tag{3.27}$$

for some constant $C$ independent of $j$. By (3.24) and Jensen's inequality, $c_j$ is bounded above. On the other hand, by (3.22) and (3.27) we have

$$\frac{4\pi n}{\lambda^2} + o(1) \le e^{c_j}\int e^{u_0+w_j} \le Ce^{c_j} \quad \text{as } j\to\infty\,,$$

which shows that $c_j$ is also bounded below uniformly in $j$. Consequently, the sequence $\{v_j\}$ is bounded in $H^2(\Omega)$.

It follows that we can find a subsequence $w_{j_k}+c_{j_k}$ such that $w_{j_k} \rightharpoonup w_0 \in X_2$ weakly in $H^2(\Omega)$, strongly in $W^{1,p}(\Omega)\ \forall p\ge 1$, and uniformly on $\Omega$, and $c_{j_k}\to c_0$.

Hence,

$$
\begin{aligned}
o(1) &= \langle I'(v_{j_k}), w_{j_k} - w_0 \rangle \\
&= \frac{1}{\mu^2}\left( \int |\Delta w_{j_k}|^2 - \int \Delta w_{j_k} \Delta w_0 \right) - \frac{\lambda}{\mu} \int \Delta(w_{j_k} - w_0) e^{u_0 + v_{j_k}} \\
&\quad + \frac{\lambda}{\mu} \int (4\pi n - \Delta v_{j_k}) e^{u_0 + v_{j_k}} (w_{j_k} - w_0) + \int |\nabla w_{j_k}|^2 - \int \nabla w_{j_k} \cdot \nabla w_0 \\
&\quad + \lambda^2 \int e^{u_0 + v_{j_k}} (e^{u_0 + v_{j_k}} - 1)(w_{j_k} - w_0) + 4\pi n (c_{j_k} - c_0) \\
&= \frac{1}{2\mu^2} (\|\Delta w_{j_k}\|_2^2 - \|\Delta w_0\|_2^2) + o(1),
\end{aligned}
$$

as $j \to \infty$. That is,

$$
\|\Delta(w_{j_k} - w_0)\|_2^2 = o(1) \quad \text{as } j \to \infty,
$$

and the conclusion follows. □

PROPOSITION 3.7 *Let* $\mu > 0$ *and* $\lambda > 0$ *be such that* $\lambda - 4\pi n/\mu > 1/\kappa_1$; *then the functional* $I$ *admits a* second *critical point* (*other than the local minimum established in Proposition* 3.5).

PROOF: Set $v_* = w^+ + c_+(w^+)$ the local minimum as obtained in Proposition 3.5, and let $\rho_0 > 0$ be such that $\forall v \in H^2(\Omega) : \|v - v_*\|_{H^2} < \rho_0$ there holds $I(v) \geq I(v_*)$. We have the following alternatives:

1. $\inf_{\|v - v^*\|_{H^2} = \rho} I(v) = I(v^*)$ $\forall \rho \in (0, \rho_0)$ or
2. there exists $\rho_1 \in (0, \rho_0)$ such that $\inf_{\|v - v_*\|_{H^2} = \rho_1} I(v) > I(v_*)$.

In the case where alternative 1 holds, by Ekeland's lemma (see [12]) we find for every $0 < \rho < \rho_0$ there exists a $v_\rho \in X_2$ such that $\|v_\rho - v_*\|_{H^2} = \rho$ and $I(v_\rho) = I(v_*)$. Thus, $v_\rho$ defines another local minimum for $I$, and we actually obtain a continuum of critical points in this situation. Therefore assume that alternative 2 holds. In this case, we show that $I$ admits a mountain pass structure which, in view of the result of Ambrosetti and Rabinowitz [1], leads to a second critical point of $I$ of mountain pass type. Therefore in any case we may conclude the existence of a second critical point for $I$.

In order to construct the mountain pass critical value, note that

$$
\begin{aligned}
I(v_* + c) = \frac{1}{2\mu^2} \|\Delta v_*\|_2^2 + \frac{\lambda}{\mu} e^c \int (4\pi n - \Delta v_*) e^{u_0 + v_*} + \frac{1}{2} \|\nabla v_*\|_2^2 \\
+ \frac{\lambda^2}{2} \int (e^{u_0 + v_* + c} - 1)^2 + 4\pi n \int v_* + 4\pi n c \to -\infty
\end{aligned}
$$

as $c \to -\infty$. Hence we may find a sufficiently large negative constant $c_1 < 0$ such that, setting $v_1 = v_* + c_1$ and using alternative 2, we have

$$
\|v_* - v_1\|_{H^2} > \delta_1 \quad \text{and} \quad I(v_1) < I(v_*) < \inf_{\|v - v_*\|_{H^2} = \rho_1} I. \tag{3.28}
$$

Lemma 3.6 together with (3.28) guarantees that all assumptions of the mountain pass lemma are satisfied. Thus, by setting

$$\mathcal{P} = \{\gamma : [0,1] \to H^2(\Omega) \text{ continuous such that } \gamma(0) = v_* \text{ and } \gamma(1) = v_1\},$$

we can conclude that

$$c_1 = \inf_{\gamma\in\mathcal{P}} \sup_{t\in[0,1]} I(\gamma(t)) \geq \inf_{\|v-v_*\|_{H^2}=\rho_1} I > I(v_*)$$

defines a critical value for $I$ other than $I(v_*)$. □

In order to conclude the proof of Theorem 3.1, we need only establish the following:

LEMMA 3.8 *Let $n = 1$. There exists $\kappa_2 > 0$ (independent of $\mu$ and $\lambda$) such that if $\lambda - 4\pi n/\mu > 1/\kappa_2$, then the minimum $w^-$ of $J_-$ on $\mathcal{A}$ satisfies* (3.8); *namely,*

$$\int \left(\lambda - \frac{4\pi n - \Delta w^-}{\mu}\right) e^{u_0+w^-} > 4\sqrt{\pi n} \int e^{2(u_0+w^-)}.$$

PROOF: Once more we appeal to Lemma 3.4 and compare the minimal value of $J_-$ on $\Gamma$ with $J_-(0)$. To this end, note that

$$\int |\nabla u_0|^2 e^{u_0} = 4\pi n \int e^{u_0}$$

and

$$\begin{aligned} e^{c_-(0)} &= \frac{\int e^{u_0} - \frac{1}{\lambda\mu}\int|\nabla u_0|^2 e^{u_0} - \sqrt{(\int e^{u_0} - \frac{1}{\lambda\mu}\int|\nabla u_0|^2 e^{u_0})^2 - \frac{16\pi n}{\lambda^2}\int e^{2u_0}}}{2\int e^{2u_0}} \\ &= \frac{8\pi n}{\lambda^2} \frac{1}{\int e^{u_0}\left[1 - \frac{4\pi n}{\lambda\mu} + \sqrt{\left(1 - \frac{4\pi n}{\lambda\mu}\right)^2 - \frac{16\pi n}{\lambda^2}\frac{\int e^{2u_0}}{(\int e^{u_0})^2}}\,\right]}. \end{aligned}$$

Hence, in this case we have

$$\frac{4\pi n}{\lambda^2 \int e^{u_0}} \leq e^{c_-(0)} \leq \frac{8\pi n}{\lambda^2(1 - \frac{4\pi n}{\lambda\mu})\int e^{u_0}}.$$

Consequently,

$$\begin{aligned} J_-(0) &= -\frac{\lambda^2}{2} e^{2c_-(0)} \int e^{2u_0} + 4\pi n c_-(0) \\ &\leq -\frac{8\pi^2 n^2}{\lambda^2} \frac{\int e^{2u_0}}{(\int e^{u_0})^2} - 4\pi n \ln\lambda\left(\lambda - \frac{4\pi n}{\mu}\right) + 4\pi n \ln\frac{8\pi n}{\int e^{u_0}} \\ &\leq -4\pi n\ln\lambda - 4\pi n\ln\left(\lambda - \frac{4\pi n}{\mu}\right) + 4\pi n\ln(8\pi n) \end{aligned}$$

and for $n = 1$ we conclude that

$$J_-(0) \leq -4\pi\ln\lambda - 4\pi\ln\left(\lambda - \frac{4\pi}{\mu}\right) + 4\pi\ln(8\pi). \tag{3.29}$$

On the other hand, in view of Lemma 3.4, from (3.16) with $n = 1$ and (3.29), we find $\kappa_2 > 0$ sufficiently small such that if $\lambda - 4\pi/\mu > 1/\kappa_2$, then

$$\inf_{\Gamma} J_- \geq -4\pi \ln \lambda - C_1 > J_-(0) \geq \inf_{\mathcal{A}} J_- = J_-(w^-)\,.$$

Consequently, $w^- \in \mathcal{A} \setminus \Gamma$, and the desired conclusion follows. ☐

PROOF OF THEOREM 3.1: Take $\kappa_* = \min\{\kappa_1, \kappa_2\}$. Then the existence of the first solution of (3.1) follows by Proposition 3.5 together with Lemma 3.2 just by taking $v_1 = v_* = w^+ + c_+(w^+)$. The existence of the second solution is contained in Proposition 3.7. Finally, the case $n = 1$ follows by Lemma 3.8 and Lemma 3.2 with $v_2 = w^- + c_-(w^-)$. ☐

PROOF OF THEOREM 1.2: Recalling the transformations of (1.15), we derive Theorem 1.2 as a consequence of (1.14) and the multiple existence of solutions for (1.13), which follows by Theorem 3.1 and Lemma 2.2. ☐

## 4 The CS Limit and the Proof of Theorem 1.4

Recall that in the CS model of Hong, Kim, and Pac [13] and Jackiw and Weinberg [14], the Maxwell term $F_{\alpha\beta}F^{\alpha\beta}$ is neglected and the electrodynamics of the systems is governed solely by the Chern-Simons term $\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma$. This approximation is sensible at large distances, where the second-order Maxwell term is dominated by the first-order Chern-Simons term.

The Lagrangean density is given by

$$\mathcal{L}^{\mathrm{CS}}(A,\phi) = \frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma - D_\alpha\phi(D^\alpha\phi)^* - \frac{1}{\kappa^2}|\phi|^2(|\phi|^2-1)^2$$

with $\kappa$ the Chern-Simons coupling constant. Note that in this case the modified Gauss law reduces to

$$F_{12} = \frac{1}{\kappa}J^0 = -\frac{2}{\kappa}A_0|\phi|^2\,. \tag{4.1}$$

A periodic CS-vortex $(A,\phi)$ in $\Omega$ is defined as a time-independent configuration of $\mathcal{L}^{\mathrm{CS}}$ satisfying the first three in the set of boundary conditions (1.2). As above we may define its vortex number $n$ and find that the *magnetic flux* $\Phi = \int_\Omega F_{12} = -2\pi n$, while, by (4.1), the total *electric charge* $Q = \int_\Omega J^0 = \kappa\Phi = -2\pi\kappa n$.

Furthermore, (4.1) also enables us to obtain the following form for the energy:

$$E^{\mathrm{CS}}(A,\phi) = \int_\Omega \left\{ |(D_1 \pm iD_2)\phi|^2 + \frac{1}{4}\left[\frac{\kappa}{|\phi|}F_{12} \mp \frac{2}{\kappa}|\phi|(|\phi|^2-1)\right]^2 \mp F_{12}\right\} dx\,.$$

Thus, in the class of periodic $n$-vortices we have

$$E^{\mathrm{CS}} \geq 2\pi|n|\,,$$

and an energy minimizer $(A,\phi)$ corresponds to a solution of the Bogomol′nyĭ equations

$$\begin{cases} (D_1 \pm iD_2)\phi = 0 \\ \pm F_{12} = \frac{2}{\kappa^2}|\phi|^2(|\phi|^2-1) \\ F_{12} = -\frac{2}{\kappa}A_0|\phi|^2 . \end{cases} \tag{4.2}$$

It is possible to check rigorously that solutions for (4.2) indeed lead to periodic $n$-vortices for $\mathcal{L}^{\mathrm{CS}}$ with $n = \deg(\phi,\Omega,0)$, $E = 2\pi|n|$, and $\Phi = \frac{1}{\kappa}Q = -2\pi n$. As before, we shall limit our attention to the case $n > 0$, for which we must consider (4.2) with the upper sign. Furthermore, by Taubes's approach [23], we can use doubly periodic solutions for the elliptic problem

$$\Delta u = \frac{4}{\kappa^2}e^u(e^u-1) + 4\pi\sum_{j=1}^{s} n_j\delta_{p_j} \tag{4.3}$$

with $p_j \in \Omega$, $n_j \in \mathbb{N}$, $j = 1,\dots,s$, and $n = \sum_{j=1}^{s} n_j$, to construct periodic $n$-vortices $(A,\phi)$ as solutions of (4.2) (with the upper sign) by setting

$$\begin{cases} \phi(x) = \exp\{\frac{1}{2}u + i\sum_{j=1}^{s} n_j \operatorname{Arg}(\frac{x-p_j}{|x-p_j|})\} \\ A_1 + iA_2 = 2i\overline{\partial}\ln\phi \\ A_0 = \frac{1}{\kappa}(1-|\phi|^2), \end{cases} \tag{4.4}$$

and $\phi$ vanishes exactly at $p_j$ with multiplicity $n_j$, $j = 1,\dots,s$. Using the notation of Section 1, we identify $\Omega$ with the flat torus obtained as the quotient of $\mathbb{R}^2$ by the lattice generated by $\mathbf{a}_1$ and $\mathbf{a}_2$, and let

$$\kappa = \frac{2}{\lambda} . \tag{4.5}$$

Furthermore, after scaling, we can always assume $|\Omega| = 1$. So, if we decompose $u = u_0 + v$, with $u_0$ uniquely defined by (2.1), in view of (4.3) and (4.5) we obtain $v$ as a solution for the equation

$$\Delta v = \lambda^2 e^{u_0+v}(e^{u_0+v}-1) + 4\pi n \tag{4.6}$$

in the compact manifold $\Omega$.

In order to obtain Theorem 1.4 we start with the following:

PROPOSITION 4.1 *Let $(v_j,N_j)$ be solutions for* (2.2) *with $\mu = \mu_j \to +\infty$ and $\lambda > 0$ fixed. There exists a solution $\tilde{v}$ of equation* (4.6) *such that, for a subsequence* (*still denoted by $(v_j,N_j)$*), *the following hold*:

(i) *$v_j \to \tilde{v}$, strong $H^1$ and uniformly,*
(ii) *$N_j \to \lambda e^{u_0+\tilde{v}}$, strong $L^p$ $\forall p \geq 1$,*
(iii) *$\frac{N_j}{\mu_j} \to 0$, strong $H^1$, and*
(iv) *$v_j + \frac{N_j}{\mu_j} \to \tilde{v}$ in $C^{1,\alpha}$.*

PROOF: By Lemmas 2.1 and 2.3, we have that $v_j$ and $N_j/\mu_j$ are uniformly bounded in $H^1(\Omega)$ and $N_j$ is uniformly bounded in $L^\infty$. Therefore, there exist functions $\tilde{v} \in H^1(\Omega)$ and $T \in L^\infty(\Omega)$ such that, up to subsequences,

- $v_j \rightharpoonup \tilde{v}$ weakly in $H^1$, strongly in $L^p$ $\forall p \geq 1$, and pointwise a.e.,
- $N_j \rightharpoonup T$ weakly in $L^p$ $\forall p > 1$, and
- $\frac{N_j}{\mu_j} \rightharpoonup 0$ weakly in $H^1$, strong $L^p$ $\forall p \geq 1$.

Since (by Lemma 2.1) $u_0 + v_j \leq 0$, by the dominated convergence theorem we also have

- $e^{u_0+v_j} \to e^{u_0+\tilde{v}}$ strong $L^p$ $\forall p \geq 1$.

Using the first equation in (2.2), we have

$$\int \frac{\nabla v_j}{\mu_j} \cdot \nabla \varphi - \int (\lambda e^{u_0+v_j} - N_j)\varphi + \frac{4\pi n}{\mu_j} \int \varphi = 0 \quad \forall \varphi \in H^1(\Omega),$$

and taking limits we find

$$T = \lambda e^{u_0+\tilde{v}} \quad \text{a.e. in } \Omega. \tag{4.7}$$

From (2.3) we have

$$\int \nabla \left( v_j + \frac{N_j}{\mu_j} \right) \cdot \nabla \varphi + \lambda \int e^{u_0+v_j}(N_j - \lambda)\varphi + 4\pi n \int \varphi = 0 \quad \forall \varphi \in H^1(\Omega),$$

and so, passing to the limit, we see that $\tilde{v}$ is a weak solution for (4.3). By elliptic regularity, $\tilde{v}$ is smooth in $\Omega$.

We are left to show that the convergences are strong. To this end, we multiply the second equation in (2.2) by $\mu_j^{-2} N_j$ and integrate to obtain

$$\int \left| \nabla \frac{N_j}{\mu_j} \right|^2 = \int (\lambda e^{u_0+v_j} - N_j) N_j + \lambda \mu_j^{-1} \int e^{u_0+v_j}(\lambda - N_j) N_j$$

and thus

$$\left\| \nabla \frac{N_j}{\mu_j} \right\|_2^2 = -\|N_j\|_2^2 + \int \lambda^2 e^{2(u_0+\tilde{v})} + o(1).$$

Consequently, in view of (4.7),

$$\limsup_{j\to\infty} \left\| \nabla \frac{N_j}{\mu_j} \right\|_2^2 = \int \lambda^2 e^{2(u_0+\tilde{v})} - \liminf_{j\to\infty} \|N_j\|_2^2 \leq \lambda^2 \int e^{2(u_0+\tilde{v})} - \|T\|_2^2 = 0,$$

and we conclude

$$\left\| \nabla \frac{N_j}{\mu_j} \right\|_2 \to 0 \quad \text{and} \quad \|N_j - \lambda e^{u_0+\tilde{v}}\|_2 \to 0.$$

Since $N_j$ is uniformly bounded in $L^\infty(\Omega)$, in fact we have

$$N_j \to \lambda e^{u_0+\tilde{v}} \quad \text{strong } L^p \ \forall p \geq 1.$$

Inserting into (2.3), we see that

$$\Delta\left(v_j - \tilde{v} + \frac{N_j}{\mu_j}\right) = \lambda e^{u_0+v_j}(N_j - \lambda e^{u_0+\tilde{v}}) + \lambda^2 e^{u_0+\tilde{v}}(e^{u_0+v_j} - e^{u_0+\tilde{v}}).$$

Therefore, by the Schauder estimates we may conclude $\nabla(v_j - \tilde{v} + N_j/\mu_j) \to 0$ in $C^{1,\alpha}$ for every $\alpha \in (0,1)$ and, in particular, $\|\nabla(\tilde{v} - v_j)\|_2 \to 0$ as $j \to +\infty$. $\square$

While Proposition 4.1 will allow us to derive immediately part (i) of Theorem 1.4, in order to show that, as $\kappa \to 0$, the limiting vortices admit the asymptotic behavior as discussed in (ii) of Theorem 1.4, we start by showing that the minimality properties of $w^{\pm}$ for $J_{\pm}$ (see Lemma 3.3) pass to the limit as $\mu \to +\infty$.

To this purpose, let us recall the variational properties of (4.6) (see [4, 22]). The solutions for (4.6) correspond to critical points for the functional

$$\tilde{I}(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{\lambda^2}{2}\int (e^{u_0+v} - 1)^2 + 4\pi n \int v,$$

defined on $H^1(\Omega)$. Integration of (4.6) over $\Omega$ yields the constraint

$$\int e^{2(u_0+v)} - \int e^{u_0+v} + \frac{4\pi n}{\lambda^2} = 0.$$

Therefore, setting

$$X_1 = \left\{ w \in H^1(\Omega) : \int w = 0 \right\},$$

if $\tilde{v}$ satisfies (4.6), then writing $\tilde{v} = \tilde{w} + \tilde{c}$ with $\tilde{w} \in X_1$, we see that $\tilde{w}$ belongs to the set

$$\tilde{\mathcal{A}} = \left\{ w \in X_1 : \left(\int e^{u_0+w}\right)^2 \geq \frac{16\pi n}{\lambda^2}\int e^{2(u_0+w)} \right\}$$

and $\tilde{c}$ takes one of the following values:

$$e^{\tilde{c}} = e^{\tilde{c}_{\pm}(\tilde{w})} = \frac{\int e^{u_0+\tilde{w}} \pm \sqrt{(\int e^{u_0+\tilde{w}})^2 - \frac{16\pi n}{\lambda^2}\int e^{2(u_0+\tilde{w})}}}{2\int e^{2(u_0+\tilde{w})}}.$$

Thus, as above, for $\tilde{w} \in \tilde{\mathcal{A}}$ we may define the constrained functional $\tilde{J}_{\pm}$ by setting

$$\begin{aligned}\tilde{J}_{\pm}(w) &= \tilde{I}(w + \tilde{c}_{\pm}(w)) - \frac{\lambda^2}{2} + 4\pi n \\ &= \frac{1}{2}\|\nabla w\|_2^2 - \frac{\lambda^2}{2}e^{2\tilde{c}_{\pm}(w)}\int e^{2(u_0+w)} + 4\pi n \tilde{c}_{\pm}(w).\end{aligned}$$

Since every $w \in \tilde{\mathcal{A}}$ satisfies (3.13), as in the proof of Lemma 3.3 we can show that for $\lambda > 4\sqrt{\pi n}(\|e^{u_0}\|_1/\|e^{u_0}\|_2)$, both functionals $\tilde{J}_{\pm}$ are bounded from below and attain their infimum in $\tilde{\mathcal{A}}$.

Minimizers $\tilde{w} \in \tilde{\mathcal{A}}$ for $\tilde{J}_\pm$ satisfying the strict inequality

$$\left(\int e^{u_0+\tilde{w}}\right)^2 > \frac{16\pi n}{\lambda^2}\int e^{2(u_0+\tilde{w})} \tag{4.8}$$

lead to critical points $\tilde{v} = \tilde{w} + \tilde{c}_\pm(\tilde{w})$ for $\tilde{I}$ (see [22]). We have the following:

PROPOSITION 4.2 *Let $\lambda > 4\sqrt{\pi n}(\|e^{u_0}\|_1/\|e^{u_0}\|_2)$. Along a sequence $\mu_n \to +\infty$ the minimizer $w_n^\pm$ for $J_\pm$ in $\mathcal{A}$ converges in $H^1$ to a minimizer of $\tilde{J}_\pm$ in $\tilde{\mathcal{A}}$.*

From now on we take $\lambda > 4\sqrt{\pi n}(\|e^{u_0}\|_1/\|e^{u_0}\|_2)$, $\mu > 0$ sufficiently large, and denote by $w^\pm$ the minimizer of $J_\pm$ in $\mathcal{A}$ as given by Lemma 3.3. The mere assumption $\lambda > 4\sqrt{\pi n}(\|e^{u_0}\|_1/\|e^{u_0}\|_2)$ does not necessarily guarantee $w^\pm \in \mathcal{A}\setminus\Gamma$, and so we cannot use the a priori estimates established above to obtain the following:

LEMMA 4.3

$$\frac{1}{\mu}\int (4\pi n - \Delta v)e^{u_0+v} = \frac{1}{\mu}\int |\nabla(u_0+w^\pm)|^2 e^{u_0+w^\pm} \to 0 \quad \text{as } \mu \to +\infty\,.$$

PROOF: Recall that the given identity is established in (3.11). By part (ii) of Lemma 3.3, there exists a constant $C > 0$, independent of $\mu$ and $\lambda$, such that

$$\frac{1}{\mu}\|\Delta w^\pm\|_2 + \|\nabla w^\pm\|_2 \le C(\lambda+1)\,. \tag{4.9}$$

In particular, from (4.9), it follows that $\|\Delta\frac{w^\pm}{\mu}\|_2$ is bounded uniformly in $\mu$, and so for every $p \ge 1$ we may find a constant $C_p > 0$, independent of $\mu$, such that

$$\left\|\nabla\frac{w^\pm}{\mu}\right\|_p \le C_p\,.$$

Furthermore, from (4.9) we also have that $\|\nabla w^\pm\|_2$ is bounded uniformly in $\mu$, and by the Moser-Trudinger inequality (2.17), for every $q \ge 1$ we find a constant $C_q$, independent of $\mu$, such that

$$\|e^{w^\pm}\|_q \le C_q\,. \tag{4.10}$$

For fixed $\alpha \in (0,1)$ and $p > 2$, take $q > 1$ to satisfy

$$(1-\alpha)\left(\frac{1}{2}-\frac{1}{p}\right) = \frac{1}{q}$$

so that

$$\begin{aligned}\frac{1}{\mu}\int |\nabla w^\pm|^2 e^{u_0+w^\pm} &\le \frac{\max e^{u_0}}{\mu^\alpha}\int \left|\nabla\frac{w^\pm}{\mu}\right|^{1-\alpha}|\nabla w^\pm|^{1+\alpha}e^{w\pm}\\ &\le \frac{\max e^{u_0}}{\mu^\alpha}\left(\int\left|\nabla\frac{w^\pm}{\mu}\right|^p\right)^{\frac{1-\alpha}{p}}\left(\int|\nabla w^\pm|^2\right)^{\frac{1+\alpha}{2}}\|e^{w^\pm}\|_q \le \frac{C_1}{\mu^\alpha}\end{aligned}$$

for a suitable constant $C_1 > 0$ independent of $\mu$.

Recalling that $e^{u_0}|\nabla u_0|$ is continuous and $e^{u_0}|\nabla u_0|^2$ belongs to $L^q(\Omega)$, for all $q \in [1,2)$, using (4.9) and (4.10) we obtain a suitable constant $C_2 > 0$, independent of $\mu$, such that

$$\begin{aligned}
&\frac{1}{\mu}\int |\nabla(u_0+w^\pm)|^2 e^{u_0+w^\pm} \\
&\quad \le \frac{1}{\mu}\int |\nabla u_0|^2 e^{u_0+w^\pm} + \frac{2}{\mu}\int |\nabla u_0| e^{u_0} |\nabla w^\pm| e^{w^\pm} + \frac{1}{\mu}\int |\nabla w^\pm|^2 e^{u_0+w^\pm} \\
&\quad \le \frac{C_2}{\mu} + \frac{C_1}{\mu^\alpha} \to 0 \quad \text{as } \mu \to +\infty\,,
\end{aligned}$$

and the assertion of the lemma follows. ☐

LEMMA 4.4

$$\lim_{\mu\to+\infty} J_\pm(w_\pm) = \inf_{\tilde{\mathcal{A}}} \tilde{J}_\pm\,.$$

PROOF: By Lemma 4.3 we have that $c_\pm(w_\pm) = \tilde{c}_\pm(w^\pm) + o(1)$ as $\mu \to +\infty$. Therefore,

$$J_\pm(w_\pm) \ge \tilde{J}_\pm(w_\pm) + o(1) \quad \text{as } \mu \to +\infty\,,$$

and we conclude that

$$\liminf_{\mu\to+\infty} J_\pm(w^\pm) \ge \inf_{\tilde{\mathcal{A}}} \tilde{J}_\pm\,.$$

To obtain the reverse inequality, we use an approximation procedure for the minimum of $\tilde{J}_\pm$ on $\tilde{\mathcal{A}}$ (which a priori belongs to $X_1$ but not necessarily to $X_2$).

For every $\varepsilon > 0$ we take $w_\varepsilon^\pm \in \tilde{\mathcal{A}} \cap X_2$ satisfying (4.8) and such that

$$\tilde{J}_\pm(w_\varepsilon^\pm) \le \inf_{\tilde{\mathcal{A}}} \tilde{J}_\pm + \varepsilon\,.$$

By taking $\mu > 0$ sufficiently large, we can ensure that $w_\varepsilon^\pm \in \mathcal{A}$ and

$$\inf_{\mathcal{A}} J_\pm \le J_\pm(w_\varepsilon^\pm) = \tilde{J}_\pm(w_\varepsilon^\pm) + o(1) \le \inf_{\tilde{\mathcal{A}}} \tilde{J}_\pm + \varepsilon + o(1)\,.$$

Hence, by letting $\mu \to +\infty$ and using the arbitrariness of $\varepsilon > 0$, we obtain

$$\limsup_{\mu\to+\infty} J_\pm(w^\pm) \le \inf_{\tilde{\mathcal{A}}} \tilde{J}_\pm\,,$$

and the statement follows. ☐

PROOF OF PROPOSITION 4.2: From (4.9) we find a sequence $\mu_n \to +\infty$ such that the corresponding minima $w_n^\pm \rightharpoonup \tilde{w}^\pm$ weakly in $H^1$, strongly in $L^p$, and p.w.a.e.

in $\Omega$. In particular, $\tilde{w}^{\pm} \in \tilde{\mathcal{A}}$ and using Lemma 4.4, it follows that

$$\begin{aligned}\inf_{\tilde{\mathcal{A}}} \tilde{J}_{\pm} + o(1) = J_{\pm}(w_n^{\pm}) &= \frac{1}{2\mu_n^2}\|\Delta w_n^{\pm}\|_2^2 + \tilde{J}_{\pm}(w_n^{\pm}) + o(1)\\ &\geq \frac{1}{2\mu_n^2}\|\Delta w_n^{\pm}\|_2^2 + \tilde{J}_{\pm}(\tilde{w}^{\pm}) + o(1)\\ &\geq \frac{1}{2\mu_n^2}\|\Delta w_n^{\pm}\|_2^2 + \inf_{\tilde{\mathcal{A}}} \tilde{J}_{\pm} + o(1)\,.\end{aligned}$$

Consequently,

$$\frac{1}{\mu_n^2}\|\Delta w_n^{\pm}\|_2^2 \to 0 \quad \text{as } n \to +\infty\,, \qquad \tilde{J}_{\pm}(\tilde{w}^{\pm}) = \inf_{\tilde{\mathcal{A}}} \tilde{J}_{\pm}\,,$$

and necessarily $w_n^{\pm} \to \tilde{w}^{\pm}$ strongly in $H^1(\Omega)$. □

At this point we are ready to complete the proof of Theorem 1.4 by means of the following:

THEOREM 4.5 *There exists $\lambda_0 > 0$ sufficiently large such that $\forall \lambda > \lambda_0$, along a sequence $\mu_j \to +\infty$ the corresponding minimizers $w_j^{\pm}$ for $J_{\pm}$ in $\mathcal{A}$ satisfy the following*:

(i) $v_j^+ = w_j^+ + c_+(w_j^+) \to \tilde{v}^+$ *strongly in $H^1$ with $\tilde{v}^+$ a solution for* (4.6) *such that $e^{u_0+\tilde{v}^+} \to 1$ as $\lambda \to +\infty$ in $W^{1,q}(\Omega)$ and uniformly on compact subsets of $\Omega \setminus \{p_1, \dots, p_s\}$.*

(ii) *If $n = 1$, $v_j^- = w_j^- + c_-(w_j^-) \to \tilde{v}^-$ strongly in $H^1$, with $\tilde{v}^-$ a solution for* (4.6) *such that $e^{u_0+\tilde{v}^-} \to 0$ as $\lambda \to +\infty$, $C^k(\Omega)$-uniformly $\forall k \geq 0$.*

PROOF: In view of Proposition 3.5 and Lemma 3.8, if $\lambda_0 > 0$ is sufficiently large, then $\forall \lambda > \lambda_0$ we have that $v^{\pm} = w^{\pm} + c_{\pm}(w^{\pm})$ defines a solution for (3.1). In turn, we obtain a solution for (1.16) by setting $N^{\pm} = \frac{1}{\mu}(\lambda\mu e^{u_0+v^{\pm}} + 4\pi n - \Delta w^{\pm})$. Thus, by Proposition 4.1, along a sequence $\mu_j \to +\infty$ we may establish the strong convergence in $H^1$ to a solution $\tilde{v}^{\pm}$ of (4.6). In addition, we may use Proposition 4.2 and Lemma 4.3 to conclude that

$$\tilde{v}^{\pm} = \tilde{w}^{\pm} + \tilde{c}_{\pm}(\tilde{w}^{\pm}) \quad \text{with } \tilde{w}^{\pm} = \inf_{\tilde{\mathcal{A}}} \tilde{J}^{\pm}\,.$$

At this point, taking $\lambda_0 > 0$ larger if necessary, we may combine Proposition 3.2 and Corollary 3.1 in [22] to conclude (ii).

In order to obtain (i), we use for $v^+$ the lower bound (iii)(b) of Proposition 3.5, which at the limit $\mu_j \to +\infty$ leads to

$$\int e^{2(u_0+\tilde{v}^+)} \geq C\,, \tag{4.11}$$

with a constant $C > 0$ independent of $\lambda$. By lemma 3.1 in [22], we have that, for every $q \in (1,2)$, $|\nabla(u_0 + \tilde{v}^+)|$ is bounded in $L^q$ uniformly in $\lambda$. Notice also that the maximum principle and (4.6) imply $u_0 + \tilde{v}^+ \leq 0$ in $\Omega$. Hence, along a sequence

$\lambda_j \to +\infty$ we have $\tilde{w}^+ \rightharpoonup w^*$ weakly in $W^{1,q}$, pointwise a.e., and strongly in $L^p$, $\forall p \geq 1$, and $e^{\tilde{c}_+} \to A \leq 1$.

In addition, the dominated convergence theorem also guarantees, as $\lambda_j \to +\infty$,

$$e^{u_0+\tilde{v}^+} \to Ae^{u_0+w^*} \leq 1 \quad \text{p.w.a.e. and strongly in } L^p,\ \forall p \geq 1\,.$$

In particular, by (4.11) we have

$$A\int e^{u_0+w^*} \geq C > 0$$

and necessarily $A > 0$. Set $c^* = \ln A$ and $u^* = w^* + c^* + u_0$. Integrating (4.6) over $\Omega$ and passing to the limit as $\lambda_j \to +\infty$, we find $\int e^{u^*}(1-e^{u^*}) = 0$, and so $u^* \equiv 0$ a.e. in $\Omega$.

At this point we can argue exactly as in the proof of proposition 3.1 in [22] to conclude that for every $q \in (1,2)$,

$$u^+ := u_0 + \tilde{v}^+ \to 0 \quad \text{as } \lambda \to +\infty, \text{ strongly in } W^{1,q}(\Omega)\,.$$

Finally, for every domain $\Omega' \subset \Omega \setminus \{p_1, \dots, p_s\}$ we have

$$-\Delta u^+ = \lambda^2 e^{u^+}(1-e^{u^+}) \geq 0 \quad \text{in } \Omega'\,.$$

So we can use the mean value theorem to conclude that for every compact set $K \subset \Omega' \subset \Omega \setminus \{p_1, \dots, p_s\}$ there exists a constant $C > 0$ (depending on $K$ only) such that

$$0 \geq \max_K u^+ \geq \min_K u^+ \geq C\int_{\Omega'} u^+ \geq -C\|u^+\|_{L^1(\Omega)} \to 0 \quad \text{as } \lambda \to +\infty\,,$$

that is, $u^+ \to 0$ uniformly on $K$ as $\lambda \to +\infty$. □

PROOF OF THEOREM 1.4: Recalling the transformations (1.14) and (1.15) for the MCS Bogomol′nyĭ equations (1.9) and the corresponding transformations (4.4) and (4.5) for the CS Bogomol′nyĭ equations (4.2) (with the upper sign), we see that part (i) of Theorem 1.4 follows immediately by the convergence results in Proposition 4.1; analogously, part (ii) is obtained by Theorem 4.5. □

## 5 The AH Limit and the Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5, namely, to the description of the asymptotic behavior of the periodic $n$-vortices of Theorem 1.2 in terms of the AH limit.

To start, recall that the AH model was originally introduced by Ginzburg and Landau (see [15]) to give a phenomenological description of superconductivity. It is defined on the Minkowski space $\mathbb{R}^{1+2}$ by the Lagrangean density

$$\mathcal{L}^{\mathrm{AH}}(A,\phi) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} - D_\alpha\phi(D^\alpha\phi)^* - \frac{q^2}{2}(|\phi|^2-1)^2\,.$$

The constant $q > 0$ represents the electric charge. As above, a periodic $n$-vortex in $\Omega$ is given by a static configuration of $\mathcal{L}^{\mathrm{AH}}$ satisfying the first three of the boundary conditions (1.2) together with (1.3). For a periodic $n$-vortex $(A,\phi)$, the energy over $\Omega$ takes the form

$$E^{\mathrm{AH}}(A,\phi)=\int_\Omega\Big\{|(D_1\pm iD_2)\phi|^2+\frac{1}{2q^2}[F_{12}\mp q^2(|\phi|^2-1)]^2 \\ +\frac{1}{2q^2}(F_{01}^2+F_{02}^2)+|D_0\phi|^2\mp F_{12}\Big\},$$

and consequently

$$E^{\mathrm{AH}}\geq 2\pi|n|\,.$$

The lower bound for $E^{\mathrm{AH}}$ is attained by configurations $(A,\phi)$ satisfying the Bogomol′nyĭ equations

$$\begin{cases}(D_1\pm iD_2)\phi=0\\ F_{12}\mp q^2(|\phi|^2-1)=0\\ F_{01}=F_{02}=D_0\phi=0\,.\end{cases}\tag{5.1}$$

Solutions to (5.1), together with the boundary conditions (1.2) and (1.3), define AH-periodic $n$-vortices that are "electrically neutral," because the total electric charge $Q=\int_\Omega -2A_0|\phi|^2=0$, and carry a quantized magnetic flux $\Phi=\int_\Omega F_{12}=-2\pi n$ and energy $E^{\mathrm{AH}}=2\pi|n|$. As above, without loss of generality, we shall limit our attention to the case $n>0$ and consider (5.1) with the upper sign.

By Taubes's reduction, from system (5.1) (with the upper sign) we obtain the equation

$$\Delta u=2q^2(e^u-1)+4\pi\sum_{j=1}^s n_j\delta_{p_j}\,.\tag{5.2}$$

Every doubly periodic solution for (5.2) enables us to construct a periodic $n$-vortex $(A,\phi)$ solution for (5.1) by setting $n=\sum_{j=1}^s n_j$,

$$\begin{cases}\phi(x)=\exp\{\frac12 u+i\sum_{j=1}^s n_j\operatorname{Arg}(\frac{x-p_j}{|x-p_j|})\}\\ A_1+iA_2=2i\bar\partial\ln\phi\\ A_0=0\,,\end{cases}\tag{5.3}$$

and so $\phi$ vanishes at $p_j$ with multiplicity $n_j$ (see [15, 25]).

Using (5.2), Jaffe and Taubes [15] extensively studied system (5.1) on the whole space $\mathbb{R}^2$ and obtained a complete characterization of finite action configurations for $\mathcal{L}^{\mathrm{AH}}$. The periodic case was considered by Wang and Yang in [25]. They showed that the condition $2q^2>4\pi n/|\Omega|$ is necessary and sufficient for the existence of a doubly periodic solution for (5.2), which is *unique*. Thus, for $2q^2>4\pi n/|\Omega|$, to each configuration of zeroes $p_1,\dots,p_s\in\Omega$ and relative multiplicities $n_1,\dots,n_s\in\mathbb{N}$, there corresponds a *unique* (up to gauge transformations) periodic $n$-vortex $(A,\phi)_0$ solution for (5.1) with $n=\sum_{j=1}^s n_j$ and $\phi$ vanishing exactly at $p_j$ with multiplicity $n_j$.

By the dilation $\partial_{x'} = \frac{1}{2q^2}\partial_x$, we may take $2q^2 = 1$ and therefore, throughout this section, we assume

$$\frac{4\pi n}{|\Omega|} < 1\,. \tag{5.4}$$

Furthermore, decomposing $u = u_0 + v$ with $u_0$ uniquely defined in (2.1), we derive a *unique* solution for the equation

$$\Delta v = e^{u_0+v} - 1 + \frac{4\pi n}{|\Omega|}\,, \tag{5.5}$$

which we denote by $\overline{v}$.

In the notation of the previous sections, the normalization $2q^2 = 1$ amounts to taking $\lambda\mu = 1$ in (2.2), and we reduce the system to

$$\begin{cases} \Delta v = e^{u_0+v} - \mu N + \frac{4\pi n}{|\Omega|} \\ \Delta(\mu v + N) = e^{u_0+v}(N - \frac{1}{\mu}) + \mu\frac{4\pi n}{|\Omega|}\,. \end{cases} \tag{5.6}$$

Our goal will be to analyze the behavior of the solutions $(v,N)^{\pm}_{\mu}$ of (5.6) (given by Theorem 3.1) as $\mu \to 0$. In this direction, the a priori estimates valid for $v^+$ (see (i) and (ii), Theorem 3.1) are very useful and permit us to prove the following:

PROPOSITION 5.1 *For $\mu > 0$ sufficiently small, let $(v,N)^+_{\mu}$ be the solution for* (5.6) *as established in Theorem* 3.1. *Set $S^+ = N^+ - ⨍ N^+$ and $2A_0^+ = \frac{1}{\mu} - N^+$ as $\mu \to 0$ we have*

(i) *$v^+ \to \overline{v}$, $A_0^+ \to 0$, $S^+ \to 0$, $C^k$-uniformly, $\forall k \geq 0$, and*
(ii) *$\mu ⨍ N^+ \to 1$.*

To see what happens to the second solution $(v,N)^-_{\mu}$ given by Theorem 3.1, we argue differently. For $\mu > 0$ sufficiently small, we give a rather precise description of the solution set for (5.6) when $\mu$ is sufficiently small as follows:

PROPOSITION 5.2 *There exists $\mu^* > 0$ sufficiently small such that for $\mu \in (0,\mu^*)$ the solution set of* (5.6) *is formed of exactly two smooth curves $(v,N)^{\pm}_{\mu}$ that are parametrized by $\mu$ and are connected in $\mathbb{R} \times [C^{2,\alpha}(\Omega)]^2$. Moreover, as $\mu \to 0$, we have that*

(i) *$v^+$ and $N^+$ satisfy the convergence properties* (i) *and* (ii) *of Proposition* 5.1.
(ii) *Let $w^- = v^- - ⨍ v^-$, $S^- = N^- - ⨍ N^-$ and $2A_0^- = 1/\mu - N^-$; then $w^- \to 0$ and $S^- \to 0$ $C^k$-uniformly, $\forall k \geq 0$ and $⨍ v^- \to -\infty$, $\mu ⨍ N^- \to 4\pi n/|\Omega|$, $2\mu ⨍ A_0 \to 1 - 4\pi n/|\Omega|$.*

We start with the following:

PROOF OF PROPOSITION 5.1: Set $v^+ = w^+ + c_+$ with $c_+ = ⨍ v^+$. In view of estimate (i) of Theorem 3.1 (with $\lambda\mu = 1$) we have

$$\|\Delta w^+\|_2 \leq C \tag{5.7}$$

with $C > 0$ independent of $\mu$. To estimate $c_+$, recall first that $c_+ \le 0$ (by Lemma 2.1). On the other hand, by estimate (ii) of Theorem 3.1 (with $\lambda\mu = 1$),

$$e^{2c_+}\int e^{2(u_0+w^+)} \ge C\left(\left(1-\frac{4\pi n}{|\Omega|}\right)^2 + \mu^2 \ln\left(1-\frac{4\pi n}{|\Omega|}\right)\right) \tag{5.8}$$

which holds with a suitable $C > 0$ independent of $\mu$. Thus, by means of (5.7), estimate (5.8) immediately yields a lower bound (independent of $\mu$) for $c_+$ when $\mu \to 0^+$. So $v^+$ is uniformly bounded in $H^2(\Omega)$ as $\mu \to 0^+$. Hence along any sequence $\mu_j \to 0^+$, there exists a subsequence and $v \in H^2(\Omega)$ such that, for the corresponding solutions $(v_j^+, N_j^+)$, we have $v_j^+ \rightharpoonup v$ weakly in $H^2(\Omega)$ and uniformly in $\Omega$ and $\mu_j N_j^+ \rightharpoonup e^{u_0+v} + 4\pi n/|\Omega| - \Delta v =: V$ weakly in $L^2(\Omega)$.

Since for $\lambda\mu = 1$, $0 < \mu N^+ < 1$ (see Lemma 2.1), we have $0 \le V \le 1$ a.e. in $\Omega$. On the other hand, integrating over $\Omega$ the second equation in (5.6) and passing to the limit as $j \to +\infty$, we find $\int e^{u_0+v}(1-V) = 0$. Thus $V = 1$ a.e. in $\Omega$, and by the first equation in (5.6), the limit $v$ must correspond to the *unique* solution of (5.5), that is, $v = \overline{v}$. Next, writing $N^+ = S^+ + ⨍ N^+$ with $\int S^+ = 0$, we have established that $\mu_j ⨍ N_j^+ \to 1$ as $j \to +\infty$. To estimate $S^+$, notice that by the second equation in (5.6) we have $\Delta\mu_j N_j^+ \rightharpoonup 0$ weakly in $L^2(\Omega)$, and so (up to subsequences) $\mu_j N_j^+ \to 1$ strongly in $L^p(\Omega)$, $\forall p \ge 1$. Furthermore, if we multiply the same equation by $N^+ - 1/\mu$ and integrate over $\Omega$, we get

$$\|\nabla S^+\|_2^2 = \\ -\int e^{u_0+v^+}\left(N^+ - \frac{1}{\mu}\right)^2 - \mu\frac{4\pi n}{|\Omega|}\int\left(N^+ - \frac{1}{\mu}\right)\mu\int \Delta w^+\left(N^+ - \frac{1}{\mu}\right).$$

Consequently, as $j \to +\infty$, $\|\nabla S_j^+\|_2^2 \to 0$ and $2 ⨍ A_{0,j}^+ = 1/\mu_j - ⨍ N_j^+ \to 0$. In other words, $N_j^+ - 1/\mu_j \to 0$ strongly in $H^1$, and from (5.6) we can use a bootstrap argument to conclude the desired stronger convergence as claimed in (i). Since the convergences hold along any sequence $\mu_j \to 0$, we obtain (i) and (ii) as $\mu \to 0^+$. □

PROOF OF PROPOSITION 5.2: Write $v = w + c$ and $N = S + ⨍ N$ with $\int w = 0 = \int S$ and notice that the quadratic equation (2.10) takes the following form when $\lambda\mu = 1$:

$$e^{2c}\left(⨍ e^{u_0+w}\right)^2 - \left(1 - \frac{4\pi n}{|\Omega|} - \mu\frac{⨍ e^{u_0+w}S}{⨍ e^{u_0+w}}\right)e^c ⨍ e^{u_0+w} + \mu^2\frac{4\pi n}{|\Omega|} = 0. \tag{5.9}$$

Equation (5.9) yields that $(w, S)$ belongs to the set

$$\mathcal{A}_\mu = \left\{ \begin{matrix} (w,S) \in [C^{2,\alpha}(\Omega)]^2 \\ \int w = 0 = \int S \end{matrix} : \frac{⨍ e^{u_0+w}S}{⨍ e^{u_0+w}} \le \frac{1 - \frac{4\pi n}{|\Omega|} - 4\mu\sqrt{\frac{\pi n}{|\Omega|}}}{\mu} \right\},$$

and $e^c$ satisfies

$$e^c = \frac{1 - \frac{4\pi n}{|\Omega|} - \mu \frac{\int e^{u_0+w} S}{⨍ e^{u_0+w}} \pm \sqrt{\left(1 - \frac{4\pi n}{|\Omega|} - \mu \frac{⨍ e^{u_0+w} S}{⨍ e^{u_0+w}}\right)^2 - \mu^2 \frac{16\pi n}{|\Omega|}}}{2 ⨍ e^{u_0+w}}. \tag{5.10}$$

On the other hand, from the first equation in (5.6) we have that

$$⨍ N = \frac{e^c ⨍ e^{u_0+w} + \frac{4\pi n}{|\Omega|}}{\mu}. \tag{5.11}$$

We insert the values for $c$ and $⨍ N$ given by (5.10) and (5.11) into (5.6) to obtain the system

$$\begin{cases} \Delta w = a_\mu^\pm(w,S)\left(\frac{e^{u_0+w}}{⨍ e^{u_0+w}} - 1\right) - \mu S \\ \Delta(\mu w + S) = a_\mu^\pm(w,S)\left(\frac{e^{u_0+w} S}{⨍ e^{u_0+w}} - \frac{e^{u_0+w} \int e^{u_0+w} S}{(⨍ e^{u_0+w})^2}\right) - \mu \frac{4\pi n}{|\Omega|}\left(\frac{e^{u_0+w}}{⨍ e^{u_0+w}} - 1\right) \\ (w,S) \in \mathcal{A}_\mu, \end{cases} \tag{5.12}$$

where $a_\mu^\pm$ is defined by

$$a_\mu^\pm(w,S) = e^{c_\pm(w,S)} ⨍ e^{u_0+w},$$

and $e^{c_\pm(w,S)}$ is defined by (5.10). System (5.12) together with (5.10) and (5.11) is equivalent to (5.6). We point out that (5.12) is well-defined for $\mu = 0$, even though $⨍ N$ in (5.11) is not. This simple fact is crucial, because it allows us to obtain solution curves by the implicit function theorem.

To this purpose, consider the smooth maps $f_\mu^\pm : \mathcal{A}_\mu \to [C^{0,\alpha}(\Omega)]^2$ defined by

$$f_\mu^\pm(w,S) = (f_{1,\mu}^\pm(w,S), f_{2,\mu}^\pm(w,S))$$

with

$$\begin{aligned} f_{1,\mu}^\pm(w,S) &= -\Delta w + a_\mu^\pm(w,S)\left(\frac{e^{u_0+w}}{⨍ e^{u_0+w}} - 1\right) - \mu S, \\ f_{2,\mu}^\pm(w,S) &= -\Delta(\mu w + S) + a_\mu^\pm(w,S)\left(\frac{e^{u_0+w} S}{⨍ e^{u_0+w}} - \frac{e^{u_0+w} \int e^{u_0+w} S}{(⨍ e^{u_0+w})^2}\right) \\ &\qquad - \mu \frac{4\pi n}{|\Omega|}\left(\frac{e^{u_0+w}}{⨍ e^{u_0+w}} - 1\right). \end{aligned}$$

When $\mu = 0$, we have

$$a_0^+(w,S) = 1 - \frac{4\pi n}{|\Omega|},$$

and therefore $f_0^+(w,S) = 0$ if and only if $(w,S)$ satisfies the system

$$\begin{cases} \Delta w = \left(1 - \frac{4\pi n}{|\Omega|}\right)\left(\frac{e^{u_0+w}}{⨍ e^{u_0+w}} - 1\right) \\ \Delta S = \left(1 - \frac{4\pi n}{|\Omega|}\right)\left(\frac{e^{u_0+w} S}{⨍ e^{u_0+w}} - \frac{e^{u_0+w} ⨍ e^{u_0+w} S}{(⨍ e^{u_0+w})^2}\right) \\ \int w = 0 = \int S. \end{cases} \tag{5.13}$$

The first equation is uncoupled from $S$ and has a unique solution $w_0$ corresponding to the minimum of the strictly convex, coercive functional

$$\frac{1}{2}\|\nabla w\|_2^2 + \left(1 - \frac{4\pi n}{|\Omega|}\right) \ln \fint e^{u_0+w}$$

defined on $X_1 = \{w \in H^1(\Omega) : \int w = 0\}$. The (linear) equation $f^+_{2,0}(w_0, S) = 0$ has the unique solution $S = 0$. Hence $f_0^+$ has a unique zero, namely, $(w_0, 0)$. Since $w_0 + c_{+,0}(w_0, 0)$ satisfies (5.5), by uniqueness we conclude $\overline{v} = w_0 + c_{+,0}(w_0, 0)$. It is easy to check that $(w_0, 0)$ is a nondegenerate zero for $f_0^+$. Furthermore, the obvious inequality

$$\frac{\fint e^{u_0+w} S}{\fint e^{u_0+w}} \le \max_\Omega |S| \tag{5.14}$$

implies that for all $\mu \in (0, \frac{|\Omega|-4\pi n}{8\sqrt{\pi n|\Omega|}})$, the set $\mathcal{A}_\mu$ includes the strip

$$U = \left\{ \begin{matrix} (w,S) \in [C^{2,\alpha}(\Omega)]^2 \\ \int w = 0 = \int S \end{matrix} : \max_\Omega |S| < \frac{4\sqrt{\pi n}}{|\Omega|} \right\}.$$

An application of the implicit function theorem to $f^+$ that is restricted to the set

$$\left[0, \frac{|\Omega| - 4\pi n}{8\sqrt{\pi n|\Omega|}}\right) \times U$$

provides the first curve of solutions

$$(v,N)_\mu^+ = (w,S)_\mu^+ + \left(c^+((w,S)_\mu^+), \fint N^+((w,S)_\mu^+)\right).$$

The second curve of solutions $(v,N)_\mu^-$ is obtained similarly by considering the map $f_\mu^-$. The procedure is actually even simpler, since $a_0^-(w,S) = 0$, which implies that

$$f^-_{\mu=0} = \begin{pmatrix} \Delta w \\ \Delta S \end{pmatrix}$$

and consequently the equation $f^-_{\mu=0} = 0$ is trivial in $X_1 \times X_1$. The implicit function theorem allows us to extend the unique nondegenerate zero $(0,0)$ of $f_0^-$ to a smooth curve $(w,S)_\mu^-$ of zeroes of $f_\mu^-$.

Using (5.10) and (5.11), it is simple to check that

$$\lim_{\mu \to 0^+} e^{c_-} = 0, \quad \lim_{\mu \to 0^+} \mu \fint N^+ = 1, \quad \lim_{\mu \to 0^+} \mu \fint N^- = \frac{4\pi n}{|\Omega|}.$$

We are left to analyze the behavior of the mean values

$$-2 \fint A_0^\pm = \fint N^\pm - \frac{1}{\mu} = \frac{e^{c_\pm} \fint e^{u_0+w} - 1 + \frac{4\pi n}{|\Omega|}}{\mu}$$

as $\mu \to 0$ along the curves $(w,S)^{\pm}_{\mu}$. To this purpose, notice that along solution curves, we have

$$\frac{\fint e^{u_0+w}S}{\fint e^{u_0+w}} = o(1) \quad \text{as } \mu \to 0\,.$$

Consequently, by (5.10) we get

$$e^{c_\pm(w,S)} = \frac{1-\frac{4\pi n}{|\Omega|}-\mu\frac{\fint e^{u_0+w}S}{\fint e^{u_0+w}} \pm \left(1-\frac{4\pi n}{|\Omega|}-\mu\frac{\fint e^{u_0+w}S}{\fint e^{u_0+w}}\right)+o(\mu)}{2\fint e^{u_0+w}}\,. \tag{5.15}$$

Thus, we easily conclude that

$$\fint A_0^+ = o(1) \ \text{ as } \mu \to 0 \quad \text{and} \quad 2\mu \fint A_0^- \to 1-\frac{4\pi n}{|\Omega|} \ \text{ as } \mu \to 0\,.$$

The uniqueness of the two curves $(v,N)^{\pm}_{\mu}$ for (5.6) when $\mu \in (0,\mu^*)$ is a consequence of the relative compactness of the solution set of the (equivalent) system (5.12) (as it follows by part (i) of Lemma (2.3) with $\lambda\mu = 1$) and the uniqueness of (5.5).

To obtain the connectedness of the solution curves, we use a degree theoretical argument of Rabinowitz [19] and the estimates of Corollary 2.4 with $\lambda\mu = 1$. We consider the map $g_\mu : [X_1 \times \mathbb{R}]^2 \to [X_1 \times \mathbb{R}]^2$ defined by

$$g_\mu = \begin{pmatrix} g_\mu^{11} & g_\mu^{12} \\ g_\mu^{21} & g_\mu^{22} \end{pmatrix},$$

where

$$g_\mu^{11}(w+c,\, S+d) = -w + \left[e^{u_0+w+c} - \mu(S+d) + \frac{4\pi n}{|\Omega|}\right] * G,$$

$$g_\mu^{12}(w+c,\, S+d) = \fint \left[e^{u_0+w+c} - \mu(S+d) + \frac{4\pi n}{|\Omega|}\right],$$

$$g_\mu^{21}(w+c,\, S+d) = -(\mu w + S) + \left[e^{u_0+w+c}(S+d) + \mu\frac{4\pi n}{|\Omega|}\right] * G,$$

$$g_\mu^{22}(w+c,\, S+d) = \fint \left[e^{u_0+w+c}(S+d) + \mu\frac{4\pi n}{|\Omega|}\right].$$

$G = G(x,y)$ is the Green function for $\Delta$ on the compact manifold $\Omega$ with $\int_\Omega G(x,y)\, dx = 0 = \int_\Omega G(x,y)dy$ (see [2]), and $*$ denotes convolution. The zeroes of $g$ correspond to the solutions of (5.6) subject to the (natural) constraints (5.10) and (5.11). For every fixed $\mu$, $g$ is a compact perturbation of a boundedly invertible linear operator; therefore the Leray-Schauder degree (see [17]) is well-defined. Since $(w_0,0)$ is a nondegenerate zero for $f^+$, then, for $\bar{\mu} \in (0,\mu^*)$ sufficiently small and a sufficiently small neighborhood $B$ of $(v,N)^+_{\mu=\bar{\mu}}$, we have that

$$|d(g_{\bar{\mu}}, B, 0)| = 1\,.$$

Now the global bifurcation results of Rabinowitz [19] provide an unbounded maximal continuum $\mathcal{C}$ of solutions to (5.6) containing the curve $(v,N)^+$. By uniqueness near $\mu = 0$ and the bounds of Corollary 2.4 (with $\lambda\mu = 1$), we conclude that necessarily $\mathcal{C}$ intersects the curve of solutions $(v,N)^-$. The connectedness in $[C^{2,\alpha}(\Omega)]^2$ follows by standard elliptic regularity. $\square$

PROOF OF THEOREM 1.5: Recalling the transformations (1.15), the normalization $2q^2 = 1$ and (5.2), we can easily derive the conclusion of Theorem 1.5 by applying the results of Propositions 5.1 and 5.2 to (1.14). $\square$

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