Sharp Sobolev inequalities involving scalar curvature

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Abstract

Some sharp Sobolev inequalities on Riemannian manifolds are presented, emphasizing the role of scalar curvature, on the line of our joint work with Y.Y. Li [13]. Proofs, which are based on a fine blow-up analysis of solutions to a nonlinear elliptic equation with critical growth, are outlined. The main estimate is obtained in a new and more general form.

1 Introduction and preliminaries

It is well-known that sharp Sobolev inequalities on Riemannian manifolds are of interest in many problems from geometry and physics, see e.g., Aubin [2, 3], Brezis and Lieb [5], Brezis and Nirenberg [6], Carlen and Loss [7], Druet [8], Escobar [10], Hebey and Vaugon [12], Li and Zhu [15], Moser [16], Talenti [19], Trudinger [20], and references therein.

To begin, let us recall a classical result by Aubin [1] and Talenti [19]. For $n \geq 3$, $2^* = 2n/(n-2)$, there holds

$$\inf\left\{\frac{\|\nabla u\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \ :\ u \in L^{2^*}(\mathbb{R}^n)\setminus\{0\}, |\nabla u| \in L^2(\mathbb{R}^n)\right\} = K^{-1} \tag{1}$$

with K defined by

$$K^2 = \frac{4}{n(n-2)\sigma_n^{2/n}},$$

where σ_n is the volume of the standard n-sphere . The set of minimizers for (1) is given by $\{t\,U_{y,\lambda}\ ;\ y \in \mathbb{R}^n, \lambda > 0, t \neq 0\}$, where

$$U_{y,\lambda}(x) = \lambda^{(n-2)/2}U(\lambda(x-y))$$
$$U(x) = U_{0,1}(x) = \big(\frac{1}{1+\bar\lambda^2|x|^2}\big)^{(n-2)/2},$$

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$\bar{\lambda}^2 = [n(n-2)]^{-1}K^{-2}$. In this notation, $U_{y,\lambda}$ *concentrates* at y (in the sense of L^{2^*}) as $\lambda \to +\infty$. We are interested in the sharp extensions of (1) to Riemannian manifolds. Let (M, g) denote a smooth, compact Riemannian manifold without boundary, of dimension $n \geq 3$. Henceforth, we use the following notation:

$$\|\cdot\|_{L^p(M,g)} := \left(\int_M |\cdot|^p \, dv_g\right)^{1/p}, \qquad 1 \leq p < +\infty,$$

and we recall that in local coordinates:

$$g = g_{ij}(x)\, \mathrm{d}x^i \mathrm{d}x^j, \quad dv_g = \sqrt{\det g(x)}\, \mathrm{d}x, \quad |\nabla_g u|^2 = g^{ij}(x)\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j},$$

where indices are lowered and raised in the usual way. We also recall that in geodesic normal coordinates centered at fixed point $P \in M$, a Riemannian metric differs from the Euclidean metric by an error of the second order depending on curvature, namely:

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{iklj}(P)x^i x^j + O(|x|^3), \tag{2}$$

where R_{iklj} are the components of the Riemann curvature tensor, see [3]. In our notation, the scalar curvature of M is the contraction $R_g = \sum_{i,k} R^i_{kik}$. Using (2) together with partitions of unity and interpolation, it is not difficult to derive the following inequality from (1):

$$\|u\|^2_{L^{2^*}(M,g)} \leq (K^2 + \varepsilon)\|\nabla_g u\|^2_{L^2(M,g)} + A_\varepsilon \|u\|^2_{L^1(M,g)} \quad \forall u \in H^1(M), \tag{3}$$

where $A_\varepsilon > 0$ depends on (M, g) and on ε, but not on u. It is also clear that K in (3) may not be replaced by any smaller constant. At this point, a natural question is whether or not inequality (3) holds true with $\varepsilon = 0$ (more precisely, whether or not $\sup_{\varepsilon > 0} A_\varepsilon < +\infty$). It turns out that the answer depends on (M, g). To see this, following an idea of Aubin [1], we fix $P \in M$ and we construct test functions $\xi_{P,\lambda}$ concentrating at P using the minimizers $U_{y,\lambda}$ defined above:

$$\xi_{P,\lambda}(Q) := \left(\frac{\lambda}{1 + \lambda^2 \mathrm{dist}^2(P,Q)}\right)^{(n-2)/2}, \quad Q \in M$$

($\xi_{P,\lambda}$ is the pullback of $U_{0,\lambda}$ by the exponential map centered at P). When $n \geq 7$, using (2), it is not difficult to verify that

$$\frac{\|\nabla_g \xi_{P,\lambda}\|^2_{L^2(M,g)}}{\|\xi_{P,\lambda}\|^2_{L^{2^*}(M,g)}} = K^{-2} - \frac{n-2}{4(n-1)}R(P)\lambda^{-2} + O(\lambda^{-4}) \tag{4}$$

and

$$C^{-1}\lambda^{-2} \leq \|\xi_{P,\lambda}\|^2_{L^2(M,g)} \leq C\lambda^{-2},$$
$$\|\xi_{P,\lambda}\|^2_{L^p(M,g)}\| = \circ(\lambda^{-2}) \qquad \forall 1 \leq p < 2.$$

Consequently: If there exists some $P \in M$ such that $R_g(P) > 0$, then (3) does *not* hold with $\varepsilon = 0$. In fact, by the work of Aubin [1] and Hebey and Vaugon [12], inequality (3) has been sharpened in the form:

$$\|u\|^2_{L^{2^*}(M,g)} \leq K^2\|\nabla_g u\|^2_{L^2(M,g)} + A\|u\|^2_{L^2(M,g)} \qquad \forall u \in H^1(M), \tag{5}$$

where $A > 0$ depends on (M, g) only. If the scalar curvature of M is positive at some point, then the L^2-norm in (5) may not be replaced by any L^p-norm with $1 \le p < 2$.

Motivated by the above arguments, in a joint work with Y.Y. Li [13] we proved the following inequality, which clarifies the role of scalar curvature in the context of sharp Sobolev-type inequalities.

Theorem 1 ([13]). *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \ge 6$. There exists a constant $A > 0$, depending on (M, g) only, such that for all $u \in H^1(M)$ there holds:*

$$\|u\|^2_{L^{2^*}(M,g)} \le K^2 \int_M \{|\nabla_g u|^2 + c(n) R_g u^2\}\, dv_g + A\|u\|^2_{L^{\bar r}(M,g)}, \tag{6}$$

where 2^* and K are defined above, $c(n) = (n-2)/[4(n-1)]$, $\bar r = 2n/(n+2)$, R_g is the scalar curvature of g.

Remark 1. *The case $n = 6$ is the "limit" case for inequality* (6) *and requires a more delicate proof.*

Remark 2. *The quantity $\int_M \{|\nabla_g u|^2 + c(n) R_g u^2\}\, dv_g$ is conformally invariant. It also appears in the Yamabe problem, see, e.g., [3, 17]. We recall that the Yamabe functional Y_M for (M, g) is defined by*

$$Y_M(u) := \frac{\int_M \{|\nabla_g u|^2 + c(n) R_g u^2\}\, dv_g}{\|u\|^2_{L^{2^*}(M,g)}}, \qquad u \in H^1(M) \setminus \{0\},$$

and that (1) *is equivalent to* $\inf Y_{\mathbb{S}^n} = K^{-2}$, *where $\mathbb{S}^n$ denotes the Euclidean n-sphere. Hence, inequality* (6) *is equivalent to the lower bound*

$$\inf Y_{\mathbb{S}^n} \le \inf Y_M + A \frac{\|\cdot\|^2_{L^{\bar r}(M,g)}}{\|\cdot\|^2_{L^{2^*}(M,g)}}. \tag{7}$$

Concerning sharpness of (6), we have

Remark 3. *K and R_g are* sharp, *in the sense that K may not be replaced by any smaller constant and R_g may not be replaced by any smaller function.*

On the other hand, the sharpness of $\bar r$ depends on (M, g). This fact is a consequence of the following expansion, also due to Aubin [1], which sharpens the expansion (4):

$$Y_M(\xi_{P,\lambda}) = K^{-2} - \gamma_n |W(P)|^2 \lambda^{-4} + \circ(\lambda^{-4}), \tag{8}$$

where $\gamma_n > 0$ is a dimensional constant and W denotes the Weyl tensor of g. We recall that $W \equiv 0$ if and only if (M, g) is locally conformally flat. Since

$$C^{-1}\lambda^{-4} \le \|\xi_{P,\lambda}\|^2_{L^{\bar r}(M,g)} \le C\lambda^{-4},$$

we conclude that

Remark 4. *If (M, g) is not locally conformally flat, then the $L^{\bar r}$-norm in* (6) *may not be replaced by any L^p-norm, with $1 \le p < \bar r$.*

For locally conformally flat manifolds, we have the following Sobolev-Poincaré-type inequality:

Theorem 2 ([13]). *Let (M,g) be a smooth compact locally conformally flat Riemannian manifold without boundary of dimension $n \geq 3$. There exists a constant $A > 0$, depending on (M,g) only, such that for all $u \in H^1(M)$ there holds:*

$$\|u\|^2_{L^{2^*}(M,g)} \leq K^2 \int_M \left\{ |\nabla_g u|^2 + c(n) R_g u^2 \right\} dv_g + A \|u\|^2_{L^1(M,g)}.$$

The proofs of Theorem 1 and of Theorem 2 are based on a fine blow-up analysis of "approximate minimizers", solutions to a nonlinear elliptic equation with critical growth. We shall (briefly) outline the proofs in Section 3, and we refer to [13], [14] for the details. The key step is a *pointwise estimate* as in Lemma 3, which is based on an a priori estimate for solutions to a class of elliptic equations with coefficients of a particular form, see Proposition 1 below. We believe that Proposition 1 is of its own interest, and therefore in Section 2 we present it in a new and more general form.

2 An elliptic problem

In this section we present an a elliptic estimate, which is the main step towards obtaining the crucial Lemma 3 below.

Proposition 1. *Let $\rho_i \geq 0$, $i \to +\infty$, and f be measurable functions defined on M, with $f \in L^\infty$, and let $A_i > 0$, $A_i \to +\infty$, $1 \leq q < 2$. Consider the functions V_i defined by*

$$V_i := \begin{cases} \min\left\{ f + A_i \left(\frac{\|\rho_i\|_q}{\rho_i} \right)^{2-q}, 1 \right\} & \text{when } \rho_i \neq 0 \\ 1 & \text{when } \rho_i = 0 \end{cases}.$$

Then the operators $-\Delta_g + V_i$ are coercive on $H^1(M)$ for sufficiently large i, with coercivity constant uniform in i. Consequently, for every i sufficiently large there exists a unique (distributional) solution G_i to the equation:

$$(9) \qquad -\Delta_g G_i + V_i G_i = \delta_{P_i}, \qquad \textit{on } M.$$

Furthermore, the first nonzero eigenvalue of $-\Delta_g + V_i$ is bounded away from zero and therefore G_i satisfies, for some constant $C > 0$ independent of i,

(i) $G_i \in C^2_{loc}(M \setminus \{P_i\})$;

(ii) $C^{-1} \mathrm{dist}_g(x, P_i)^{2-n} \leq G_i(x) \leq C \mathrm{dist}_g(x, P_i)^{2-n} \quad \forall\, x \in M$.

Note that V_i is Lipschitz on M (with Lipschitz constant depending on i) and it is uniformly bounded:

$$(10) \qquad -\|f\|_\infty \leq V_i \leq 1.$$

In order to prove Proposition 1 we need the following

Lemma 1. *The functions V_i satisfy:*

$$\lim_{i\to+\infty} \mathrm{vol}_g\{V_i < \frac{1}{2}\} = 0.$$

Proof. Note that for every measurable set E such that $\overline{E} \subset M \cap \{\rho_i > 0\}$ we have the lower bound:

$$\|\rho_i\|_{L^q(E)} \|\rho_i^{-1}\|_{L^q(E)} \geq (\mathrm{vol}_g E)^{2/q}.$$

Indeed, using the Hölder inequality we find:

$$\mathrm{vol}_g E = \int_E dv_g = \int_E \rho_i^{q/2} \rho_i^{-q/2} \, dv_g \leq \|\rho_i\|_{L^q(E)}^{q/2} \|\rho_i^{-1}\|_{L^q(E)}^{q/2}.$$

It follows that

$$\begin{aligned} (11) \qquad \|(\|\rho_i\|_{L^q(M)}\rho_i^{-1})^{2-q}\|_{L^{q/(2-q)}(E)} =& \|\rho_i\|_{L^q(M)}^{2-q} \|\rho_i^{-(2-q)}\|_{L^{q/(2-q)}(E)} \\ \geq& \|\rho_i\|_{L^q(E)}^{2-q} \|\rho_i^{-1}\|_{L^q(E)}^{2-q} \geq |E|^{(2-q)2/q}. \end{aligned}$$

Let $E_i := \{V_i < 1/2\}$. Then $\overline{E}_i \subset M \cap \{\rho_i > 0\}$ and therefore, by (11),

$$(\mathrm{vol}_g E_i)^{(2-q)2/q} \leq \|(\|\rho_i\|_{L^q(M)}\rho_i^{-1})^{2-q}\|_{L^{q/(2-q)}(E_i)}.$$

On the other hand, since

$$A_i(\|\rho_i\|_{L^q(M)}\rho_i^{-1})^{2-q} < \frac{1}{2} + |f|, \qquad \text{on } E_i,$$

we have

$$A_i\|(\|\rho_i\|_{L^q(M)}\rho_i^{-1})^{2-q}\|_{L^{q/(2-q)}(E_i)} \leq (\frac{1}{2} + \|f\|_{L^\infty(M)})(\mathrm{vol}_g M)^{(2-q)/q},$$

and consequently,

$$A_i(\mathrm{vol}_g E_i)^{(2-q)/q} \leq C,$$

for some $C > 0$ independent of i. Recalling that $A_i \to +\infty$, Lemma 1 follows immediately. □

Proof of Proposition 1. Proof of the coercivity. For $\tilde{\gamma} = 1/2$ and $u \in H^1(M)$, by the Sobolev inequality and a straightforward computation we have:

$$\begin{aligned} &\int_M \{|\nabla_g u|^2 + V_i u^2\} \, dv_g = \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2 + (V_i - \tilde{\gamma})u^2\} \, dv_g \\ &\geq \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2 - (V_i - \tilde{\gamma})_- u^2\} \, dv_g \\ &\geq \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2\} \, dv_g - \|(V_i - \tilde{\gamma})_-\|_{L^{n/2}(M)} \|u\|_{L^{2^*}(M)}^2 \\ &\geq \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2\} \, dv_g - C\mathrm{vol}_g^{2/n}\{V_i < 1/2\} \int_M \{|\nabla_g u|^2 + u^2\} \, dv_g, \end{aligned}$$

where $(V_i - \tilde{\gamma})_- \geq 0$ denotes the negative part of $V_i - \tilde{\gamma}$. The coercivity and its uniformity in i follow from the above and Lemma 1.
Proof of (i) and (ii). Because of the coercivity of $-\Delta_g + V_i$, the Lipschitz regularity and the uniform L^∞-bound for V_i, it follows from standard elliptic theories (see e.g., [11], [18] and [9]) and the maximum principle that G_i is uniquely defined by (9) and it satisfies (i) and (ii). □

3 Proof of Theorem 1

In the rest of this note we outline the proof of Theorem 1 in the (simpler) case $n \geq 7$.

Since the "correction term" $\|\tilde{\xi}_{P,\lambda}\|_{\bar{r}}$ is of the order λ^{-4}, the expansion (8) implies that inequality (6) holds for the family $\{t\widetilde{\xi}_{P,\lambda}\}$ defined above, uniformly in $t \neq 0, P \in M, \lambda > 0$. To obtain (6) for general $u \in H^1(M)$, we argue by contradiction, and we take a *global* approach. Negating (6) in the equivalent form (7), we assume that for some sequence $\alpha \to +\infty$ there holds:

$$\inf \left\{ Y_M(u) + \alpha \frac{\|u\|_{\bar{r}}^2}{\|u\|_{2^*}^2} : \ u \in H^1(M) \setminus \{0\} \right\} < K^{-2}. \tag{12}$$

By standard arguments, (12) implies the existence of a minimizer $u_\alpha \in H^1(M)$ for the functional $Y_M + \alpha\|\cdot\|_{\bar{r}}^2/\|\cdot\|_{2^*}^2$ with $u_\alpha \geq 0$, $\int_M u_\alpha^{2^*}\, dv_g = 1$, which satisfies the Euler-Lagrange equation:

$$-\Delta_g u_\alpha + c(n) R_g u_\alpha + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} u_\alpha^{r-1} = \ell_\alpha u_\alpha^{2^*-1} \qquad \text{on } M. \tag{13}$$

Note that (13) is a nonlinear elliptic equation with critical growth. The idea of the proof is to obtain the contradiction $\alpha \leq C$ by blowup analysis of u_α as $\alpha \to +\infty$. We denote:

$$x_\alpha \in M : u_\alpha(x_\alpha) = \max_M u_\alpha, \qquad \mu_\alpha^{(n-2)/2} := u_\alpha(x_\alpha)^{-1},$$

and without loss of generality we assume that x_α converges to a point in M. It is a standard fact that u_α concentrates "in energy" at a single point. Namely:

Lemma 2. *As $\alpha \to +\infty$, we have:*

$$\begin{aligned}
&\mu_\alpha \to 0 \\
&\|\nabla_g u_\alpha\|_2^2 \to K^{-2}, \qquad \alpha \|u_\alpha\|_{\bar{r}}^2 \to 0 \\
&\|\nabla_g (u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}})\|_{L^2(B_{\delta_0}(x_\alpha))} + \|u_\alpha - \xi_{x_\alpha,\mu_\alpha^{-1}}\|_{L^{2^*}(B_{\delta_0}(x_\alpha))} \to 0 \\
&\mu_\alpha^{(n-2)/2} u_\alpha(\exp_{x_\alpha}(\mu_\alpha\ \cdot\ )) \to U(\cdot) \qquad \text{in } C^2_{\rm loc}(\mathbb{R}^n),
\end{aligned}$$

where $\delta_0 > 0$ is small and fixed and depends only on (M, g).

In order to derive a contradiction from (12), we estimate the rates of the asymptotic behaviors stated in Lemma 2. Towards this goal, the following *pointwise estimate* is a key step, which allows to neglect "boundary values" of u_α, and thus to "localize" the blow-up analysis at x_α. It is a direct consequence of Proposition 1.

Lemma 3 (Pointwise estimate). *The following estimate holds:*

$$u_\alpha(x) \leq C \mu_\alpha^{(n-2)/2} \mathrm{dist}_g^{2-n}(x, x_\alpha), \qquad \forall x \in M. \tag{14}$$

Estimate (14) implies that the boundary values of u_α on $B_{\delta_\alpha}(x_\alpha)$ for some suitable $\delta_\alpha \in [\delta_0/2, \delta_0]$ decay in the sense of L^∞ and of H^1 with the rate μ_α^{n-2}, and therefore they are negligible .

Outline of the proof of Lemma 3. Using Proposition 1 with $A_i := \alpha$, $\rho_i := u_\alpha$, $f := c(n)R_g$, $q := \bar r$, we have that the operators $-\Delta_g + V_\alpha$, where

$$V_\alpha := \begin{cases} \min\left\{c(n)R_g + \alpha\big(\frac{\|u_\alpha\|_{\bar r}}{u_\alpha}\big)^{2-\bar r}, 1\right\} & \text{when } u_\alpha > 0 \\ 1 & \text{when } u_\alpha = 0, \end{cases}$$

are coercive on $H^1(M)$, with coercivity constant independent of α. Consequently, φ_α is well-defined by the equation

$$-\Delta_g\varphi_\alpha + V_\alpha\varphi_\alpha = \mu_\alpha^{(n-2)/2}\delta_{x_\alpha} \quad \text{on } M$$

and it satisfies

$$C^{-1}\mu_\alpha^{(n-2)/2}\mathrm{dist}_g^{2-n}(x, x_\alpha) \le \varphi_\alpha(x) \le C\mu_\alpha^{(n-2)/2}\mathrm{dist}_g^{2-n}(x, x_\alpha). \tag{15}$$

Setting $\widehat g = \varphi_\alpha^{4/(n-2)} g$, we see that u_α/φ_α satisfies

$$\begin{cases} -\Delta_{\widehat g}\frac{u_\alpha}{\varphi_\alpha} \le \ell_\alpha\left(\frac{u_\alpha}{\varphi_\alpha}\right)^{2^*-1} & \text{on } M\setminus\{x_\alpha\} \\ \int_{B_{R\mu_\alpha}(x_\alpha)}\left(\frac{u_\alpha}{\varphi_\alpha}\right)^{2^*} dv_{\widehat g} \le \varepsilon_0 \end{cases}$$

where ε_0 is chosen small and R is large. The metrics φ_α are singular at x_α, nevertheless, since the singularity is uniformly of the order $\mu_\alpha^{(n-2)/2}\mathrm{dist}_g^{2-n}(x, x_\alpha)$, a Sobolev inequality holds for φ_α *independently* of α, namely:

$$\left(\int_M |u|^{2^*}\, dv_{\widehat g}\right)^{2/2^*} \le C\int_M |\nabla_{\widehat g} u|^2\, dv_{\widehat g} \qquad \forall u \in H^1(M):\ u \equiv 0 \text{ near } 0.$$

Therefore, the Moser iteration scheme may be applied on $M \setminus B_{R\mu_\alpha}(x_\alpha)$. We conclude $u_\alpha/\varphi_\alpha \le C$ in $M\setminus B_{R\mu_\alpha}(x_\alpha)$, which suffices to obtain (14). This version of the Moser iterations was used by Li and Zhu in [15]. □

Several well-known techniques may now be applied to estimate the decay rates in Lemma 2. In particular, we adopt a technique of Bahri and Coron [4] to estimate the distance of u_α to the family $\{t\xi_{P,\lambda}/t > 0, \lambda > 0, P \in M\}$ defined in Section 1. We obtain:

Lemma 4 (Energy estimate). *As $\alpha \to +\infty$, we have:*

$$\mathrm{dist}_{H^1(M)}(u_\alpha, \{t\xi_{P,\lambda}\}_{t,\lambda,P}) \le C\Big(\mu_\alpha^2 + (1 + \mu_\alpha^{-2+\beta})\alpha\|u_\alpha\|_{\bar r}^2\Big), \tag{16}$$

where $\beta = (n-6)(n-2)/[2(n+2)] > 0$ is strictly positive*, since $n \ge 7$.*

At this point we can conclude the proof by inserting (16) into the contradiction assumption (12), and using (8) and (16). Alternatively, we could use a Pohozhaev identity to balance lower-order terms. In either way, we obtain $\alpha \le C$, a contradiction.

The case $n = 6$ is more delicate than the case $n \ge 7$. Indeed, it is in some sense the "limit case" of Theorem 1. To treat this case we use a pointwise *lower* bound, which we obtain by the maximum principle, adapting an idea in [15].

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