ASYMPTOTICS FOR SELFDUAL VORTICES ON THE TORUS AND ON THE PLANE: A GLUING TECHNIQUE

MARTA MACRÌ†, MARGHERITA NOLASCO‡, AND TONIA RICCIARDI†

Abstract. We consider multivortex solutions for the selfdual Abelian Higgs model, as the ratio of the vortex core size to the separation distance between vortex points (the scaling parameter) tends to zero. To this end, we use a gluing technique (a shadowing lemma) for solutions to the corresponding semilinear elliptic equation on the plane, allowing any number (finite or countable) of arbitrarily prescribed singular sources. Our approach is particularly convenient and natural for the study of the asymptotics. In particular, in the physically relevant cases where the vortex points are either finite or periodically arranged in the plane, we prove that a frequently used factorization ansatz for multivortex solutions is rigorously satisfied, up to an error which is exponentially small.

Key words. selfdual Abelian Higgs model, elliptic equation, shadowing lemma

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1. Introduction. We consider the energy density for the static, two-dimensional selfdual Abelian Higgs model in the following form:

\[ \mathcal{E}_\delta(A, \phi) = \delta^2 |dA|^2 + |D\phi|^2 + \frac{1}{4\delta^2} |(\phi|^2 - 1|^2, \]

where \( A = A_1 dx_1 + A_2 dx_2, A_1(x), A_2(x) \in \mathbb{R} \) is a gauge potential (a connection over a principal \( U(1) \) bundle), \( \phi, \phi(x) \in \mathbb{C} \) is a Higgs matter field (a section over an associated complex line bundle), \( D = d - iA \) is the covariant derivative, and \( \delta > 0 \) is the scaling parameter. \( \mathcal{E}_\delta \) is a rescaling of \( \mathcal{E}_1 = \mathcal{E}_\delta|_{\delta=1} \), which coincides with the two-dimensional Ginzburg–Landau energy density in the so-called Bogomol’nyi limit. Such a limit describes the borderline between type I and type II superconductors; see, e.g., Jaffe and Taubes [12]. By the selfdual structure, solutions to the Euler–Lagrange equations of \( \mathcal{E}_\delta \) may be obtained from solutions to the first order system:

\[ (D_1 \pm iD_2)\phi = 0, \]
\[ F_{12} = \partial_1 A_2 - \partial_2 A_1 = \pm \frac{1}{2\delta^2} (1 - |\phi|^2). \]

The vortex-type critical points for the energy associated with \( \mathcal{E}_\delta \), namely, the solutions of (1.1)–(1.2), have received considerable attention in recent years, in view of both their physical and geometrical interest; see, e.g., García-Prada [8], Hong, Jost, and Struwe [10], Stuart [17], Taubes [19], Wang and Yang [20], and the references therein. In particular, Hong, Jost, and Struwe [10] consider (1.1)–(1.2) on a compact
Riemannian surface and perform a detailed analysis of the asymptotics as \( \delta \to 0^+ \). Indeed, the small \( \delta > 0 \) regime, corresponding to the limit of small vortex core size with respect to the separation distance between vortices, is an appropriate approximation for the analysis near the vortex points of solutions of (1.1)–(1.2) with \( \delta = 1 \). This type of asymptotics is also relevant in the context of Ginzburg–Landau vortices; see, e.g., Aftalion, Sandier, and Serfaty [2], Alama and Brousard [3], André, Bauman, and Phillips [4], Bethuel, Brezis, and Hélein [6], Lin [13], Rubinstein and Sternberg [16], just to mention a few. It has also been widely investigated in the context of other selfdual gauge theories; see the monographs of Tarantello [18] and Yang [22].

The fundamental results concerning finite energy solutions of (1.1)–(1.2) on \( \mathbb{R}^2 \) were obtained by Taubes [12, 19]. In particular, Taubes showed that such solutions are completely determined by solutions to the singular elliptic problem

\[
\begin{aligned}
-\Delta u &= \delta^{-2}(1 - e^u) - 4\pi \sum_{j=1}^s m_j \delta_{p_j} \quad \text{on } \mathbb{R}^2, \\
&\quad u(x) \to 0 \quad \text{as } |x| \to +\infty.
\end{aligned}
\]

Here \( s \in \mathbb{N} \), and for \( j = 1, 2, \ldots, s \), \( p_j \in \mathbb{R}^2 \) are the vortex points, \( m_j \in \mathbb{N} \) is the multiplicity of \( p_j \), and \( \delta_{p_j} \) is the Dirac measure at \( p_j \). By variational methods, Taubes proved that, for any \( \delta > 0 \), there exists a unique solution of (1.3) such that the configuration \((A, \phi)\) defined in complex notation by

\[
\phi(z) = \exp\left\{ \frac{i}{2} u(z) \pm i \sum_{j=1}^s m_j \arg(z - p_j) \right\},
\]

\[
A_1 + iA_2 = -i(\partial_1 \pm i\partial_2) \ln \phi
\]

is a smooth, finite energy solution of (1.1)–(1.2) on \( \mathbb{R}^2 \), satisfying \( \phi(p_j) = 0 \) (with the corresponding multiplicity \( m_j \in \mathbb{N} \)) and \( E = \int_{\mathbb{R}^2} \mathcal{E}(A, \phi) = \pm \int_{\mathbb{R}^2} F_{12} = 2\pi \sum_{j=1}^s m_j = 2\pi N \), where \( F_{12} = \partial_1 A_2 - \partial_2 A_1 \) is the magnetic field (the curvature of \( A \)).

In connection with Abrikosov’s mixed states in superconductivity [1], it is also of physical interest to analyze (1.1)–(1.2) on the flat torus \( T^2 = \mathbb{R}^2/(a\mathbb{Z} \times b\mathbb{Z}) \), where \( a, b > 0 \). Such a case has been considered by Wang and Yang in [20]. It is shown in [20] that solutions of (1.1)–(1.2) with \( \delta = 1 \) exist for any given set of vortex points \( p_j \in T^2 \), \( j = 1, 2, \ldots, s \), with multiplicity \( m_j \in \mathbb{N} \), if and only if \( N = \sum_{j=1}^s m_j < |T^2|/(4\pi) \). In particular, on \( T^2 \) the total number of vortices \( N \) cannot be arbitrarily large. Similarly as on \( \mathbb{R}^2 \), denoting \( \Omega = (0, a) \times (0, b) \), solutions of (1.1)–(1.2) on \( T^2 \) correspond to solutions for the singular elliptic problem

\[
\begin{aligned}
-\Delta u &= \delta^{-2}(1 - e^u) - 4\pi \sum_{j=1}^s m_j \delta_{p_j} \quad \text{in } \Omega, \\
&\quad u \text{ doubly periodic on } \partial\Omega.
\end{aligned}
\]

The periodic boundary conditions are justified by certain more general gauge invariant conditions on the configuration \((A, \phi)\) introduced by ’t Hooft [11]. Such conditions force the magnetic flux through a lattice cell to be a “quantized” value proportional to the number of vortices confined. Namely, the ’t Hooft boundary conditions imply the topological constraint \( \pm \int_{\Omega} F_{12} = 2\pi \sum_{j=1}^s m_j = 2\pi N \) on the solutions of (1.5), exactly as for finite energy solutions on \( \mathbb{R}^2 \). Integrating (1.5) on the periodic cell \( \Omega \), we obtain that a necessary condition to the solvability of (1.5) is \( \delta^2 < |\Omega|/(4\pi N) \).

This is obviously satisfied for any finite vortex number \( N \) provided \( \delta > 0 \) is sufficiently small.

Our aim in this note is to show that a “shadowing-type lemma” as introduced in the context of elliptic PDEs by Angenent [5] (see also Nolasco [15]) may be adapted to
elliptic equations with singular sources in order to construct solutions for the following more general equation:

$$-\Delta u = \delta^{-2}(1 - e^u) - 4\pi \sum_{j \in \mathcal{P}} m_j \delta_{p_j} \quad \text{in } \mathbb{R}^2,$$

where the set of indices $\mathcal{P}$ may be either finite or countable, and the vortex points $p_j$, $j \in \mathbb{N}$, are arbitrarily distributed in the plane with the only constraint that

$$d := \inf_{k \neq j} |p_j - p_k| > 0 \quad \text{and} \quad m := \sup_{j \in \mathcal{P}} m_j < +\infty.$$ 

The solution we obtain for (1.6) coincides with the solution obtained by Taubes for problem (1.3) when $\mathcal{P}$ is finite and with the solution obtained by Wang and Yang for problem (1.5) when $\mathcal{P}$ is infinite and the vortex points are periodically arranged in $\mathbb{R}^2$. In fact, unlike the previous approaches, our method provides a unified analysis of (1.3) and (1.5). It should be mentioned that suitable modifications to the method described in [5] are necessary due to the singular sources appearing in (1.6). The case where $\mathcal{P}$ is countable and the vortex points are arbitrarily arranged in $\mathbb{R}^2$ does not seem to have been considered before. Of course, if $\mathcal{P}$ is countable, the energy of such a solution is infinite and only locally bounded. On the other hand, our “gluing” technique is, particularly, convenient and natural to analyze the asymptotics as $\delta \to 0^+$. In particular, as a by-product of our construction, we derive a rigorous proof of the following approximate product formula:

$$\phi(x) = \prod_{j \in \mathcal{P}} \Phi_{m_j} \left(\frac{x - p_j}{\delta}\right) + \eta_\delta,$$

where $\|\eta_\delta\|_{L^\infty(\mathbb{R}^2)} \leq C e^{-c/\delta}$ with $C, c > 0$ independent of $\delta$. Here $(A_{m_j}, \Phi_{m_j})$ is the unique, up to gauge transformation, single vortex (or antivortex) solution with multiplicity $m_j$ to (1.1)–(1.2) with $\delta = 1$ on $\mathbb{R}^2$. We note that in the small $\delta > 0$ regime, a product formula of the form (1.8) is a widely used ansatz in the physics literature, in particular in the study of the dynamics of vortices in the Ginzburg–Landau model; see, e.g., E [7], Neu [14], and Weinstein and Xin [21]. However, we have found a rigorous proof of (1.8) only for the case $N = 2$ on $\mathbb{R}^2$ in Stuart [17]. The asymptotic behavior of solutions of (1.1)–(1.2) as $\delta \to 0^+$ is readily derived from formula (1.8) as well as the convergence rates. In fact, in the case of $\mathbb{T}^2$, our asymptotic description improves the previous result obtained (for general compact Riemann surfaces) by Hong, Jost, and Struwe [10] (see Corollary 2.1).

Although we have chosen to consider the Abelian Higgs model for the sake of simplicity, we will show in a forthcoming note that our method may be adapted to other selfdual gauge theories as considered, e.g., in the monographs [18, 22].

2. Main results and outline of the proof. In order to state precisely our results, we denote by $U_N$ the unique radial solution for the problem (see [12])

$$\begin{cases} 
-\Delta U_N = 1 - e^{U_N} - 4\pi N \delta_0 & \text{in } \mathbb{R}^2, \\
U_N(x) \to 0 & \text{as } |x| \to +\infty.
\end{cases}$$

Our main result is the following theorem.

**Theorem 2.1.** Let $p_j \in \mathbb{R}^2$, $m_j \in \mathbb{N}$, $j \in \mathcal{P} \subseteq \mathbb{N}$, and assume that conditions (1.7) hold. Then there exists a constant $\delta_1 > 0$ (depending on $d$ and $m$ only) such
that for every \( \delta \in (0, \delta_1) \) there exists a solution \( u_\delta \) for (1.6). Furthermore, \( u_\delta \) satisfies the approximate superposition rule

\[
(2.2) \quad u_\delta(x) = \sum_{j \in \mathcal{P}} U_{m_j} \left( \frac{|x - p_j|}{\delta} \right) + \omega_\delta,
\]

where the error term \( \omega_\delta \) satisfies \( \|\omega_\delta\|_{\infty} \leq C e^{-c/\delta} \) for some \( C, c > 0 \) independent of \( \delta \). In particular, \( u_\delta \) satisfies the following properties:

(i) \( 0 \leq e^{u_\delta} < 1 \), \( e^{u_\delta} \) vanishes exactly at \( p_j, j \in \mathcal{P} \);

(ii) for every compact subset \( K \) of \( \mathbb{R}^2 \setminus \bigcup_{j \in \mathcal{P}} \{p_j\} \), there exist \( C, c > 0 \) such that

\[
\sup_K (1 - e^{u_\delta}) \leq C e^{-c/\delta} \quad \text{as} \quad \delta \to 0^+;
\]

(iii) \( \pm F_{12} = \frac{1}{2\pi^2} (1 - e^{u_\delta}) \to 2\pi \sum_{j \in \mathcal{P}} m_j \delta_{p_j} \) in the sense of distributions as \( \delta \to 0^+ \).

In the case that \( \mathcal{P} \) is countable, we say that the vortex points \( p_j, j \in \mathcal{P} \), are doubly periodically arranged in \( \mathbb{R}^2 \) if there exists \( s \in \mathbb{N} \) such that

\[
(2.3) \quad \{p_k\}_{k \in \mathcal{P}} = \{p_j + m_1 \epsilon_1 + n_2 \epsilon_2 : j = 1, \ldots, s; m, n \in \mathbb{Z}\},
\]

where \( \epsilon_1 \) and \( \epsilon_2 \) are the unit vectors in \( \mathbb{R}^2 \) defining the periodic cell domain \( \Omega \) (for simplicity, we assume \( a = b = 1 \)). Under this condition, solving (1.6) is equivalent to solving (1.5). Namely, we deal with the physically relevant case of a finite number of vortex points \( p_1, \ldots, p_s \in \Omega \), with the corresponding multiplicity \( m_j, j = 1, \ldots, s \), such that \( \sum_{j=1}^s m_j = N \), where \( N \) is the vortex number and \( \Omega \) is the periodic cell domain. As a consequence of Theorem 2.1, and proving in addition that if (2.3) is satisfied, then the solution \( u_\delta \) for (1.6) is in fact doubly periodic with periodic domain \( \Omega \), we derive the following result.

**Corollary 2.1.** If the \( p_j \)'s are doubly periodically arranged in \( \mathbb{R}^2 \), there exists a constant \( \delta_1 > 0 \) (depending on \( N \) only) such that for every \( \delta \in (0, \delta_1) \) the solution \( u_\delta \), given in Theorem 2.1, is a solution for (1.5). Furthermore, the corresponding vortex configurations \( (A_\delta, \phi_\delta) \) satisfy the approximate factorization rule

\[
\phi_\delta(x) = \prod_{j=1}^s \Phi_{m_j} \left( \frac{x - p_j}{\delta} \right) + \eta_\delta, \quad x \in \Omega,
\]

where the error term \( \eta_\delta \) satisfies \( \|\eta_\delta\|_{\infty} \leq C e^{-c/\delta} \) for some \( C, c > 0 \) independent of \( \delta \), and \( (A_\delta, \Phi_{m_j}) \) is the unique, up to gauge transformation, single vortex (or antivortex) solution with multiplicity \( m_j \), to (1.1)–(1.2) with \( \delta = 1 \) on \( \mathbb{R}^2 \). In particular, we have the following:

(i) \( 0 \leq |\phi_\delta|^2 < 1 \), \( \phi_\delta \) vanishes exactly at \( p_j, j = 1, \ldots, s \);

(ii) for every compact subset \( K \) of \( \Omega \setminus \{p_1, \ldots, p_s\} \), there exist \( C, c > 0 \) such that

\[
0 \leq \sup_K (1 - |\phi_\delta|^2) \leq C e^{-c/\delta} \quad \text{as} \quad \delta \to 0^+;
\]

(iii) \( \pm F_{12}(A_\delta, \phi_\delta) = \frac{1}{2\pi^2} (1 - |\phi_\delta|^2) \to 2\pi \sum_{j=1}^s m_j \delta_{p_j} \) in the sense of distributions (on \( \Omega \)) as \( \delta \to 0^+ \);

(iv) \( \int_{\Omega} E_\delta(A_\delta, \phi_\delta) = \pm \int_{\Omega} F_{12}(A_\delta, \phi_\delta) = 2\pi N \).

An outline of the proof is as follows. Our starting point in proving Theorem 2.1 is to consider \( \delta > 0 \) as a scaling parameter. Setting \( \hat{u}(x) = u(\delta x) \), we have that \( \hat{u} \) satisfies

\[
(2.4) \quad -\Delta \hat{u} = 1 - e^{\hat{u}} - 4\pi \sum_{j \in \mathcal{P}} m_j \delta_{p_j} \quad \text{in} \quad \mathbb{R}^2,
\]
where \( \hat{p}_j = p_j / \delta \). Note that the vortex points \( \hat{p}_j \) “separate” as \( \delta \to 0^+ \). Section 3 contains some properties of the radial solutions \( U_N \) to (2.1). We rely on the results of Taubes [19] for the existence and uniqueness of \( U_N \) as well as for the exponential decay properties at infinity. We also prove a nondegeneracy property of \( U_N \). The exponential decay of solutions justifies the following approximate superposition picture for small values of \( \delta \), i.e., for vortex points \( \hat{p}_j \) which are “far apart”:

\[
\hat{u}(x) \approx \sum_{j \in \mathcal{P}} U_{m_j}(|x - \hat{p}_j|).
\]

In fact, we take the following preliminary form of the superposition rule:

\[
\hat{u} = \sum_{j \in \mathcal{P}} \hat{\phi}_j U_{m_j}(x - \hat{p}_j) + z
\]

as an ansatz for \( \hat{u}_\delta \). Here, radial solutions centered at \( \hat{p}_j \) are “glued” together by the functions \( \hat{\phi}_j \), which belong to a suitable locally finite partition of unity. Section 4 contains the definition and the main properties of the partition as well as of the appropriate functional spaces \( \hat{X}_\delta, \hat{Y}_\delta \), which are also obtained by “gluing” \( H^2(R^2) \) and \( L^2(R^2) \), respectively. Hence, we are reduced to show that for small values of \( \delta > 0 \), there exists an exponentially small “error” \( z \) such that \( \hat{u} \) defined by (2.5) is a solution of (2.4). The existence of such a \( z \in \hat{X}_\delta \) is the aim of section 5 (see Proposition 5.1). To this end, we use the shadowing lemma. We characterize \( z \) by the property

\[
F_{\delta}(z) = 0,
\]

where \( F_{\delta} : \hat{X}_\delta \to \hat{Y}_\delta \) is suitably defined. The nondegeneracy property of \( U_N \) is essential in order to prove that the operator \( DF_{\delta}(0) \) is invertible, and that its inverse is bounded independently of \( \delta > 0 \) (Lemma 5.3). At this point, the Banach fixed point argument applied to \( I - (DF_{\delta}(0))^{-1} F_{\delta} \) yields the existence of the desired error term \( z \). In section 6 we show that (2.5) implies (2.2) and we derive the asymptotic behavior of solutions, thus concluding the proof of Theorem 2.1. Finally, we derive Corollary 2.1 by showing that periodically arranged vortex points lead to periodic solutions.

Henceforth, unless otherwise stated, we denote by \( C, c > 0 \) general constants independent of \( \delta > 0 \) and of \( j \in \mathcal{P} \).

3. Single vortex point solutions. In this section, we consider the solution \( U_N \) to the radially symmetric equation (2.1). For every \( r > 0 \), we denote \( B_r = \{ x \in \mathbb{R}^2 : |x| < r \} \). The following lemma contains some properties of \( U_N \) that will be needed in the following. The proof is a consequence of the results of Taubes [12, 19] on the existence, uniqueness, and the exponential decay of \( U_N \) together with standard elliptic theory as in, e.g., [9]. Therefore, it is omitted.

**Lemma 3.1.** The following properties hold:

(i) \( e^{U_N(x)} < 1 \) for any \( x \in \mathbb{R}^2 \);

(ii) for every \( r > 0 \) there exist constants \( C_N > 0 \) and \( \alpha_N > 0 \), depending on \( r \) and \( N \) only, such that

\[
|1 - e^{U_N(x)}| + |\nabla U_N(x)| + |U_N(x)| \leq C_N e^{-\alpha_N |x|}
\]

for all \( x \in \mathbb{R}^2 \setminus B_r \).

We consider the bounded linear operator

\[
L_N = -\Delta + e^{U_N} : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2).
\]
In order to apply the shadowing lemma, we also need the following nondegeneracy property of $U_N$.

**Lemma 3.2.** The operator $L_N$ is invertible and for every $N > 0$ there exists $C_N > 0$ such that $\|L_N^{-1}\| \leq C_N$.

**Proof.** It is readily seen that $L_N$ is injective. Indeed, suppose $L_N u = 0$ for some $u \in H^2(\mathbb{R}^2)$. Multiplying by $u$ and integrating on $\mathbb{R}^2$, we have

$$\int |\nabla u|^2 + \int e^{U_N} u^2 = 0.$$

Therefore, $u = 0$. Now, we claim that $L_N$ is a Fredholm operator. Indeed, we write

$$L_N = (-\Delta + 1)(\mathbb{I} - T)$$

with $T = (-\Delta + 1)^{-1}(1 - e^{U_N}) : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$. Clearly, $T$ is continuous. Let us check that $T$ is compact. To this end, let $u_n \in H^2(\mathbb{R}^2)$, $\|u_n\|_{H^2} = 1$. We have to show that $T u_n$ has a convergent subsequence. Note that by the Sobolev embedding

\begin{equation}
\|u\|_{L_\infty(\mathbb{R}^2)} \leq C_S \|u\|_{H^2(\mathbb{R}^2)},
\end{equation}

for all $u \in H^2(\mathbb{R}^2)$, we have $\|u_n\|_\infty \leq C'$, for some $C' > 0$ independent of $n$, and there exists $u_\infty$, $\|u_\infty\|_{H^2} \leq 1$, such that $u_{n_k} \to u_\infty$ strongly in $L^2_{\text{loc}}$ for a subsequence $u_{n_k}$. Now, by Lemma 3.1, for any fixed $\varepsilon > 0$, there exists $R > 0$ such that $\|1 - e^{U_N}\|_{L^2(\mathbb{R}^2 \setminus B_R)} \leq \varepsilon$. Consequently, $\|(1 - e^{U_N})(u_{n_k} - u_\infty)\|_{L^2(\mathbb{R}^2 \setminus B_R)} \leq 2C'\varepsilon$. On the other hand, $\|(1 - e^{U_N})(u_{n_k} - u_\infty)\|_{L^2(B_R)} \to 0$. We conclude that $(1 - e^{U_N})(u_{n_k} - u_\infty) \to 0$ in $L^2$. In turn, we have $T(u_{n_k} - u_\infty) = (-\Delta + 1)^{-1}(1 - e^{U_N})(u_{n_k} - u_\infty) \to 0$ in $H^2$, which implies that $T$ is compact. It follows that $L_N$ is a Fredholm operator. Consequently, $L_N$ is also surjective. At this point, the open mapping theorem concludes the proof. \hfill \qed

**4. A partition of unity.** In this section, we introduce a partition of unity and we prove some technical results which will be needed in the following. Let $p_j \in \mathbb{R}^2$ ($j \in \mathcal{P} \subseteq \mathbb{N}$) be the vortex points. By assumption (1.7), $r_0 = d/8 = \inf_{j \neq k} |p_j - p_k|/8 > 0$. We consider the set $K = (\frac{-3}{4}r_0, \frac{-3}{4}r_0) \times (\frac{-3}{4}r_0, \frac{3}{4}r_0)$. Then, for any $n \in \mathbb{Z}^2$, we introduce $K_n = K + nr_0$. The collection of sets $\{K_n\}_{n \in \mathbb{Z}^2}$ is a locally finite covering of $\mathbb{R}^2$. We consider an associated partition of unity defined as follows: let $0 \leq \zeta \in C_0^\infty(K)$ be such that $\sum_{n \in \mathbb{Z}^2} \zeta_n = 1$ pointwise on $\mathbb{R}^2$, where $\zeta_n(x) = \zeta(x - nr_0)$ for all $x \in \mathbb{R}^2$. Then, for any $j \in \mathcal{P}$, we introduce the set

$$\mathcal{P}_j = \left\{ n \in \mathbb{Z}^2 : d(p_j, K_n) < \frac{r_0}{4} \right\}.$$

Note that the cardinality of $\mathcal{P}_j$ is uniformly bounded, namely, $|\mathcal{P}_j| \leq 4$ for any $j \in \mathcal{P}$. We set

$$\mathcal{P}_j = \bigcup_{n \in \mathcal{P}_j} K_n, \quad \varphi_j = \sum_{n \in \mathcal{P}_j} \zeta_n$$

for any $j \in \mathcal{P}$. Since, by choice of $r_0$, $\mathbb{Z}^2 \setminus \bigcup_{j \in \mathcal{P}} \mathcal{P}_j$ is a countable set (even in the case that $\mathcal{P}$ is countable), there exists a bijection $T : \mathbb{N} \to \mathbb{Z}^2 \setminus \bigcup_{j \in \mathcal{P}} \mathcal{P}_j$. For convenience of notation, we denote by $\mathcal{Q}$ the countable set of indices defined by

$$\mathcal{Q} = T^{-1}\left( \mathbb{Z}^2 \setminus \bigcup_{j \in \mathcal{P}} \mathcal{P}_j \right).$$
We set
\[ Q_k = K_{I(k)}, \quad \psi_k = \zeta_{I(k)} \quad \forall k \in Q. \]

Then, \( \{P_j, Q_k\}_{(j,k) \in P \times Q} \) is a locally finite open covering of \( \mathbb{R}^2 \) with the property that \( P_j \cap P_{j'} = \emptyset \) for every \( j' \neq j \). Moreover, \( \{\varphi_j, \psi_k\}_{(j,k) \in P \times Q} \) is a partition of unity associated with \( \{P_j, Q_k\}_{(j,k) \in P \times Q} \) such that
\[ \text{supp} \varphi_j \subset P_j, \quad \text{supp} \psi_k \subset Q_k \]
and such that
\[ \sup_{j \in P}\left\{ \|\nabla \varphi_j\|_\infty, \|D^2 \varphi_j\|_\infty \right\} < +\infty, \quad \sup_{k \in Q}\left\{ \|\nabla \psi_k\|_\infty, \|D^2 \psi_k\|_\infty \right\} < +\infty. \]

In particular,
\[ 0 \leq \varphi_j, \psi_k \leq 1 \quad \text{and} \quad \sum_{j \in P} \varphi_j + \sum_{k \in Q} \psi_k = \sum_{n \in \mathbb{Z}^2} \zeta_k = 1. \]

We define a rescaled covering
\[ \hat{P}_j = P_j / \delta, \quad \hat{Q}_k = Q_k / \delta. \]

Then, \( \{\hat{\varphi}_j, \hat{\psi}_k\}_{(j,k) \in P \times Q} \) defined by
\[ \hat{\varphi}_j(x) = \varphi_j(\delta x), \quad \hat{\psi}_k(x) = \psi_k(\delta x) \]
is a partition of unity associated with \( \{\hat{P}_j, \hat{Q}_k\}_{(j,k) \in P \times Q} \). It will also be convenient to define the sets
\[ \hat{C}_j = \{x \in \hat{P}_j : \hat{\varphi}_j(x) = 1\}, \quad j \in P. \]

Note that
\[ \text{supp}\{\nabla \hat{\varphi}_j, D^2 \hat{\varphi}_j\} \subset \hat{P}_j \setminus \hat{C}_j \]
and
\[ (4.1) \quad \sup_{(j,k) \in P \times Q}\left\{ \|\nabla \hat{\varphi}_j\|_\infty + \|\nabla \hat{\psi}_k\|_\infty \right\} \leq C\delta, \quad \sup_{(j,k) \in P \times Q}\left\{ \|D^2 \hat{\varphi}_j\|_\infty + \|D^2 \hat{\psi}_k\|_\infty \right\} \leq C\delta^2. \]

For every fixed \( x \in \mathbb{R}^2 \), we define the following subsets of indices:
\[ J(x) = \{ j \in P : \hat{\varphi}_j(x) \neq 0 \}, \quad K(x) = \{ k \in Q : \hat{\psi}_k(x) \neq 0 \}. \]

Note that, for every \( x \in \mathbb{R}^2 \),
\[ (4.2) \quad |J(x)| \leq 1, \quad |K(x)| \leq 4, \]
where \( |J(x)| \) and \( |K(x)| \) denote the cardinality of \( J(x) \) and \( K(x) \), respectively. We shall use the following Banach spaces:
\[ \hat{X}_\delta = \left\{ u \in H^2_{\text{loc}}(\mathbb{R}^2) : \sup_{(j,k) \in P \times Q}\left\{ \|\hat{\varphi}_j u\|_{H^2(\mathbb{R}^2)}, \|\hat{\psi}_k u\|_{H^2(\mathbb{R}^2)} \right\} < +\infty \right\}, \]
\[ \hat{Y}_\delta = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^2) : \sup_{(j,k) \in P \times Q}\left\{ \|\hat{\varphi}_j f\|_{L^2(\mathbb{R}^2)}, \|\hat{\psi}_k f\|_{L^2(\mathbb{R}^2)} \right\} < +\infty \right\}. \]

We collect in the following lemma some estimates that will be used in what follows.
LEMMA 4.1. There exists a constant $C > 0$ such that for any $u \in \dot{X}_κ$ and $j \in \mathcal{P}$, we have

(i) $\|u\|_{H^2(\hat{P}_j)} \leq C\|u\|_{\dot{X}_κ}$,

(ii) $\|u\|_{L^∞(\mathbb{R}^2)} \leq C\|u\|_{\dot{X}_κ}$.

Proof. (i) For every fixed $j \in \mathcal{P}$, let $\mathcal{J}(j) = \{k \in \mathcal{Q} : \text{supp } \hat{\varphi}_j \cap \text{supp } \hat{\psi}_k \neq \emptyset\}$. Then, $\sup_{j \in \mathcal{P}} |\mathcal{J}(j)| < +\infty$, and we estimate

$$\|u\|_{H^2(\hat{P}_j)} = \left\|\hat{\varphi}_j u + \sum_{k \in \mathcal{J}(j)} \hat{\psi}_k u\right\|_{H^2(\hat{P}_j)} \leq \|\hat{\varphi}_j u\|_{H^2(\hat{P}_j)} + \sum_{k \in \mathcal{J}(j)} \|\hat{\psi}_k u\|_{H^2(\hat{P}_j)} \leq (1 + |\mathcal{J}(j)|) \|\varphi\|_{\dot{X}_κ} \leq C \|\varphi\|_{\dot{X}_κ}.$$

(ii) For any fixed $x \in \mathbb{R}^2$, we have in view of (3.1) and (4.2)

$$|u(x)| = \sum_{j \in \mathcal{P}} \hat{\varphi}_j(x)|u(x)| + \sum_{k \in \mathcal{Q}} \hat{\psi}_k(x)|u(x)| = \sum_{j \in \mathcal{J}(x)} \hat{\varphi}_j(x)|u(x)| + \sum_{k \in \mathcal{K}(x)} \hat{\psi}_k(x)|u(x)| \leq \sum_{j \in \mathcal{J}(x)} CS\|\hat{\varphi}_j u\|_{H^2(\mathbb{R}^2)} + \sum_{k \in \mathcal{K}(x)} CS\|\hat{\psi}_k u\|_{H^2(\mathbb{R}^2)} \leq \sup_{x \in \mathbb{R}^2} (|J(x)| + |K(x)|)CS\|u\|_{\dot{X}_κ} = C\|u\|_{\dot{X}_κ}.$$

Hence, (ii) is established. □

We shall also need the following family of functions:

$$\hat{g}_j(x) = \frac{\hat{\varphi}_j(x)}{\left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j^2 + \sum_{k \in \mathcal{Q}} \hat{\psi}_k^2\right)^{1/2}}, \quad \hat{h}_k(x) = \frac{\hat{\psi}_k(x)}{\left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j^2 + \sum_{j \in \mathcal{Q}} \hat{\psi}_j^2\right)^{1/2}}.$$

In view of (4.1), it is readily checked that the following lemma follows.

LEMMA 4.2. The family $\{\hat{g}_j, \hat{h}_k\}_{(j,k) \in \mathcal{P} \times \mathcal{Q}}$ satisfies supp $\hat{g}_j \subset \hat{P}_j$, supp $\hat{h}_k \subset \hat{Q}_k$ and furthermore,

(i) $\sum_{j \in \mathcal{P}} \hat{g}_j^2(x) + \sum_{k \in \mathcal{Q}} \hat{h}_k^2(x) = 1 \forall x \in \mathbb{R}^2$;

(ii) $C^{-1}\hat{\varphi}_j(x) \leq \hat{g}_j(x) \leq C\hat{\varphi}_j(x)$ and $C^{-1}\hat{\psi}_k(x) \leq \hat{h}_k(x) \leq C\hat{\psi}_k(x) \forall x \in \mathbb{R}^2$;

(iii) $\sup_{(j,k) \in \mathcal{P} \times \mathcal{Q}} \{\|\nabla \hat{g}_j\|_\infty + \|\nabla \hat{h}_k\|_\infty\} \leq C\delta$ and $\sup_{(j,k) \in \mathcal{P} \times \mathcal{Q}} \{\|D^2 \hat{g}_j\|_\infty + \|D^2 \hat{h}_k\|_\infty\} \leq C\delta^2$.

5. The shadowing lemma. Recall from the introduction that $\hat{p}_j = p_j/\delta$, $j \in \mathcal{P}$. For every $j \in \mathcal{P}$ we define

$$\hat{U}_j(x) = U_{m_j}(x - \hat{p}_j).$$

We make the following ansatz for solutions $\tilde{u}$ to (2.4):

$$\tilde{u} = \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z.$$

Our aim in this section is to prove the following.
Proposition 5.1. There exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1)$ there exists $z_\delta \in \hat{X}_\delta$ such that $\hat{u}_\delta$ defined by $\hat{u}_\delta = \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z_\delta$ is a solution of (2.4). Moreover, $\|z_\delta\|_{\hat{X}_\delta} \leq C e^{-c/\delta}$.

We note that the functional $F_\delta : \hat{X}_\delta \to \hat{Y}_\delta$ given by

$$F_\delta(z) = -\Delta z + \sum_{j \in \mathcal{P}} \hat{\varphi}_j (1 - e^{\hat{U}_j}) - (1 - e^{\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z}) - \sum_{j \in \mathcal{P}} [\hat{\varphi}_j, \Delta] \hat{U}_j$$

is well defined as well as $C^1$. Here $[\Delta, \hat{\varphi}_j] = \Delta \hat{\varphi}_j + 2 \nabla \hat{\varphi}_j \nabla$. Moreover, if $z \in \hat{X}_\delta$ satisfies $F_\delta(z) = 0$, then $\hat{u}$ defined by (5.1) is a solution of (2.4).

Lemma 5.1. For $\delta > 0$ sufficiently small, we have

$$\|F_\delta(0)\|_{\hat{Y}_\delta} \leq C e^{-c/\delta} \quad \text{as } \delta \to 0^+$$

for some constants $C, c > 0$ independent of $\delta$.

Proof. Let

$$\mathcal{R} = \sum_{j \in \mathcal{P}} \hat{\varphi}_j (1 - e^{\hat{U}_j}) - (1 - e^{\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j})$$

and

$$\mathcal{C} = \sum_{j \in \mathcal{P}} [\hat{\varphi}_j, \Delta] \hat{U}_j.$$

Note that $\{\supp \mathcal{R}, \supp \mathcal{C}\} \subset \cup_{j \in \mathcal{P}} \hat{P}_j \setminus \hat{C}_j$. We fix $x \in \cup_{j \in \mathcal{P}} \hat{P}_j$. We estimate

$$|\mathcal{R}(x)| \leq \sup_{j \in \mathcal{P}} \|\hat{\varphi}_j (1 - e^{\hat{U}_j})\|_{L^\infty(\hat{P}_j \setminus \hat{C}_j)} + \sup_{j \in \mathcal{P}} \|1 - e^{\hat{\varphi}_j \hat{U}_j}\|_{L^\infty(\hat{P}_j \setminus \hat{C}_j)}$$

$$\leq C \sup_{j \in \mathcal{P}} \|\hat{U}_j\|_{L^\infty(\hat{P}_j \setminus \hat{C}_j)} \leq C_1 e^{-c_1/\delta}.$$

On the other hand, in view of (4.1) and Lemma 3.1, for $x \in \cup_{j \in \mathcal{P}} \hat{P}_j$, we have

$$|\mathcal{C}(x)| \leq \sup_{j \in \mathcal{P}} \|[\Delta, \hat{\varphi}_j] \hat{U}_j\|_{L^\infty(\hat{P}_j \setminus \hat{C}_j)}$$

$$\leq C \left( \sup_{j \in \mathcal{P}} \|\hat{U}_j \Delta \hat{\varphi}_j\|_{L^\infty(\hat{P}_j \setminus \hat{C}_j)} + \sup_{j \in \mathcal{P}} \|\nabla \hat{U}_j\|_{L^\infty(\hat{P}_j \setminus \hat{C}_j)} \right) \leq C_2 e^{-c_2/\delta}.$$

Here and above, $c_1, C_1, c_2, C_2 > 0$ are positive constants independent of $\delta > 0$. Hence, we conclude that, as $\delta \to 0^+$,

$$\|F_\delta(0)\|_{\hat{Y}_\delta} \leq C \sup_{j \in \mathcal{P}} \left( \|\mathcal{R}\|_{L^2(\hat{P}_j)} + \|\mathcal{C}\|_{L^2(\hat{P}_j)} \right) \leq C e^{-c/\delta}$$

for some constants $C, c > 0$ independent of $\delta > 0$. \qed

Now, we consider the operator $L_\delta \equiv DF_\delta(0) : \hat{X}_\delta \to \hat{Y}_\delta$ given by

$$L_\delta = -\Delta + e^{\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j}.$$

For every $j \in \mathcal{P}$, we define the operators

$$\hat{L}_j = -\Delta + e^{\hat{U}_j}.$$
It will also be convenient to define
\[ \hat{L}_0 = -\Delta + 1. \]

The following lemma holds.

**Lemma 5.2.** There exist \( C, c > 0 \) such that for any \( u \in \hat{X}_\delta \), we have
\[
\| (L_\delta - \hat{L}_j) \hat{\varphi}_j u \|_{L^2} \leq C e^{-c/\delta} \| \hat{\varphi}_j u \|_{L^2}, \quad j \in \mathcal{P},
\]
\[
\| (L_\delta - \hat{L}_0) \hat{\psi}_k u \|_{L^2} \leq C e^{-c/\delta} \| \hat{\psi}_k u \|_{L^2}, \quad k \in \mathcal{Q}.
\]

**Proof.** For any \( j \in \mathcal{P} \), by Lemma 3.1, we have, as \( \delta \to 0^+ \),
\[
\| (L_\delta - \hat{L}_j) \hat{\varphi}_j u \|_{L^2} \leq \left( \| 1 - e^{\hat{L}_j} \|_{L^\infty (\hat{P}_j \setminus \hat{C}_j)} + \| 1 - e^{\hat{L}_j U} \|_{L^\infty (\hat{P}_j \setminus \hat{C}_j)} \right) \| \hat{\varphi}_j u \|_{L^2}
\leq C \| 1 - e^{\hat{L}_j} \|_{L^\infty (\hat{P}_j \setminus \hat{C}_j)} \| \hat{\varphi}_j u \|_{L^2} \leq C e^{-c/\delta} \| \hat{\varphi}_j u \|_{L^2}.
\]

Similarly, as \( \delta \to 0^+ \),
\[
\| (L_\delta - \hat{L}_0) \hat{\psi}_k u \|_{L^2} \leq \| (1 - e^{\hat{L}_0}) \hat{\psi}_k u \|_{L^2}
\leq \sup_{j \in \mathcal{P}} \| 1 - e^{\hat{L}_j} \|_{L^\infty (\hat{P}_j \setminus \hat{C}_j)} \| \hat{\psi}_k u \|_{L^2} \leq C e^{-c/\delta} \| \hat{\psi}_k u \|_{L^2}.
\]

Now, we prove an essential nondegeneracy property of \( L_\delta \).

**Lemma 5.3.** There exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), the operator \( L_\delta \) is invertible. Moreover, \( L_\delta^{-1} : \hat{Y}_\delta \to \hat{X}_\delta \) is uniformly bounded with respect to \( \delta \in (0, \delta_0) \).

**Proof.** Following a gluing technique introduced in [5], we construct an “approximate inverse” \( S_\delta : \hat{Y}_\delta \to \hat{X}_\delta \) for \( L_\delta^{-1} \) as follows:
\[
S_\delta = \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} \hat{g}_j + \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} \hat{h}_k,
\]
where \( \hat{g}_j \) and \( \hat{h}_k \) are the functions introduced in section 4. We claim that the operator \( S_\delta \) is well defined and uniformly bounded with respect to \( \delta \). That is, we claim that
\[
(5.3) \quad \| S_\delta f \|_{\hat{X}_\delta} \leq C \| f \|_{\hat{Y}_\delta}
\]
for some \( C > 0 \) independent of \( f \in \hat{X}_\delta \) and of \( \delta > 0 \).

Indeed, for any \( f \in \hat{Y}_\delta \), we have
\[
\| S_\delta f \|_{\hat{X}_\delta} = \sup_{(j, k) \in \mathcal{P} \times \mathcal{Q}} \{ \| \hat{\varphi}_j S_\delta f \|_{H^2}, \| \hat{\psi}_k S_\delta f \|_{H^2} \}
\]
and
\[
\| \hat{\varphi}_j S_\delta f \|_{H^2} \leq \| \hat{\varphi}_j \hat{g}_j \hat{L}_j^{-1} \hat{g}_j f \|_{H^2} + \left\| \hat{\varphi}_j \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} \hat{h}_k f \right\|_{H^2},
\]
\[
\| \hat{\psi}_k S_\delta f \|_{H^2} \leq \| \hat{\psi}_k \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} \hat{g}_j f \|_{H^2} + \left\| \hat{\psi}_k \sum_{j \in \mathcal{Q}} \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f \right\|_{H^2}.
\]
We estimate, recalling the properties of $\hat{\phi}_j$ and $\hat{g}_j$ as in Lemma 4.2, and in view of Lemma 3.2

$$\|\hat{\phi}_j \hat{g}_j \hat{L}_j^{-1} \hat{g}_j f\|_{H^2} \leq C \|\hat{L}_j^{-1} \hat{g}_j f\|_{H^2} \leq C \|\hat{g}_j f\|_{L^2} \leq C \|\hat{\phi}_j f\|_{L^2} \leq C \|f\|_{\tilde{Y}_s}.$$  

We have

$$\left\| \hat{\phi}_j \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} \hat{h}_k f \right\|_{H^2} \leq \left\| \hat{\phi}_j \sum_{k \in \mathcal{J}(j)} \hat{h}_k \hat{L}_0^{-1} \hat{h}_k f \right\|_{H^2} \leq \sum_{k \in \mathcal{J}(j)} \|\hat{\phi}_j \hat{h}_k \hat{L}_0^{-1} \hat{h}_k f\|_{H^2},$$

where $\mathcal{J}(j) = \{k \in \mathcal{Q} : \text{supp } \hat{\psi}_k \cap \text{supp } \hat{\phi}_j = \emptyset\}$ satisfies $\sup_{j \in \mathcal{P}} |\mathcal{J}(j)| < +\infty$. In view of Lemmas 4.2 and 3.2, we estimate

$$\sum_{k \in \mathcal{J}(j)} \|\hat{\phi}_j \hat{h}_k \hat{L}_0^{-1} \hat{h}_k f\|_{H^2} \leq C \sum_{k \in \mathcal{J}(j)} \|\hat{L}_0^{-1} \hat{h}_k f\|_{H^2} \leq C \sum_{k \in \mathcal{J}(j)} \|\hat{h}_k f\|_{L^2} \leq C \sum_{k \in \mathcal{J}(j)} \|\hat{\psi}_k f\|_{L^2} \leq C \|f\|_{\tilde{Y}_s}.$$  

Therefore,

$$\sup_{j \in \mathcal{P}} \left\| \hat{\phi}_j \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} \hat{h}_k f \right\|_{H^2} \leq C \|f\|_{\tilde{Y}_s}.$$  

Similarly, we obtain that

$$\sup_{k \in \mathcal{Q}} \left\| \hat{\psi}_k \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} \hat{g}_j f \right\|_{H^2} \leq C \|f\|_{\tilde{Y}_s}, \quad \sup_{k \in \mathcal{Q}} \left\| \hat{\psi}_k \sum_{j \in \mathcal{Q}} \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f \right\|_{H^2} \leq C \|f\|_{\tilde{Y}_s},$$

and (5.3) follows.

Now, we claim that there exists $\delta_0$ such that for any $\delta \in (0, \delta_0)$, the operator $S_\delta L_\delta : \hat{X}_s \to \hat{X}_s$ is invertible, and furthermore, $\|S_\delta L_\delta\| \leq C$ for some $C > 0$ independent of $\delta > 0$. We note that $(L_\delta - \hat{L}_j) \hat{g}_j : \hat{X}_s \to \hat{Y}_s$ and $(L_\delta - \hat{L}_0) \hat{h}_k : \hat{X}_s \to \hat{Y}_s$ are well-defined bounded linear operators. Thus, we decompose

$$(5.4) \quad S_\delta L_\delta = \mathbb{1}_{\hat{X}_s} + \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} (\hat{g}_j L_\delta - \hat{L}_j \hat{g}_j) + \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} (\hat{h}_k L_\delta - \hat{L}_0 \hat{h}_k) \ni \hat{X}_s + \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} (L_\delta - \hat{L}_j) \hat{g}_j + \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} (L_\delta - \hat{L}_0) \hat{h}_k + \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} [\Delta, \hat{g}_j] + \sum_{k \in \mathcal{Q}} \hat{h}_k \hat{L}_0^{-1} [\Delta, \hat{h}_k].$$

Hence, it suffices to prove that the last four terms in (5.4) are sufficiently small, in the operator norm, provided $\delta > 0$ is sufficiently small. By Lemmas 5.2 and 4.2, we have, for any $u \in \hat{X}_s$,

$$\left\| \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1} (L_\delta - \hat{L}_j) \hat{g}_j u \right\|_{\hat{X}_s} \leq C \sup_{j \in \mathcal{P}} \|\hat{L}_j^{-1} (L_\delta - \hat{L}_j) \hat{g}_j u\|_{H^2} \leq C \sup_{j \in \mathcal{P}} \|L_\delta - \hat{L}_j\| \|\hat{g}_j u\|_{L^2} \leq C e^{-c/\delta} \|\hat{\phi}_j u\|_{L^2} \leq C e^{-c/\delta} \|u\|_{\hat{X}_s}.$$
Similarly, for \( u \in \hat{X}_\delta \), we have
\[
\left\| \sum_{j \in \mathcal{P}} \hat{g}_j \hat{L}_j^{-1}[\Delta, \hat{g}_j] u \right\|_{\hat{X}_\delta} \leq C \sup_{j \in \mathcal{P}} \| \hat{L}_j^{-1}[\Delta, \hat{g}_j] u \|_{H^2} \leq C \sup_{j \in \mathcal{P}} \| [\Delta, \hat{g}_j] u \|_{L^2}.
\]
Recalling that \( [\Delta, \hat{g}_j] u = 2 \nabla u \nabla \hat{g}_j + u \Delta \hat{g}_j \), by Lemmas 4.2 and 4.1(i) we derive that
\[
\| [\Delta, \hat{g}_j] u \|_{L^2} \leq C \delta \| u \|_{H^1(\hat{P}_j)} \leq C \delta \| u \|_{\hat{X}_\delta}.
\]
The remaining terms are estimated similarly. Hence, \( \| S_\delta L_\delta - \hat{I}_{\hat{X}_\delta} \| \to 0 \) as \( \delta \to 0^+ \).

Now, we observe that \( L_\delta^{-1} = (S_\delta L_\delta)^{-1} S_\delta \). It follows that for any \( f \in \hat{Y}_\delta \), we have
\[
\| L_\delta^{-1} f \|_{\hat{X}_\delta} = \| (S_\delta L_\delta)^{-1} S_\delta f \|_{\hat{X}_\delta} \leq C \| S_\delta f \|_{\hat{Y}_\delta} \leq C \| f \|_{\hat{Y}_\delta}
\]
with \( C > 0 \) independent of \( \delta \). Hence, \( L_\delta \) is invertible and its inverse is bounded independently of \( \delta \), as asserted. \( \square \)

Now we can provide the following proof.

**Proof of Proposition 5.1.** We use the Banach fixed point argument. For any \( \delta \in (0, \delta_0) \), with \( \delta_0 > 0 \) given by Lemma 5.3, we introduce the nonlinear map \( G_\delta \in C^1(\hat{X}_\delta, \hat{X}_\delta) \) defined by
\[
G_\delta(z) = z - L_\delta^{-1} F_\delta(z).
\]
Then, fixed points of \( G_\delta \) correspond to solutions of the functional equation \( F_\delta(z) = 0 \).

First, note that \( DG_\delta(0) = 0 \) and that
\[
DF(z) = -\Delta + e^{\sum_{j \in \mathcal{P}} \hat{g}_j U_j} z.
\]
By Lemma 5.3, for any \( z \in \hat{X}_\delta \) and \( u \in \hat{X}_\delta \), we have
\[
\| DG_\delta(z) u \|_{\hat{X}_\delta} = \| (DG_\delta(z) - DG_\delta(0)) u \|_{\hat{X}_\delta} = \| L_\delta^{-1}(DF_\delta(z) - L_\delta) u \|_{\hat{X}_\delta}
\]
\[
\leq C \| (DF_\delta(z) - L_\delta) u \|_{\hat{Y}_\delta} = C \| e^{\sum_{j \in \mathcal{P}} \hat{g}_j U_j} (e^z - 1) u \|_{\hat{Y}_\delta} \leq C \| (e^z - 1) u \|_{\hat{Y}_\delta}.
\]
By the elementary inequality \( e^t - 1 \leq C t e^t \), for all \( t > 0 \), where \( C > 0 \) does not depend on \( t \), and in view of Lemma 4.1, we have
\[
\| e^z - 1 \|_{\hat{Y}_\delta} \leq e\| z \|_{\hat{Y}_\delta} - 1 \leq C \| z \|_{\hat{Y}_\delta} e\| z \|_{\hat{Y}_\delta} \leq C \| z \|_{\hat{Y}_\delta} e\| z \|_{\hat{X}_\delta}.
\]
Hence,
\[
\| DG_\delta(z) u \|_{\hat{X}_\delta} \leq C \| (e^z - 1) u \|_{\hat{Y}_\delta} \leq C \| z \|_{\hat{X}_\delta} e\| z \|_{\hat{X}_\delta} \| u \|_{\hat{Y}_\delta} \leq C \| z \|_{\hat{X}_\delta} e\| z \|_{\hat{X}_\delta} \| u \|_{\hat{X}_\delta}.
\]
Consequently, there exists \( R_0 > 0 \) such that for every \( R \in (0, R_0) \), we have
\[
\| DG_\delta(z) \| < \frac{1}{2} \quad \forall z \in B_R
\]
for all \( \delta > 0 \), where
\[
B_R = \{ u \in \hat{X}_\delta : \| u \|_{\hat{X}_\delta} < R \}.\]
Now, for every $R \in (0, R_0)$,

$$\|G_\delta(z)\|_{\hat{X}_\delta} \leq \|G_\delta(z) - G_\delta(0)\|_{\hat{X}_\delta} + \|G_\delta(0)\|_{\hat{X}_\delta} \leq \frac{1}{2} \|z\|_{\hat{X}_\delta} + \|L_\delta^{-1} F_\delta(0)\|_{\hat{X}_\delta}.$$ 

By Lemmas 5.3 and 5.1, there exist $C_0, c_0 > 0$ independent of $\delta > 0$ such that

$$\|L_\delta^{-1} F_\delta(0)\|_{\hat{X}_\delta} \leq C \|F_\delta(0)\|_{\hat{Y}_\delta} \leq C_0 e^{-c_0/\delta}.$$ 

Choosing $R = R_\delta = 2C_0 e^{-c_0/\delta}$, we obtain that $G_\delta(B_{R_\delta}) \subset B_{R_\delta}$. Hence, $G_\delta$ is a strict contraction in $B_{R_\delta}$, for any $\delta \in (0, \delta_1)$, with $\delta_1 = c_0/(\ln(2C_0/R_0))$. By the Banach fixed-point theorem, for any $\delta \in (0, \delta_1)$, there exists a unique $z_\delta \in B_{R_\delta}$ such that $F_\delta(z_\delta) = 0$.

6. Proof of the main results. In this section, we finally provide the proof of Theorem 2.1 and derive Corollary 2.1. In view of Proposition 5.1, the function $\hat{u}_\delta$ defined by

$$\hat{u}_\delta = \sum_{j \in P} \hat{\varphi}_j \hat{U}_j + z_\delta$$

is a solution of (2.4). Consequently, $u_\delta$ defined by

$$(6.1) \quad u_\delta(x) = \hat{u}_\delta \left( \frac{x}{\delta} \right) = \sum_{j \in P} \varphi_j(x) U_{m_j} \left( \frac{x - p_j}{\delta} \right) + z_\delta \left( \frac{x}{\delta} \right)$$

is a solution of (1.6).

**Lemma 6.1.** The solution $u_\delta$ defined in (6.1) satisfies the approximate superposition rule

$$u_\delta(x) = \sum_{j \in J(x)} U_{m_j} \left( \frac{x - p_j}{\delta} \right) + \omega_\delta(x)$$

with $\|\omega_\delta\|_\infty \leq C e^{-c/\delta}$.

**Proof.** In view of (6.1) and of the definition of $J(x)$ in section 4, we have

$$u_\delta(x) = \sum_{j \in J(x)} U_{m_j} \left( \frac{x - p_j}{\delta} \right) + \tilde{\omega}_\delta(x),$$

where

$$\tilde{\omega}_\delta(x) = - \sum_{j \in J(x)} (1 - \varphi_j(x)) U_{m_j} \left( \frac{x - p_j}{\delta} \right) + z_\delta \left( \frac{x}{\delta} \right).$$

In view of Lemma 3.1, we estimate

$$\left\| \sum_{j \in J(x)} (1 - \varphi_j(x)) U_{m_j} \left( \frac{x - p_j}{\delta} \right) \right\|_\infty \leq \sum_{j \in J(x)} \sup_{x \in \mathbb{R}^2 \setminus C_j} \left| U_{m_j} \left( \frac{x - p_j}{\delta} \right) \right| \leq C e^{-c/\delta}.$$
On the other hand, by Proposition 5.1 and Lemma 4.1(ii), we have

$$
\left\| z_{\delta} \left( \frac{\cdot}{\delta} \right) \right\|_{\infty} = \left\| z_{\delta} \right\|_{\infty} \leq Ce^{-c/\delta}.
$$

Therefore, \( \| \tilde{\omega}_{\delta} \|_{\infty} \leq Ce^{-c/\delta} \). We have to show that

$$
\left\| \sum_{j \in P \setminus J(x)} U_{m_j} \left( \frac{x - p_j}{\delta} \right) \right\|_{\infty} \leq Ce^{-c/\delta}.
$$

To this end, we fix \( x \in \mathbb{R}^2 \) and for every \( M \in \mathbb{N} \) we define \( B_M = \{ y \in \mathbb{R}^2 : |y - x| < dM \} \). Then,

$$
\sum_{j \in P \setminus J(x)} U_{m_j} \left( \frac{x - p_j}{\delta} \right) = \sum_{M \in \mathbb{N}} \sum_{p_j \in B_{M+1} \setminus B_M} U_{m_j} \left( \frac{x - p_j}{\delta} \right).
$$

Since \( \inf_{j \neq k} |p_j - p_k| = d > 0 \), there exists \( C > 0 \) independent of \( M \in \mathbb{N} \) and of \( x \in \mathbb{R}^2 \) such that

$$
|\{ p_j \in B_{M+1} \setminus B_M \}| \leq CM.
$$

Hence, we estimate

$$
\left| \sum_{j \in P \setminus J(x)} U_{m_j} \left( \frac{x - p_j}{\delta} \right) \right| \leq C \sum_{M \in \mathbb{N}} Me^{-cM/\delta} \leq Ce^{-c/\delta}.
$$

This implies the statement of the lemma.

**Lemma 6.2.** Let \( u_{\delta} \) be given by (6.1). The following properties hold:

(i) \( e^{u_{\delta}} < 1 \) on \( \mathbb{R}^2 \) and vanishes exactly at \( p_j \) with multiplicity \( 2m_j, j \in P \);

(ii) for every compact subset \( K \) of \( \mathbb{R}^2 \setminus \bigcup_{j \in P} \{ p_j \} \), there exist \( C, c > 0 \) such that

$$
1 - e^{u_{\delta}} \leq Ce^{-c/\delta} \text{ as } \delta \to 0^+;
$$

(iii) \( \delta^2 (1 - e^{u_{\delta}}) \to 4\pi \sum_{j \in P} m_j \delta_{p_j} \) in the sense of distributions as \( \delta \to 0^+ \).

**Proof.** (i) Since \( u_{\delta} \) is a solution of (1.6), \( e^{u_{\delta}} < 1 \) follows by the maximum principle. Moreover, since

$$
U_{m_j} \left( \frac{x - p_j}{\delta} \right) = \ln |x - p_j|^{2m_j} + v_j
$$

with \( v_j \) a continuous function (see [12]), we have near \( p_j \) that \( e^{u_{\delta}} = |x - p_j|^{2m_j} f_{j,\delta}(x) \) with \( f_{j,\delta}(x) \) a continuous strictly positive function. Hence, (i) is established.

(ii) Let \( K \) be a compact subset of \( \mathbb{R}^2 \setminus \bigcup_{j \in P} \{ p_j \} \). In view of Lemma 3.1 and Proposition 5.1, we have as \( \delta \to 0^+ \)

$$
\sup_{x \in K \cap P_j} 1 - e^{\varphi_j(x)U_{m_j}(x-p_j)/\delta} \leq Ce^{-c/\delta} \left\| z_{\delta} \left( \frac{\cdot}{\delta} \right) \right\|_{\infty}
$$

$$
\leq C \left\| z_{\delta} \right\|_{\tilde{X}_{\delta}} \leq CR_{\delta} \leq Ce^{-c/\delta}.
$$
Therefore, we have that for any compact set $K \subset \mathbb{R}^2 \setminus \bigcup_{j \in P} \{ p_j \}$,

$$0 \leq \sup_{x \in K} (1 - e^{u_\delta}) \leq C \sup_{j \in P} \sup_{x \in K \cap p_j} (1 - e^{u_\delta}) \leq Ce^{-c/\delta}.$$  

(iii) Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then,

$$- \int_{\mathbb{R}^2} u_\delta \Delta \varphi = \delta^{-2} \int_{\mathbb{R}^2} (1 - e^{u_\delta}) \varphi - 4\pi \sum_{j \in P} m_j \varphi(p_j).$$

We claim that

$$\int_{\mathbb{R}^2} u_\delta \Delta \varphi \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0. \quad (6.3)$$

Indeed, let $j_k \in P$, $k = 1, \ldots, n$, be such that $\text{supp} \varphi \subset \bigcup_{k=1}^n P_{j_k} \cup K$ with $K$ a compact subset of $\mathbb{R}^2 \setminus \bigcup_{j \in P} \{p_j\}$. Since $\sup_K |u_\delta| \leq Ce^{-c/\delta}$, we have

$$\left| \int_K u_\delta \Delta \varphi \right| \leq C \| \Delta \varphi \|_\infty e^{-c/\delta} \rightarrow 0.$$  

On the other hand, in view of Lemma 6.1, in $P_{j_k}$ we have $u_\delta(x) = U_{m_{j_k}}(|x - p_{j_k}|/\delta) + O(e^{-c/\delta})$. Note that $U_{m_{j_k}} \in L^1(\mathbb{R}^2)$ in view of (6.2) and Lemma 3.1. Therefore,

$$\sup_{1 \leq k \leq n} \left| \int_{P_{j_k}} u_\delta \Delta \varphi \right| \leq \sup_{1 \leq k \leq n} \left| \int_{P_{j_k}} U_{m_{j_k}} \left( \frac{x - p_{j_k}}{\delta} \right) \Delta \varphi \right| + O(e^{-c/\delta})

\leq \delta^2 \sup_{1 \leq k \leq n} \| \Delta \varphi \|_\infty \| U_{m_{j_k}} \|_{L^1} + O(e^{-c/\delta}) \leq C\delta^2 \rightarrow 0.$$  

Hence, (6.3) follows, and (iii) is established.  

**Proof of Theorem 2.1.** For every $\delta \in (0, \delta_1)$, where $\delta_1$ is given in Proposition 5.1, we obtain a solution $u_\delta$ of (1.6). Furthermore, $u_\delta$ satisfies (2.2) in view of Lemma 6.1. Finally, $u_\delta$ satisfies the asymptotic behavior as in (i)–(iii) in view of Lemma 6.2. Hence, Theorem 2.1 is completely established.  

**Proof of Corollary 2.1.** To begin, we want to prove that if $p_j$‘s are doubly periodically arranged in $\mathbb{R}^2$, then $u_\delta$ is in fact a doubly periodic solution of (1.5). Recall that the $p_j$‘s are doubly periodically arranged in $\mathbb{R}^2$ if (2.3) holds. We define $\hat{\delta}_k = \delta_k/\delta$, $k = 1, 2$. Equivalently, we show $\hat{u}_\delta(x + \hat{\delta}_k) = \hat{u}_\delta(x)$ for any $x \in \mathbb{R}^2$ and for $k = 1, 2$. Indeed, we may assume that $\hat{\varphi}_j(x + \hat{\delta}_k) = \hat{\varphi}_j(x)$, $\hat{\psi}_j(x + \hat{\delta}_k) = \hat{\psi}_j(x)$ for any $j \in \mathbb{N}$, $x \in \mathbb{R}^2$, $k = 1, 2$. Then,

$$\hat{u}_\delta(x + \hat{\delta}_k) = \sum_{j \in \mathbb{N}} \hat{\varphi}_j(x) \hat{U}_j(x) + \hat{z}_\delta(x + \hat{\delta}_k).$$

Hence, it is sufficient to prove that $z_\delta(x + \hat{\delta}_k) = z_\delta(x)$ for every $x \in \mathbb{R}^2$ and for $k = 1, 2$. First, we claim that $z_\delta(\cdot + \hat{\delta}_k) \in B_{R_\delta}$. Indeed, for every $j \in \mathbb{N}$ there exists exactly one $j' \in \mathbb{N}$ such that

$$\hat{\varphi}_j z_\delta(\cdot + \hat{\delta}_k) \|_{H^2} = \| \hat{\varphi}_j z_\delta \|_{H^2}. \quad (6.4)$$

Hence, we obtain

$$\| z_\delta(\cdot + \hat{\delta}_k) \|_{\hat{X}_\delta} = \| z_\delta \|_{\hat{X}_\delta} \leq R_\delta \quad (6.5)$$.
Moreover, if $F_\delta(z_\delta) = 0$ we also have $F_\delta(z_\delta(\cdot + \hat{e}_k)) = 0$. Therefore, $z_\delta(\cdot + \hat{e}_k)$ is a fixed point of $G_\delta$ in $B_{R_\delta}$. By uniqueness, we conclude that $z_\delta(\cdot + \hat{e}_k) = z_\delta$, $k = 1, 2$, as asserted. At this point, the remaining statements follow recalling that in the periodic cell domain $\Omega$, $(A_\delta, \phi_\delta)$ is given, up to gauge transformations, by (1.4).

REFERENCES