A shadowing lemma for abelian Higgs vortices

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Abstract

We use a shadowing-type lemma in order to analyze the singular, semilinear elliptic equation describing static self-dual abelian Higgs vortices. Such an approach allows us to construct new solutions having an *infinite* number of arbitrarily prescribed vortex points. Furthermore, we obtain the precise asymptotic profile of the solutions in the form of an approximate superposition rule, up to an error which is exponentially small.

Key Words: Abelian Higgs model, elliptic equation, shadowing lemma

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1 Introduction

We consider the energy density for the static two-dimensional self-dual abelian Higgs model in the following form:

$$\mathcal{E}_\delta(A,\phi) = \delta^2|\mathrm{d}A|^2 + |D\phi|^2 + \frac{1}{4\delta^2}\left(|\phi|^2-1\right)^2,$$

where $A = A_1\mathrm{d}x_1 + A_2\mathrm{d}x_2$, $A_1(x), A_2(x) \in \mathbb{R}$ is a gauge potential (a connection over a principal $U(1)$ bundle), ϕ, $\phi(x) \in \mathbb{C}$ is a Higgs matter field (a section over an associated complex line bundle), $D = \mathrm{d} - iA$ is the covariant derivative and $\delta > 0$ is the coupling constant. It corresponds to the two-dimensional Ginzburg-Landau energy density in the so-called "Bogomol'nyi limit", denoting the borderline between type I and type II superconductors. In recent years, $\mathcal{E}_\delta$ has received considerable attention, in view of both its physical and geometrical interest, see, e.g., [2, 4, 7, 8, 9] and the references therein.

The smooth, finite action critical points for the action functional corresponding to $\mathcal{E}_\delta$ on $\mathbb{R}^2$ have been completely classified by Taubes [5, 8]. It is shown in [5] that such critical points are completely determined by the distributional solutions to the elliptic problem

$$-\Delta u = \delta^{-2}(1-\mathrm{e}^u) - 4\pi\sum_{j=1}^{s} m_j\delta_{p_j} \qquad \text{on } \mathbb{R}^2, \tag{1.1}$$

which decay in the sense of the Sobolev space $H^1(\mathbb{R}^2)$ at infinity. Here $s \in \mathbb{N}$, and for $j = 1, 2, \ldots, s$, $p_j \in \mathbb{R}^2$ are the vortex points, $m_j \in \mathbb{N}$ is the multiplicity of p_j, δ_{p_j} is the Dirac measure at p_j. By variational methods, Taubes proved that there exists a unique solution to (1.1) leading to a smooth, finite action critical point for the action functional of $\mathcal{E}_\delta$ on $\mathbb{R}^2$, for any $s \in \mathbb{N} \cup \{0\}$, $p_j \in \mathbb{R}^2$ and $m_j \in \mathbb{N}$, $j = 1, \ldots, s$, and for any value of $\delta > 0$. Such a solution satisfies the topological constraint $\int_{\mathbb{R}^2} F_{12} = 2\pi \sum_{j=1}^s m_j$, where $F_{12} = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field (the curvature of A).

The case of infinitely many vortex points arranged on a periodic lattice has been considered in [9] and, in the more general setting of a compact Riemannian 2-manifold, in [2, 4]. We say that the vortex points p_j, $j \in \mathbb{N}$ are doubly periodically arranged in $\mathbb{R}^2$ if there exists $s \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $k > s$ there exist $j \in \{1, 2 \ldots s\}$ and $m, n \in \mathbb{Z}$ such that $p_k = p_j + m\underline{e}_1 + n\underline{e}_2$, where $\underline{e}_1, \underline{e}_2$ are the unit vectors in $\mathbb{R}^2$. Similarly as in the previous case, denoting by $\Omega = \mathbb{R}^2/\mathbb{Z}^2$ the flat 2-torus, finite action critical points for the action of $\mathcal{E}_\delta$ on Ω correspond to distributional solutions to the problem

$$-\Delta u = \delta^{-2}(1 - \mathrm{e}^u) - 4\pi \sum_{j=1}^{s} m_j \delta_{p_j} \qquad \text{on } \Omega, \tag{1.2}$$

satisfying the topological constraint $\int_\Omega F_{12} = 2\pi \sum_{j=1}^s m_j$. It is shown in [9], that a unique solution for (1.2) exists if and only if $\delta \in (0, \pi^{-1})$. The asymptotics as $\delta \to 0^+$ has been considered in [4, 9].

Our aim in this note is to show that a shadowing lemma as introduced in the context of PDE's by Angenent [1], see also [6], may be adapted in order to construct solutions to the following more general equation containing *infinitely* many arbitrarily prescribed vortex points:

$$-\Delta u = \delta^{-2}(1 - \mathrm{e}^u) - 4\pi \sum_{j \in \mathbb{N}} m_j \delta_{p_j} \qquad \text{in } \mathbb{R}^2. \tag{1.3}$$

Suitable modifications to the method described in [1] are necessary, due to the singular sources appearing in (1.3). We assume that the vortex points p_j, $j \in \mathbb{N}$ are *arbitrarily* distributed in the plane, with the only constraint that

$$d := \inf_{k \neq j} |p_j - p_k| > 0 \qquad \text{and} \qquad m := \sup_{j \in \mathbb{N}} m_j < +\infty. \tag{1.4}$$

This situation does not seem to have been considered before. Furthermore, our gluing technique shows that solutions to (1.3) satisfy an *approximate superposition rule*, see (1.6) below. For a finite number of vortex points on $\mathbb{R}^2$, such a rule exists formally in the physics literature, and has been rigorously derived in [7]. In view of the representation (1.6), we can easily analyze the asymptotic behavior of solutions to (1.3) as $\delta \to 0^+$, thus obtaining more direct proofs for the asymptotics derived in [4, 9], in the special case (1.2).

In order to state our results, we denote by U_N the unique radial solution for the problem:

$$\begin{cases} -\Delta U_N = 1 - \mathrm{e}^{U_N} - 4\pi N \delta_0 & \text{in } \mathbb{R}^2 \\ U_N(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{1.5}$$

Existence, uniqueness and exponential decay as $|x| \to +\infty$ for U_N are established in [5], see Section 2 below.

Our main result is the following

Theorem 1.1. *Let $p_j \in \mathbb{R}^2$, $m_j \in \mathbb{N}$, $j \in \mathbb{N}$ satisfy* (1.4). *There exists a constant $\delta_1 > 0$ (dependending on d and m only) such that for every $\delta \in (0, \delta_1)$ there exists a solution u_δ for* (1.3). *If the p_j's are doubly periodically arranged in $\mathbb{R}^2$, then u_δ is doubly periodic. Furthermore, u_δ satisfies the approximate superposition rule:*

$$u_\delta(x) = \sum_{j\in\mathbb{N}} U_{m_j}\left(\frac{|x - p_j|}{\delta}\right) + \omega_\delta, \tag{1.6}$$

where the error term ω_δ satisfies $\|\omega_\delta\|_\infty \le C\mathrm{e}^{-c/\delta}$, for some $c > 0$ independent of δ. In particular, u satisfies the following properties:

(i) $0 \le \mathrm{e}^{u_\delta} < 1$, *$\mathrm{e}^{u_\delta}$ vanishes exactly at p_j, $j \in \mathbb{N}$;*

(ii) For every compact subset K of $\mathbb{R}^2 \setminus \cup_{j\in\mathbb{N}}\{p_j\}$ there exist $C, c > 0$ such that $\sup_K(1 - \mathrm{e}^{u_\delta}) \le C\mathrm{e}^{-c/\delta}$ *as $\delta \to 0^+$;*

(iii) $\delta^{-2}(1 - \mathrm{e}^{u_\delta}) \to 4\pi \sum_{j\in\mathbb{N}} m_j\delta_{p_j}$ *in the sense of distributions, as $\delta \to 0^+$.*

We note that $\delta^{-2}(1 - \mathrm{e}^{u_\delta}) = 2|F_{12}|$.

An outline of this note is as follows. Our starting point in proving Theorem 1.1 is to consider δ as a scaling parameter. Setting $\hat{u}(x) = u(\delta x)$, we have that $\hat{u}$ satisfies:

$$-\Delta\hat{u} = 1 - \mathrm{e}^{\hat{u}} - 4\pi \sum_{j\in\mathbb{N}} m_j\delta_{\hat{p}_j} \qquad \text{in } \mathbb{R}^2, \tag{1.7}$$

where $\hat{p}_j = p_j/\delta$. Note that the vortex points $\hat{p}_j$ "separate" as $\delta \to 0^+$. Section 2 contains the necessary properties of the radial solutions U_N to (1.5). We rely on the results of Taubes [8] for the existence and uniqueness of U_N, as well as for the exponential decay properties at infinity. We also prove a necessary non-degeneracy property of U_N. The exponential decay of solutions justifies the following approximate superposition picture for small values of δ, i.e., for vortex points $\hat{p}_j$ which are "far apart":

$$\hat{u}(x) \approx \sum_{j\in\mathbb{N}} U_{m_j}\left(|x - \hat{p}_j|\right). \tag{1.8}$$

In fact, we take the following preliminary form of the superposition rule:

$$\hat{u} = \sum_{j\in\mathbb{N}} \hat{\varphi}_j U_{m_j}(x - \hat{p}_j) + z, \tag{1.9}$$

as an *ansatz* for $\hat{u}_\delta$. Here, radial solutions centered at $\hat{p}_j$ are "glued" together by the functions $\hat{\varphi}_j$, which belong to a suitable locally finite partition of unity. Section 3 contains the definition and the main properties of the partition, as well as of the appropriate functional spaces $\hat{X}_\delta, \hat{Y}_\delta$, which are also obtained by "gluing" $H^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$, respectively. Hence, we are reduced to show that for small values of δ there exists an exponentially small "error" z such that $\hat{u}$

defined by (1.9) is a solution for (1.7). The existence of such a $z \in \hat{X}_\delta$ is the aim of Section 4 (see Proposition 4.1). To this end we use the shadowing lemma. We characterize z by the property $F_\delta(z) = 0$, where $F_\delta : \hat{X}_\delta \to \hat{Y}_\delta$ is suitably defined. The non-degeneracy property of U_N is essential in order to prove that the operator $DF_\delta(0)$ is invertible, and that its inverse is bounded independently of $\delta > 0$ (Lemma 4.4). At this point, the Banach fixed point argument applied to $\mathbb{I} - (DF_\delta(0))^{-1} F_\delta$ yields the existence of the desired error term z. In Section 5 we show that periodically arranged vortex points lead to periodic solutions, that (1.9) implies (1.6) and we derive the asymptotic behavior of solutions, thus concluding the proof of Theorem 1.1. For the reader's convenience, following the monograph of Jaffe and Taubes [5], we outline in an appendix the derivation of equation (1.1) for smooth, finite action critical points to the action of $\mathcal{E}_\delta$ on $\mathbb{R}^2$, as well as some properties of solutions to (1.1), which imply the necessary properties of U_N.

Although we have chosen to consider the abelian Higgs model for the sake of simplicity, it will be clear from the proof that our method may be adapted to many other self-dual gauge theories as considered, e.g., in the monograph [10].

Henceforth, unless otherwise stated, we denote by $C, c > 0$ general constants independent of $\delta > 0$ and of $j \in \mathbb{N}$.

2 Single vortex point solutions

In this section we consider the solution U_N to the radially symmetric equation (1.5). We refer to [5, 8] for the proof of the existence and uniqueness of U_N (see also the Appendix). We collect in the following lemma some properties of U_N that will be needed in the sequel. For every $r > 0$, we denote $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$.

Lemma 2.1. *The following properties hold:*

(i) $\mathrm{e}^{U_N(x)} < 1$ *for any* $x \in \mathbb{R}^2$.

(ii) For every $r > 0$ *there exist constants* $C_N > 0$ *and* $\alpha_N > 0$ *depending on* r *and* N *such that*

$$|1 - \mathrm{e}^{U_N(x)}| + |\nabla U_N(x)| + |U_N(x)| \le C_N \mathrm{e}^{-\alpha_N |x|},$$

for all $x \in \mathbb{R}^2 \setminus B_r$.

Proof. Property (i) follows by the maximum principle. In order to establish (ii), we note that the estimate $|1 - \mathrm{e}^{U_N(x)}| \le C_N \mathrm{e}^{-\beta_N |x|}$ for some $\beta_N > 0$ depending on N was established by Taubes ([8], Theorem III.1.1), see the Appendix. In view of (i), it follows that for all $|x| \ge r$ we have

$$|U_N(x)| = \frac{|U_N(x)|}{1 - \mathrm{e}^{U_N(x)}} \left(1 - \mathrm{e}^{U_N(x)}\right) \le C \mathrm{e}^{-\beta |x|}.$$

In order to estimate the decay of $|\nabla U_N|$, we set $A = B_{4r} \setminus \overline{B}_r$, and for all $R \ge r$ we define $A_R = B_{4Rr} \setminus \overline{B}_{Rr}$, $A'_R = B_{3Rr} \setminus \overline{B}_{2Rr}$. For $y \in A$, we consider $u_R(y) = U_N(Ry)$. Then u_R satisfies $-\Delta u_R = f_R$ in A with f_R given by $f_R(y) = R^2(1 - \exp\{U_N(Ry)\})$. We recall the standard elliptic estimate for u_R (see, e.g., [3] Theorem 3.9):

$$\sup_A d_y |\nabla u_R(y)| \le C \left(\sup_A |u_R| + \sup_A d_y^2 |f_R(y)| \right),$$

where $d_y = \mathrm{dist}(y, \partial A)$ and $C > 0$ is independent of R. In terms of U_N, the above estimate yields

$$\sup_{A_R} d_x |\nabla U_N(x)| \le C \left(\sup_{A_R} |U_N| + \sup_{A_R} d_x^2 (1 - \exp\{U_N(x)\}) \right). \tag{2.1}$$

where $d_x = \mathrm{dist}(x, \partial A_R) = R d_y$. Hence, we have for any $x \in A'_R$

$$|\nabla U_N(x)| \le \sup_{A'_R} \frac{d_x}{R} |\nabla U_N(x)| \le C R \mathrm{e}^{-\beta R} \le C|x| \mathrm{e}^{-\frac{\beta}{3}|x|} \tag{2.2}$$

and we conclude that

$$|\nabla U_N(x)| \le C \mathrm{e}^{-\alpha |x|} \qquad \forall |x| \ge r \tag{2.3}$$

for some constant $\alpha > 0$. □

We consider the bounded linear operator

$$L_N = -\Delta + \mathrm{e}^{U_N} : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2).$$

It is known [5, 8] that U_N corresponds to the unique minimum of a strictly convex functional, and therefore it is the unique solution to (1.5), see the Appendix. In order to apply the shadowing lemma, we further have to show that U_N is non-degenerate, in the sense of the following

Lemma 2.2. *The operator L_N is invertible and for every $N > 0$ there exists $C_N > 0$ such that $\|L_N^{-1}\| \le C_N$.*

Proof. It is readily seen that L_N is injective. Indeed, suppose $L_N u = 0$ for some $u \in H^2(\mathbb{R}^2)$. Multiplying by u and integrating on $\mathbb{R}^2$ we have:

$$\int |\nabla u|^2 + \int \mathrm{e}^{U_N} u^2 = 0.$$

Therefore, $u = 0$. Now we claim that L_N is a Fredholm operator. Indeed, we write

$$L_N = (-\Delta + 1)(\mathbb{I} - T),$$

with $T = (-\Delta + 1)^{-1}(1 - \mathrm{e}^{U_N}) : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$. Clearly, T is continuous. Let us check that T is compact. To this end, let $u_n \in H^2(\mathbb{R}^2)$, $\|u_n\|_{H^2} = 1$. We have to show that $T u_n$ has a convergent subsequence. Note that by the Sobolev embedding

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le C_S \|u\|_{H^2(\mathbb{R}^2)}, \tag{2.4}$$

for all $u \in H^2(\mathbb{R}^2)$, we have $\|u_n\|_\infty \le C'$, for some $C' > 0$ independent of n, and there exists u_∞, $\|u_\infty\|_{H^2} \le 1$, such that $u_{n_k} \to u_\infty$ strongly in L^2_{loc} for a subsequence u_{n_k}. Now, by Lemma 2.1, for any fixed $\varepsilon > 0$, there exists

$R > 0$ such that $\|1 - e^{U_N}\|_{L^2(\mathbb{R}^2 \setminus B_R)} \le \varepsilon$. Consequently, $\|(1 - e^{U_N})(u_{n_k} - u_\infty)\|_{L^2(\mathbb{R}^2 \setminus B_R)} \le 2C'\varepsilon$. On the other hand, $\|(1 - e^{U_N})(u_{n_k} - u_\infty)\|_{L^2(B_R)} \to 0$. We conclude that $(1-e^{U_N})(u_{n_k}-u_\infty) \to 0$ in L^2. In turn, we have $T(u_{n_k}-u_\infty) = (-\Delta + 1)^{-1}(1 - e^{U_N})(u_{n_k} - u_\infty) \to 0$ in H^2, which implies that T is compact. It follows that L_N is a Fredholm operator. Consequently, L_N is also surjective. At this point, the Open Mapping Theorem concludes the proof. □

3 A partition of unity

In this section we introduce a partition of unity and we prove some technical results which will be needed in the sequel. Let $p_j \in \mathbb{R}^2$, $j \in \mathbb{N}$ be the vortex points. By assumption (1.4), $r_0 = d/8 = \inf_{j \neq k} |p_j - p_k|/8 > 0$. We consider the set $K = (-\frac{3}{4}r_0, \frac{3}{4}r_0) \times (-\frac{3}{4}r_0, \frac{3}{4}r_0)$. Then for any $\underline{n} \in \mathbb{Z}^2$, we introduce $K_{\underline{n}} = K + \underline{n}r_0$. The collection of sets $\{K_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^2}$ is a locally finite covering of $\mathbb{R}^2$. We consider an associated partition of unity defined as follows: let $0 \le \phi \in C_c^\infty(K)$ be such that $\sum_{\underline{n}\in\mathbb{Z}^2} \phi_{\underline{n}}(x) = 1$ pointwise, where $\phi_{\underline{n}}(x) = \phi(x - \underline{n}r_0)$. Then, for any $j \in \mathbb{N}$, we introduce the set

$$N_j = \{\underline{n} \in \mathbb{Z}^2 \,:\, d(p_j, K_{\underline{n}}) < \frac{1}{4}r_0\},$$

note that the cardinality of N_j is uniformly bounded, namely $|N_j| \le 4$ for any $j \in \mathbb{N}$. For any $j \in \mathbb{N}$, we set

$$B_j = \bigcup_{\underline{n}\in N_j} K_{\underline{n}}, \qquad \varphi_j(x) = \sum_{\underline{n}\in N_j} \phi_{\underline{n}}(x).$$

Let $\mathcal{I} : \mathbb{N} \to \mathbb{Z}^2 \setminus \bigcup_{j\in\mathbb{N}} N_j$ be a bijection. We set

$$Q_j = K_{\mathcal{I}(j)}, \qquad \psi_j(x) = \phi_{\mathcal{I}(j)}(x).$$

Then $\{B_j, Q_j\}_{j\in\mathbb{N}}$ is a locally finite open covering of $\mathbb{R}^2$ with the property that $B_j \cap B_k = \emptyset$ for every $k \neq j$. Moreover $\{\varphi_j, \psi_j\}$ is a partition of unity associated to $\{B_j, Q_j\}_{j\in\mathbb{N}}$, such that

$$\operatorname{supp} \varphi_j \subset B_j, \qquad \operatorname{supp} \psi_j \subset Q_j,$$

and such that

$$\sup_{j\in\mathbb{N}}\{\|\nabla\varphi_j\|_\infty,\ \|\nabla\psi_j\|_\infty\} < +\infty, \qquad \sup_{j\in\mathbb{N}}\{\|D^2\varphi_j\|_\infty,\ \|D^2\psi_j\|_\infty\} < +\infty.$$

In particular,

$$0 \le \varphi_j, \psi_j \le 1 \qquad \text{and} \qquad \sum_{j\in\mathbb{N}}(\varphi_j(x) + \psi_j(x)) = \sum_{\underline{n}\in\mathbb{Z}^2} \phi_{\underline{n}}(x) = 1.$$

For every $j \in \mathbb{N}$, we define a rescaled covering:

$$\hat{B}_j = B_j/\delta, \qquad \hat{Q}_j = Q_j/\delta.$$

Then $\{\hat{\varphi}_j, \hat{\psi}_j\}_{j\in\mathbb{N}}$ defined by

$$\hat{\varphi}_j(x) = \varphi_j(\delta x), \qquad \hat{\psi}_j(x) = \psi_j(\delta x)$$

is a partition of unity associated to $\{\hat{B}_j, \hat{Q}_j\}$. It will also be convenient to define the sets

$$\hat{C}_j = \{x \in \hat{B}_j \,:\, \hat{\varphi}_j(x) = 1\} \qquad j \in \mathbb{N}.$$

Note that

$$\operatorname{supp}\{\nabla\hat{\varphi}_j, D^2\hat{\varphi}_j\} \subset \hat{B}_j \setminus \hat{C}_j$$

and

$$\sup_{\mathbb{R}^2}\{|\nabla\hat{\varphi}_j| + |\nabla\hat{\psi}_j|\} \le C\delta, \qquad \sup_{\mathbb{R}^2}\{|D^2\hat{\varphi}_j| + |D^2\hat{\psi}_j|\} \le C\delta^2. \tag{3.1}$$

For every fixed $x \in \mathbb{R}^2$ we define the following subsets of $\mathbb{N}$:

$$J(x) = \{j \in \mathbb{N} \,:\, \hat{\varphi}_j(x) \ne 0\}, \qquad K(x) = \{k \in \mathbb{N} \,:\, \hat{\psi}_k(x) \ne 0\}. \tag{3.2}$$

Note that

$$\sup_{x\in\mathbb{R}^2}\{|J(x)| + |K(x)|\} < +\infty, \tag{3.3}$$

where $|J(x)|$, $|K(x)|$ denote the cardinality of $J(x)$, $K(x)$, respectively. We shall use the following Banach spaces:

$$\begin{aligned}\hat{X}_\delta =& \{u \in H^2_{\rm loc}(\mathbb{R}^2) \,:\, \sup_{j\in\mathbb{N}}\{\|\hat{\varphi}_j u\|_{H^2(\mathbb{R}^2)}, \|\hat{\psi}_j u\|_{H^2(\mathbb{R}^2)}\} < +\infty\}\\ \hat{Y}_\delta =& \{f \in L^2_{\rm loc}(\mathbb{R}^2) \,:\, \sup_{j\in\mathbb{N}}\{\|\hat{\varphi}_j f\|_{L^2(\mathbb{R}^2)}, \|\hat{\psi}_j f\|_{L^2(\mathbb{R}^2)}\} < +\infty\}.\end{aligned}$$

We collect in the following lemma some estimates that will be used in the sequel.

Lemma 3.1. *There exists a constant $C > 0$ such that for any $u \in \hat{X}_\delta$ and $j \in \mathbb{N}$ we have*

(i) $\|u\|_{H^2(\hat{B}_j)} \le C\|u\|_{\hat{X}_\delta}$

(ii) $\|u\|_{L^\infty(\mathbb{R}^2)} \le C\|u\|_{\hat{X}_\delta}$.

Proof. (i) For every fixed $k \in \mathbb{N}$, let $\mathcal{J}(k) = \{j \in \mathbb{N} \,:\, \operatorname{supp}\hat{\psi}_j \cap \operatorname{supp}\hat{\varphi}_k \ne \emptyset\}$. Then $\sup_{k\in\mathbb{N}}|\mathcal{J}(k)| < +\infty$, and we estimate:

$$\begin{aligned}\|u\|_{H^2(\hat{B}_j)} =& \|\hat{\varphi}_j u + \sum_{k\in\mathcal{J}(j)} \hat{\psi}_k u\|_{H^2(\hat{B}_j)} \le \|\hat{\varphi}_j u\|_{H^2(\hat{B}_j)} + \sum_{k\in\mathcal{J}(j)} \|\hat{\psi}_k u\|_{H^2(\hat{B}_j)}\\ \le& (1 + |\mathcal{J}(j)|)\,\|u\|_{\hat{X}_\delta} \le C\|u\|_{\hat{X}_\delta}.\end{aligned}$$

(ii) For any fixed $x \in \mathbb{R}^2$ we have, in view of (2.4) and (3.3):

$$\begin{aligned}|u(x)| =& \sum_{j\in\mathbb{N}} \hat{\varphi}_j(x)|u(x)| + \sum_{j\in\mathbb{N}} \hat{\psi}_j(x)|u(x)|\\ =& \sum_{j\in J(x)} \hat{\varphi}_j(x)|u(x)| + \sum_{j\in K(x)} \hat{\psi}_j(x)|u(x)|\\ \le& \sum_{j\in J(x)} C_S\|\hat{\varphi}_j u\|_{H^2(\mathbb{R}^2)} + \sum_{j\in K(x)} C_S\|\hat{\psi}_j u\|_{H^2(\mathbb{R}^2)}\\ \le& \sup_{x\in\mathbb{R}^2}(|J(x)| + |K(x)|)C_S\|u\|_{\hat{X}_\delta} = C\|u\|_{\hat{X}_\delta}.\end{aligned}$$

Hence, (ii) is established. ∎

We shall also need the following family of functions:

$$\hat{g}_j = \frac{\hat{\varphi}_j}{\left(\sum_{k\in\mathbb{N}}(\hat{\varphi}_k^2 + \hat{\psi}_k^2)\right)^{1/2}}, \qquad \hat{h}_j = \frac{\hat{\psi}_j}{\left(\sum_{k\in\mathbb{N}}(\hat{\varphi}_k^2 + \hat{\psi}_k^2)\right)^{1/2}}.$$

In view of (3.1), it is readily checked that

Lemma 3.2. *The family* $\{\hat{g}_j, \hat{h}_j\}_{j\in\mathbb{N}}$ *satisfies* $\mathrm{supp}\hat{g}_j \subset \hat{B}_j$, $\mathrm{supp}\hat{h}_j \subset \hat{Q}_j$ *and furthermore:*

$$\sum_{j\in\mathbb{N}}(\hat{g}_j^2 + \hat{h}_j^2) \equiv 1 \tag{3.4}$$

$$C^{-1}\hat{\varphi}_j \le \hat{g}_j \le C\hat{\varphi}_j, \qquad C^{-1}\hat{\psi}_j \le \hat{h}_j \le C\hat{\psi}_j \tag{3.5}$$

$$\sup_{\mathbb{R}^2}\{|\nabla\hat{g}_j| + |\nabla\hat{h}_j|\} \le C\delta, \qquad \sup_{\mathbb{R}^2}\{|D^2\hat{g}_j| + |D^2\hat{h}_j|\} \le C\delta^2. \tag{3.6}$$

4 The shadowing lemma

For every $j \in \mathbb{N}$ we define

$$\hat{U}_j(x) = U_{m_j}(x - \hat{p}_j).$$

We make the following ansatz for solutions $\hat{u}$ to equation (1.7):

$$\hat{u} = \sum_{j\in\mathbb{N}} \hat{\varphi}_j \hat{U}_j + z. \tag{4.1}$$

Our aim in this section is to prove:

Proposition 4.1. *There exists* $\delta_1 > 0$ *such that for all* $\delta \in (0, \delta_1)$ *there exists* $z_\delta \in \hat{X}_\delta$*, such that* $\hat{u}_\delta$ *defined by* $\hat{u}_\delta = \sum_j \hat{\varphi}_j \hat{U}_j + z_\delta$ *is a solution to* (1.7). *Moreover,* $\|z_\delta\|_{\hat{X}_\delta} \le C\mathrm{e}^{-c/\delta}$.

We note that the functional $F_\delta : \hat{X}_\delta \to \hat{Y}_\delta$ given by

$$F_\delta(z) = -\Delta z + \sum_{j\in\mathbb{N}} \hat{\varphi}_j(1 - \mathrm{e}^{\hat{U}_j}) - (1 - \mathrm{e}^{\sum_{j\in\mathbb{N}} \hat{\varphi}_j \hat{U}_j + z}) - \sum_{j\in\mathbb{N}} [\hat{\varphi}_j, \Delta]\hat{U}_j$$

is well-defined and C^1. Here $[\Delta, \hat{\varphi}_j] = \Delta\hat{\varphi}_j + 2\nabla\hat{\varphi}_j\nabla$. Moreover, if $z \in \hat{X}_\delta$ satisfies $F_\delta(z) = 0$, then $\hat{u}$ defined by (4.1) is a solution to (1.7).

Lemma 4.2. *For* $\delta > 0$ *sufficiently small, we have*

$$\|F_\delta(0)\|_{\hat{Y}_\delta} \le C\mathrm{e}^{-c/\delta} \qquad \text{as } \delta \to 0^+ \tag{4.2}$$

for some constants $C, c > 0$ *independent of* δ.

Proof. Let

$$\mathcal{R} = \sum_{j\in\mathbb{N}} \hat{\varphi}_j(1 - \mathrm{e}^{\hat{U}_j}) - (1 - \mathrm{e}^{\sum_{j\in\mathbb{N}} \hat{\varphi}_j \hat{U}_j})$$

$$\mathcal{C} = \sum_{j\in\mathbb{N}} [\hat{\varphi}_j, \Delta]\hat{U}_j$$

Note that $\{\operatorname{supp}\mathcal{R}, \operatorname{supp}\mathcal{C}\} \subset \cup_{j\in\mathbb{N}}\hat{B}_j \setminus \hat{C}_j$. We fix $x \in \cup_j \hat{B}_j$. We estimate:

$$\begin{aligned}|\mathcal{R}(x)| \leq &\sup_{j\in\mathbb{N}} \|\hat{\varphi}_j(1-\mathrm{e}^{\hat{U}_j})\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)} + \sup_{j\in\mathbb{N}}\|1-\mathrm{e}^{\hat{\varphi}_j\hat{U}_j}\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)}\\ \leq &C\sup_{j\in\mathbb{N}}\|\hat{U}_j\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)} \leq C_1\mathrm{e}^{-c_1/\delta}.\end{aligned}$$

On the other hand, in view of (3.1) and Lemma 2.1, for $x \in \cup_j\hat{B}_j$, we have

(4.3)

$$\begin{aligned}|\mathcal{C}(x)| \leq &\sup_{j\in\mathbb{N}}\|\,[\Delta,\hat{\varphi}_j]\hat{U}_j\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)}\\ \leq &C\,(\sup_{j\in\mathbb{N}}\|\hat{U}_j\,\Delta\hat{\varphi}_j\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)} + \sup_{j\in\mathbb{N}}\|\,|\nabla\hat{U}_j|\,|\nabla\hat{\varphi}_j|\,\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)}) \leq C_2\mathrm{e}^{-c_2/\delta}\end{aligned}$$

for some positive constants $c_2, C_2 > 0$ independent of δ. Hence, we conclude that, as $\delta \to 0^+$,

$$\|F_\delta(0)\|_{\hat{Y}_\delta} \leq C\sup_{j\in\mathbb{N}}(\|\mathcal{R}\|_{L^2(\hat{B}_j)} + \|\mathcal{C}\|_{L^2(\hat{B}_j)}) \leq C\mathrm{e}^{-c/\delta} \tag{4.4}$$

for some constants $C, c > 0$ independent of $\delta > 0$. □

Now, we consider the operator $L_\delta \equiv DF_\delta(0) : \hat{X}_\delta \to \hat{Y}_\delta$ given by

$$L_\delta = -\Delta + \mathrm{e}^{\sum_{j\in\mathbb{N}}\hat{\varphi}_j\hat{U}_j}.$$

For every $j \in \mathbb{N}$, we define the operators:

$$\hat{L}_j = -\Delta + \mathrm{e}^{\hat{U}_j}.$$

It will also be convenient to define:

$$\hat{L}_0 = -\Delta + 1.$$

We readily check the that the following holds:

Lemma 4.3. *There exists a constant $C > 0$ such that for any $u \in \hat{X}_\delta$ and $j \in \mathbb{N}$ we have*

$$\begin{aligned}\|(L_\delta - \hat{L}_j)\hat{\varphi}_j u\|_{L^2} &\leq C\mathrm{e}^{-c/\delta}\|\hat{\varphi}_j h\|_{L^2},\\ \|(L_\delta - \hat{L}_0)\hat{\psi}_j u\|_{L^2} &\leq C\mathrm{e}^{-c/\delta}\|\hat{\psi}_j h\|_{L^2}.\end{aligned}$$

Proof. For any $j \in \mathbb{N}$, by Lemma 2.1, we have as $\delta \to 0^+$,

(4.5)

$$\begin{aligned}\|(L_\delta - \hat{L}_j)\hat{\varphi}_j u\|_{L^2} \leq &(\|1-\mathrm{e}^{\hat{U}_j}\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)} + \|1-\mathrm{e}^{\hat{\varphi}_j\hat{U}_j}\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)})\|\hat{\varphi}_j u\|_{L^2}\\ \leq &C\|1-\mathrm{e}^{\hat{U}_j}\|_{L^\infty(\hat{B}_j\setminus\hat{C}_j)}\|\hat{\varphi}_j u\|_{L^2} \leq C\mathrm{e}^{-c/\delta}\|\hat{\varphi}_j u\|_{L^2}.\end{aligned}$$

Let $\mathcal{K}(j) = \{k \in \mathbb{N} : \operatorname{supp}\hat{\varphi}_k \cap \operatorname{supp}\hat{\psi}_j \neq \emptyset\}$. Then $\sup_{j\in\mathbb{N}}|\mathcal{K}(j)| < +\infty$ and we estimate, as $\delta \to 0^+$,

(4.6)

$$\begin{aligned}\|(L_\delta - \hat{L}_0)\hat{\psi}_j u\|_{L^2} \leq &\|(1-\mathrm{e}^{\sum_{k\in\mathcal{K}(j)}\hat{\varphi}_k\hat{U}_k})\hat{\psi}_j u\|_{L^2}\\ \leq &\sup_{k\in\mathcal{K}(j)}\|1-\mathrm{e}^{\hat{\varphi}_k\hat{U}_k}\|_{L^\infty(\hat{B}_k\setminus\hat{C}_k)}\|\hat{\psi}_j u\|_{L^2} \leq C\mathrm{e}^{-c/\delta}\|\hat{\psi}_j u\|_{L^2}.\end{aligned}$$

□

Now we prove an essential non-degeneracy property of L_δ:

Lemma 4.4. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the operator L_δ is invertible. Moreover, $L_\delta^{-1} : \hat{Y}_\delta \to \hat{X}_\delta$ is uniformly bounded with respect to $\delta \in (0, \delta_0)$.*

Proof. Following a gluing technique introduced in [1], we construct an "approximate inverse" $S_\delta : \hat{Y}_\delta \to \hat{X}_\delta$ for L_δ^{-1} as follows:

$$S_\delta = \sum_{j\in\mathbb{N}} \left(\hat{g}_j \hat{L}_j^{-1} \hat{g}_j + \hat{h}_j \hat{L}_0^{-1} \hat{h}_j \right), \tag{4.7}$$

where $\hat{g}_j, \hat{h}_j$ are the functions introduced in Section 3. We claim that the operator S_δ is well-defined and uniformly bounded with respect to δ. That is, we claim that

$$\|S_\delta f\|_{\hat{X}_\delta} \le C\|f\|_{\hat{Y}_\delta} \tag{4.8}$$

for some $C > 0$ independent of $f \in \hat{X}_\delta$ and of $\delta > 0$.

Indeed, for any $f \in \hat{Y}_\delta$ we have

$$\begin{aligned}\|S_\delta f\|_{\hat{X}_\delta} = \sup_{k\in\mathbb{N}}\{ \|\hat{\varphi}_k \sum_{j\in\mathbb{N}} (\hat{g}_j \hat{L}_j^{-1} \hat{g}_j + \hat{h}_j \hat{L}_0^{-1} \hat{h}_j) f\|_{H^2},\\ \|\hat{\psi}_k \sum_{j\in\mathbb{N}} (\hat{g}_j \hat{L}_j^{-1} \hat{g}_j + \hat{h}_j \hat{L}_0^{-1} \hat{h}_j) f\|_{H^2}\}.\end{aligned}$$

We estimate, recalling the properties of $\hat{\varphi}_j$ and $\hat{g}_j$:

$$\begin{aligned}\|\hat{\varphi}_k \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1} \hat{g}_j f\|_{H^2} &= \|\hat{\varphi}_k \hat{g}_k \hat{L}_k^{-1} \hat{g}_k f\|_{H^2}\\ &\le C\|\hat{L}_k^{-1} \hat{g}_k f\|_{H^2} \le C\|\hat{g}_k f\|_{L^2} \le C\|\hat{\varphi}_k f\|_{L^2} \le C\|f\|_{\hat{Y}_\delta}.\end{aligned}$$

We have:

$$\|\hat{\varphi}_k \sum_{j\in\mathbb{N}} \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f\|_{H^2} \le \|\hat{\varphi}_k \sum_{j\in\mathcal{J}(k)} \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f\|_{H^2} \le \sum_{j\in\mathcal{J}(k)} \|\hat{\varphi}_k \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f\|_{H^2},$$

where $\mathcal{J}(k) = \{j \in \mathbb{N} : \operatorname{supp} \hat{\psi}_j \cap \operatorname{supp} \hat{\varphi}_k \neq \emptyset\}$ satisfies $\sup_{k\in\mathbb{N}} |\mathcal{J}(k)| < +\infty$. In view of Lemma 3.2 and Lemma 2.2, we estimate:

$$\begin{aligned}\sum_{j\in\mathcal{J}(k)} \|\hat{\varphi}_k \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f\|_{H^2} &\le C \sum_{j\in\mathcal{J}(k)} \|\hat{L}_0^{-1} \hat{h}_j f\|_{H^2} \le C \sum_{j\in\mathcal{J}(k)} \|\hat{h}_j f\|_{L^2}\\ &\le \sum_{j\in\mathcal{J}(k)} \|\hat{\psi}_j f\|_{L^2} \le |\mathcal{J}(k)| \sup_{j\in\mathbb{N}} \|\hat{\psi}_j f\|_{L^2} \le C\|f\|_{\hat{Y}_\delta}.\end{aligned}$$

Therefore,

$$\sup_{k\in\mathbb{N}} \|\hat{\varphi}_k \sum_{j\in\mathbb{N}} \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f\|_{H^2} \le C\|f\|_{\hat{Y}_\delta}.$$

Similarly, we obtain that

$$\sup_{k\in\mathbb{N}} \|\hat{\psi}_k \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1} \hat{g}_j f\|_{H^2} \le C\|f\|_{\hat{Y}_\delta}, \quad \sup_{k\in\mathbb{N}} \|\hat{\psi}_k \sum_{j\in\mathbb{N}} \hat{h}_j \hat{L}_0^{-1} \hat{h}_j f\|_{H^2} \le C\|f\|_{\hat{Y}_\delta}.$$

and (4.8) follows.

Now, we claim that there exists δ_0 such that for any $\delta \in (0,\delta_0)$, the operator $S_\delta L_\delta : \hat{X}_\delta \to \hat{X}_\delta$ is invertible, and furthermore $\|S_\delta L_\delta\| \le C$ for some $C > 0$ independent of $\delta > 0$. We note that $(L_\delta - \hat{L}_j)\hat{g}_j : \hat{X}_\delta \to \hat{Y}_\delta$ and $(L_\delta - \hat{L}_0)\hat{h}_j : \hat{X}_\delta \to \hat{Y}_\delta$ are well-defined bounded linear operators. Thus, recalling (3.4) we decompose:

(4.9)
$$\begin{aligned} S_\delta L_\delta =& \mathbb{I}_{\hat{X}_\delta} + \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}(\hat{g}_j L_\delta - \hat{L}_j \hat{g}_j) + \sum_{j\in\mathbb{N}} \hat{h}_j \hat{L}_0^{-1}(\hat{h}_j L_\delta - \hat{L}_0 \hat{h}_j) \\ =& \mathbb{I}_{\hat{X}_\delta} + \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}(L_\delta - \hat{L}_j)\hat{g}_j + \sum_{j\in\mathbb{N}} \hat{h}_j \hat{L}_0^{-1}(L_\delta - \hat{L}_0)\hat{h}_j + \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}[\Delta, \hat{g}_j] \\ & + \sum_{j\in\mathbb{N}} \hat{h}_j \hat{L}_0^{-1}[\Delta, \hat{h}_j]. \end{aligned}$$

Hence, it suffices to prove that the last four terms in (4.9) are sufficiently small, in the operator norm, provided $\delta > 0$ is sufficiently small. By Lemma 4.3 and Lemma 3.2 we have, for any $u \in \hat{X}_\delta$,

$$\begin{aligned} &\|\sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}(L_\delta - \hat{L}_j)\hat{g}_j u\|_{\hat{X}_\delta} \\ &\quad = \sup_{k\in\mathbb{N}} \{\|\hat{\varphi}_k \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}(L_\delta - \hat{L}_j)\hat{g}_j u\|_{H^2}, \|\hat{\psi}_k \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}(L_\delta - \hat{L}_j)\hat{g}_j u\|_{H^2}\} \\ &\quad \le C \sup_{k\in\mathbb{N}} \|\hat{L}_j^{-1}(L_\delta - \hat{L}_k)\hat{g}_k u\|_{H^2} \le C \sup_{k\in\mathbb{N}} \|(L_\delta - \hat{L}_k)\hat{g}_k u\|_{L^2} \\ &\quad \le C\mathrm{e}^{-c/\delta} \sup_{k\in\mathbb{N}} \|\hat{\varphi}_k u\|_{L^2} \le C\mathrm{e}^{-c/\delta}\|u\|_{\hat{X}_\delta}. \end{aligned}$$

Similarly, for $u \in \hat{X}_\delta$, we have:

$$\begin{aligned} &\|\sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}[\Delta, \hat{g}_j]u\|_{\hat{X}_\delta} \\ &\quad = \sup_{k\in\mathbb{N}} \{\|\hat{\varphi}_k \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}[\Delta, \hat{g}_j]u\|_{H^2}, \|\hat{\psi}_k \sum_{j\in\mathbb{N}} \hat{g}_j \hat{L}_j^{-1}[\Delta, \hat{g}_j]u\|_{H^2}\} \\ &\quad \le C \sup_{k\in\mathbb{N}} \|\hat{L}_k^{-1}[\Delta, \hat{g}_k]u\|_{H^2} \le C \sup_{k\in\mathbb{N}} \|[\Delta, \hat{g}_k]u\|_{L^2}. \end{aligned}$$

Recalling that $[\Delta, \hat{g}_k]u = 2\nabla u \nabla \hat{g}_k + u\Delta \hat{g}_k$, by (3.6) and Lemma 3.1–(i) we derive that

$$\|[\Delta, \hat{g}_k]u\|_{L^2} \le C\delta \|u\|_{H^1(\hat{B}_k)} \le C\delta\|u\|_{\hat{X}_\delta}.$$

The remaining terms are estimated similarly. Hence, $\|S_\delta L_\delta - \mathbb{I}_{\hat{X}_\delta}\| \to 0$ as $\delta \to 0^+$. Now we observe that $L_\delta^{-1} = (S_\delta L_\delta)^{-1} S_\delta$. It follows that for any $f \in \hat{Y}_\delta$ we have

$$\|L_\delta^{-1} f\|_{\hat{X}_\delta} = \|(S_\delta L_\delta)^{-1} S_\delta f\|_{\hat{X}_\delta} \le C\|S_\delta f\|_{\hat{Y}_\delta} \le C\|f\|_{\hat{Y}_\delta} \tag{4.10}$$

with $C > 0$ independent of δ. Hence, L_δ is invertible and its inverse is bounded independently of δ, as asserted. □

Now we can provide the

Proof of Proposition 4.1. We use the Banach fixed point argument. For any $\delta \in (0, \delta_0)$, with $\delta_0 > 0$ given by Lemma 4.4, we introduce the nonlinear map $G_\delta \in C^1(\hat{X}_\delta, \hat{X}_\delta)$ defined by

$$G_\delta(z) = z - L_\delta^{-1} F_\delta(z). \tag{4.11}$$

and the set

$$\mathcal{B}_R = \{u \in \hat{X}_\delta \,:\, \|u\|_{\hat{X}_\delta} \le R\} \tag{4.12}$$

Then, fixed points of G_δ correspond to solutions of the functional equation $F_\delta(z) = 0$. First, note that $DG_\delta(0) = 0$ and that

$$DF(z) = -\Delta + \mathrm{e}^{\sum_{j\in\mathbb{N}} \hat{\varphi}_j \hat{U}_j + z}.$$

By Lemma 4.4, for any $z \in \hat{X}_\delta$ and $u \in \hat{X}_\delta$ we have

$$\begin{aligned}\|DG_\delta(z)u\|_{\hat{X}_\delta} &= \|(DG_\delta(z) - DG_\delta(0))u\|_{\hat{X}_\delta} = \|L_\delta^{-1}(DF_\delta(z) - L_\delta)u\|_{\hat{X}_\delta} \\ &\le C\|(DF_\delta(z) - L_\delta)u\|_{\hat{Y}_\delta} = C\|\mathrm{e}^{\sum_{j\in\mathbb{N}} \hat{\varphi}_j \hat{U}_j}(\mathrm{e}^z - 1)u\|_{\hat{Y}_\delta} \le C\|(\mathrm{e}^z - 1)u\|_{\hat{Y}_\delta}.\end{aligned}$$

By the elementary inequality $\mathrm{e}^t - 1 \le Ct\mathrm{e}^t$, for all $t > 0$, where $C > 0$ does not depend on t, and in view of Lemma 3.1, we have

$$\|\mathrm{e}^z - 1\|_\infty \le \mathrm{e}^{\|z\|_\infty} - 1 \le C\|z\|_\infty \mathrm{e}^{\|z\|_\infty} \le C\|z\|_{\hat{X}_\delta} \mathrm{e}^{\|z\|_{\hat{X}_\delta}}. \tag{4.13}$$

Hence,

$$\|DG_\delta(z)u\|_{\hat{X}_\delta} \le C\|(\mathrm{e}^z - 1)u\|_{\hat{Y}_\delta} \le C\|z\|_{\hat{X}_\delta} \mathrm{e}^{\|z\|_{\hat{X}_\delta}} \|u\|_{\hat{Y}_\delta} \le C\|z\|_{\hat{X}_\delta} \mathrm{e}^{\|z\|_{\hat{X}_\delta}} \|u\|_{\hat{X}_\delta}.$$

Consequently, there exists $R_0 > 0$ such that for every $R \in (0, R_0)$ we have

$$\|DG_\delta(z)\| \le \frac{1}{2}, \qquad \forall z \in \mathcal{B}_R \tag{4.14}$$

and for all $\delta > 0$. Now,

$$\begin{aligned}\|G_\delta(z)\|_{\hat{X}_\delta} &\le \|G_\delta(z) - G_\delta(0)\|_{\hat{X}_\delta} + \|G_\delta(0)\|_{\hat{X}_\delta} \\ &\le \frac{1}{2}\|z\|_{\hat{X}_\delta} + \|L_\delta^{-1} F_\delta(0)\|_{\hat{X}_\delta}.\end{aligned} \tag{4.15}$$

By Lemma 4.4 and Lemma 4.2, we have that:

$$\|L_\delta^{-1} F_\delta(0)\|_{\hat{X}_\delta} \le C\|F_\delta(0)\|_{\hat{Y}_\delta} \le C_0 \mathrm{e}^{-c_0/\delta}. \tag{4.16}$$

Choosing $R = R_\delta = 2C_0\mathrm{e}^{-c_0/\delta}$, we obtain that $G_\delta(B_{R_\delta}) \subset B_{R_\delta}$. Hence, G_δ is a strict contraction in B_{R_δ}, for any $\delta \in (0, \delta_1)$. By the Banach fixed-point theorem, for any $\delta \in (0, \delta_1)$, there exists a unique $z_\delta \in B_{R_\delta}$, such that $F_\delta(z_\delta) = 0$. □

5 Proof of Theorem 1.1

In this section we finally provide the proof of Theorem 1.1. In view of Proposition 4.1, the function $\hat{u}_\delta$ defined by

$$\hat{u}_\delta = \sum_{j\in\mathbb{N}} \hat{\varphi}_j \hat{U}_j + z_\delta \tag{5.1}$$

is a solution to equation (1.7). Consequently, u_δ defined by

$$u_\delta(x) = \hat{u}_\delta\left(\frac{x}{\delta}\right) = \sum_{j\in\mathbb{N}} \varphi_j(x) U_{m_j}\left(\frac{x-p_j}{\delta}\right) + z_\delta\left(\frac{x}{\delta}\right) \tag{5.2}$$

is a solution to (1.3). Now, we want to prove that if the p_j's are doubly periodically arranged in $\mathbb{R}^2$, then u_δ is in fact a doubly periodic solution to (1.2). Recall from Section 1 that the p_j's are doubly periodically arranged in $\mathbb{R}^2$ if there exists $s \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $k > s$ there exist $j \in \{1, 2, \ldots, s\}$ and $m, n \in \mathbb{Z}$ such that $p_k = p_j + m\underline{e}_1 + n\underline{e}_2$, where $\underline{e}_1, \underline{e}_2$ are the unit vectors in $\mathbb{R}^2$. We define $\hat{\underline{e}}_k = \underline{e}_k/\delta$, $k = 1, 2$. Equivalently, we show:

Lemma 5.1. *Suppose the vortex points p_j, $j \in \mathbb{N}$, are doubly periodically arranged in $\mathbb{R}^2$. Then $\hat{u}_\delta(x + \hat{\underline{e}}_k) = \hat{u}_\delta(x)$ for any $x \in \mathbb{R}^2$ and for $k = 1, 2$.*

Proof. We may assume that $\hat{\varphi}_j(x + \hat{\underline{e}}_k) = \hat{\varphi}_j(x)$, $\hat{\psi}_j(x + \hat{\underline{e}}_k) = \hat{\psi}_j(x)$, for any $j \in \mathbb{N}$, $x \in \mathbb{R}^2$, $k = 1, 2$. Then,

$$\hat{u}_\delta(x + \hat{\underline{e}}_k) = \sum_{j\in\mathbb{N}} \hat{\varphi}_j(x)\hat{U}_j(x) + z_\delta(x + \hat{\underline{e}}_k).$$

Hence, it is sufficient to prove that $z_\delta(x + \hat{\underline{e}}_k) = z_\delta(x)$, for every $x \in \mathbb{R}^2$ and for $k = 1, 2$. First, we claim that $z_\delta(\,\cdot\, + \hat{\underline{e}}_k) \in \mathcal{B}_{R_\delta}$. Indeed, for every $j \in \mathbb{N}$ there exists exactly one $j' \in \mathbb{N}$ such that

$$\|\hat{\varphi}_j z_\delta(\,\cdot\, + \hat{\underline{e}}_k)\|_{H^2} = \|\hat{\varphi}_{j'} z_\delta\|_{H^2}. \tag{5.3}$$

Hence, we obtain

$$\|z_\delta(\,\cdot\, + \hat{\underline{e}}_k)\|_{\hat{X}_\delta} = \|z_\delta\|_{\hat{X}_\delta} \le R_\delta. \tag{5.4}$$

Moreover, if $F_\delta(z_\delta) = 0$ we also have $F_\delta(z_\delta(\,\cdot\, + \hat{\underline{e}}_k)) = 0$. Therefore, $z_\delta(\,\cdot\, + \hat{\underline{e}}_k)$ is a fixed point of G_δ in $\mathcal{B}_{R_\delta}$. By uniqueness, we conclude that $z_\delta(\,\cdot\, + \hat{\underline{e}}_k) = z_\delta$, $k = 1, 2$, as asserted. □

Lemma 5.2. *The solution u_δ defined in* (5.2) *satisfies the approximate superposition rule:*

$$u_\delta(x) = \sum_{j\in\mathbb{N}} U_{m_j}\left(\frac{x-p_j}{\delta}\right) + \omega_\delta(x), \tag{5.5}$$

with $\|\omega_\delta\|_\infty \le C\mathrm{e}^{-c/\delta}$.

Proof. In view of (5.2) and of the definition of $J(x)$ in Section 3, we have

$$u_\delta(x) = \sum_{j\in J(x)} U_{m_j}\left(\frac{x-p_j}{\delta}\right) + \tilde{\omega}_\delta(x),$$

where

$$\tilde{\omega}_\delta(x) = -\sum_{j\in J(x)} (1-\varphi_j(x))U_{m_j}\left(\frac{x-p_j}{\delta}\right) + z_\delta\left(\frac{x}{\delta}\right).$$

We estimate:

$$\left\|\sum_{j\in J(x)} (1-\varphi_j(x))U_{m_j}\left(\frac{x-p_j}{\delta}\right)\right\|_\infty \le \sum_{j\in J(x)} \sup_{\mathbb{R}^2\setminus C_j}\left|U_{m_j}\left(\frac{x-p_j}{\delta}\right)\right| \le C\mathrm{e}^{-c/\delta}.$$

On the other hand, we readily have

$$\|z_\delta\left(\frac{\cdot}{\delta}\right)\|_\infty = \|z_\delta\|_\infty \le C\mathrm{e}^{-c/\delta}.$$

Therefore, $\|\tilde{\omega}_\delta\|_\infty \le C\mathrm{e}^{-c/\delta}$. We have to show that

$$\left\|\sum_{j\notin J(x)} U_{m_j}\left(\frac{x-p_j}{\delta}\right)\right\|_\infty \le C\mathrm{e}^{-c/\delta}. \tag{5.6}$$

To this end, we fix $x\in\mathbb{R}^2$ and for every $N\in\mathbb{N}$ we define $B_N=\{y\in\mathbb{R}^2 : |y-x|<r_0N\}$. Then,

$$\sum_{j\notin J(x)} U_{m_j}\left(\frac{x-p_j}{\delta}\right) = \sum_{N\in\mathbb{N}}\sum_{p_j\in\overline{B_{N+1}}\setminus B_N} U_{m_j}\left(\frac{x-p_j}{\delta}\right)$$

Since $\inf_{j\neq k}|p_j-p_k|>r_0$ there exists $C>0$ independent of $N\in\mathbb{N}$ and of $x\in\mathbb{R}^2$ such that

$$\left|\{p_j\in\overline{B_{N+1}}\setminus B_N\}\right| \le CN. \tag{5.7}$$

Hence, we estimate:

$$\left|\sum_{j\notin J(x)} U_{m_j}\left(\frac{x-p_j}{\delta}\right)\right| \le C\sum_{N\in\mathbb{N}} N\mathrm{e}^{-cN/\delta} \le C\mathrm{e}^{-c/\delta}.$$

This implies (5.5). ☐

We are left to analyze the asymptotic behavior of u_δ as $\delta\to0^+$. Such a behavior is a straightforward consequence of (5.2).

Lemma 5.3. *Let u_δ be given by* (5.2)*. The following properties hold:*

(i) $e^{u_\delta}<1$ *on* $\mathbb{R}^2$ *and vanishes exactly at* p_j *with multiplicity* $2m_j$, $j\in\mathbb{N}$*;*

(ii) For every compact subset K *of* $\mathbb{R}^2\setminus\cup_{j\in\mathbb{N}}\{p_j\}$ *there exist* $C,c>0$ *such that* $1-e^{u_\delta}\le C\mathrm{e}^{-c/\delta}$ *as* $\delta\to0^+$*;*

(iii) $\delta^{-2}(1-e^{u_\delta}) \to 4\pi \sum_{j\in\mathbb{N}} m_j \delta_{p_j}$ *in the sense of distributions, as* $\delta \to 0^+$.

Proof. (i) Since u_δ is a solution of equation (1.3), $\mathrm{e}^{u_\delta} < 1$ follows by the maximum principle. Moreover, since

$$U_{m_j}((x-p_j)/\delta) = \ln|x-p_j|^{2m_j} + v_j \tag{5.8}$$

with v_j a continuous function (see [5]), we have near p_j that $\mathrm{e}^{u_\delta} = |x - p_j|^{2m_j} f_{j,\delta}(x)$, with $f_{j,\delta}(x)$ a continuous strictly positive function. Hence, (i) is established.

(ii) Let K be a compact subset of $\mathbb{R}^2 \setminus \cup_{j\in\mathbb{N}}\{p_j\}$. In view of Lemma 2.1 and Proposition 4.1, we have as $\delta \to 0^+$

$$\begin{aligned} &\sup_{x\in K\cap B_j} 1 - \mathrm{e}^{\varphi_j(x)U_{m_j}((x-p_j)/\delta)} \le C\mathrm{e}^{-c/\delta} \\ &\|z_\delta(\frac{\cdot}{\delta})\|_\infty \le C\|z_\delta\|_{\hat{X}_\delta} \le CR_\delta \le C\mathrm{e}^{-c/\delta}. \end{aligned} \tag{5.9}$$

Therefore, we have that for any compact set $K \subset \mathbb{R}^2 \setminus \cup_{j\in\mathbb{N}}\{p_j\}$

$$0 \le \sup_{x\in K}(1-\mathrm{e}^{u_\delta}) \le C \sup_{j\in\mathbb{N}} \sup_{x\in K\cap B_j} (1-\mathrm{e}^{u_\delta}) \le C\mathrm{e}^{-c/\delta}. \tag{5.10}$$

(iii) Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then,

$$-\int_{\mathbb{R}^2} u_\delta \Delta\varphi = \delta^{-2}\int_{\mathbb{R}^2} (1-\mathrm{e}^u)\varphi - 4\pi m_j \varphi(p_j).$$

We claim that

$$\int_{\mathbb{R}^2} u_\delta \Delta\varphi \to 0 \qquad \text{as } \delta \to 0. \tag{5.11}$$

Indeed, let $\operatorname{supp}\varphi \subset \cup_{k=1}^N B_{j_k} \cup K$, with K a compact subset of $\mathbb{R}^2 \setminus \cup_{j\in\mathbb{N}}\{p_j\}$. Since $\sup_K |u_\delta| \le C\mathrm{e}^{-c/\delta}$, we have

$$\left|\int_K u_\delta \Delta\varphi\right| \le C\|\Delta\varphi\|_\infty \mathrm{e}^{-c/\delta} \to 0.$$

On the other hand, in view of (5.5), in B_{j_k} we have $u_\delta(x) = U_{m_{j_k}}(|x-p_{j_k}|/\delta) + O(\mathrm{e}^{-c/\delta})$. Note that $U_{m_{j_k}} \in L^1(\mathbb{R}^2)$ in view of (5.8) and Lemma 2.1. Therefore,

$$\begin{aligned} \sup_{1\le k\le N}\left|\int_{B_{j_k}} u_\delta\Delta\varphi\right| \le \sup_{1\le k\le N}&\left|\int_{B_{j_k}} U_{m_{j_k}}\left(\frac{x-p_{j_k}}{\delta}\right)\Delta\varphi\right| + O(\mathrm{e}^{-c/\delta}) \\ &\le \delta^2 \sup_{1\le k\le N} \|\Delta\varphi\|_\infty \|U_{m_{j_k}}\|_{L^1} + O(\mathrm{e}^{-c/\delta}) \le C\delta^2 \to 0. \end{aligned}$$

Hence (5.11) follows, and (iii) is established. □

Proof of Theorem 1.1. For every $\delta \in (0, \delta_1)$, where δ_1 is defined Proposition 4.1, we obtain a solution u_δ to (1.3). If the p_j's are doubly periodically arranged, then u_δ is doubly periodic in view of Lemma 5.1. Furthermore, u_δ satisfies (1.6) in view of Lemma 5.2 and of the definition of δ. Finally, u_δ satisfies the asymptotic behavior as in (i)–(ii)–(iii) in view of Lemma 5.3. Hence, Theorem 1.1 is completely established. □

6 Appendix

For the reader's convenience, we sketch in this appendix the proof of some results for smooth, finite action critical points for the action of $\mathcal{E}_\delta$, which are relevant to our discussion. The following results are due to Taubes [8]. Throughout this appendix all citations are referred to the monograph of Jaffe and Taubes [5].

6.1 Derivation of equation (1.1)

Following [5] p. 53, we consider the change of variables $A(x) = \delta^{-1}A'(x/\delta)$, $\phi(x) = \phi'(x/\delta)$ $x' = x/\delta$. We denote by D', F'_{12} the covariant derivative of A' and the curvature of A', respectively. Then, $\mathrm{d}A(x) = \delta^{-2}\mathrm{d}A'(x')$, $D\phi(x) = \delta^{-1}D'\phi'(x')$, and therefore:

$$\int_{\mathbb{R}^2} \mathcal{E}_\delta(A,\phi)\,\mathrm{d}x = \int_{\mathbb{R}^2} \mathcal{E}_1(A',\phi')\,\mathrm{d}x',$$

where $\mathcal{E}_1$ denotes $\mathcal{E}_\delta$ with $\delta = 1$. In view of Bogomol'nyi's reduction (see formula (III.1.5)), we may rewrite the action in the form:

$$\begin{aligned}\int_{\mathbb{R}^2} &\mathcal{E}_1(A',\phi')\,\mathrm{d}x' \\ &= \int_{\mathbb{R}^2} \left\{ |(D'_1 \pm iD'_2)\phi'|^2 + (F'_{12} \pm \frac{1}{2}(|\phi'|^2 - 1))^2 \pm F'_{12} \right\} \mathrm{d}x'.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^2} &\mathcal{E}_\delta(A,\phi)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left\{ |(D_1 \pm iD_2)\phi|^2 + (\delta F_{12} \pm \frac{1}{2\delta}(|\phi|^2 - 1))^2 \pm F_{12} \right\} \mathrm{d}x.\end{aligned}$$

Here and in what follows, it is understood that we either always choose upper signs, or we always choose lower signs. For smooth, finite action critical points, $(2\pi)^{-1}\int_{\mathbb{R}^2} F_{12} = N$ is an integer, defining a topological class (Theorem II.3.1 and Theorem III.8.1). Hence, the energy minimizers in a fixed topological class satisfy the following the first order equations:

$$(D_1 \pm iD_2)\phi = 0 \tag{6.1}$$

$$F_{12} = \pm\frac{1}{2\delta^2}(1 - |\phi|^2). \tag{6.2}$$

In fact, there is no loss of generality in restricting to critical points for the action in a given topological class (Theorem III.10.1). By complex analysis methods, one shows that smooth solutions to (6.1) vanish at most at isolated zeros of finite multiplicity. Hence, differentiating (6.1) we obtain

$$-\Delta \ln|\phi|^2 = \pm 2F_{12} - 4\pi\sum_{j=1}^{s} n_j\delta_{p_j},$$

in the sense of distributions. Setting $u = \ln|\phi^2|$, we obtain from the above and (6.2) that u satisfies (1.1).

In order to define the decay properties of smooth, finite action critical points, we define by $u_0(x) = -\sum_{j=1}^{s} \ln(1+\mu|x-p_j|^{-2})$ the "singular part" of u, where $\mu > 4N$. Then, $u - u_0 \in H^1(\mathbb{R}^2)$ (Theorem III.3.2). Conversely, if u satisfies (1.1) and if $u - u_0 \in H^1(\mathbb{R}^2)$, then (A, ϕ) defined by

$$\phi(z) = \exp\{\frac{1}{2}u(z) \pm i \sum_{j=1}^{s} m_j \arg(z - p_j)\}$$
$$A_1 \mp iA_2 = -i(\partial_1 \pm i\partial_2) \ln \phi$$

is a smooth, finite action critical point.

6.2 Existence and uniqueness

Without loss of generality we assume $\delta = 1$. The function $v = u - u_0$ satisfies the elliptic equation with smooth coefficients

$$-\Delta v = 1 - \mathrm{e}^{u_0+v} - g_0,$$

where $g_0 = 4\sum_{j=1}^{s} \mu(|x - p_j|^2 + \mu)^{-2}$. Solutions in $H^1(\mathbb{R}^2)$ to the equation above correspond to critical points for the functional

$$a(v) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla v|^2 + (g_0 - 1)v + \mathrm{e}^{u_0}(\mathrm{e}^v - 1) \right\},$$

which is well-defined and differentiable on $H^1(\mathbb{R}^2)$. Furthermore, a is coercive and strictly convex, and therefore it admits a unique critical point, corresponding to the absolute minimum (Theorem III.4.3). In particular, the solution to (1.1) satisfying $u - u_0 \in H^1(\mathbb{R}^2)$ is unique. Finally, the critical point (A, ϕ) obtained from $u = u_0 + v$ satisfies the following decay estimate holds, for any $\varepsilon > 0$:

$$|D\phi| \leq \frac{3}{2}(1 - |\phi|^2) \leq C_\varepsilon \mathrm{e}^{-(1-\varepsilon)|x|},$$

where $C_\varepsilon > 0$ depends on ε (Theorem III.8.1).

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