Asymptotics for some nonlinear elliptic systems in Maxwell-Chern-Simons vortex theory

Tonia Ricciardi
Dipartimento di Matematica e Applicazioni
Università di Napoli Federico II
Via Cintia
80126 Naples, Italy
fax: +39 081 675665
e-mail: tonia.ricciardi@unina.it

Abstract
We provide a unified proof of the asymptotics of the self-dual Maxwell-Chern-Simons vortices, as the Maxwell term is neglected, in both the $U(1)$ and $CP(1)$ case. This result is achieved by identifying and analyzing a suitable class of nonlinear elliptic systems with exponential type nonlinearities.

KEY WORDS: nonlinear elliptic system, Chern-Simons vortex theory
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1 Introduction and main result

The vortex solutions for the $U(1)$ Maxwell-Chern-Simons model introduced in [9], correspond to (distributional) solutions $(\tilde{u}, v)$ for the system:

\begin{align*}
-\Delta \tilde{u} &= \varepsilon^{-1}(v - e^{\tilde{u}}) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma \\
-\Delta v &= \varepsilon^{-1} \left\{ e^{\tilde{u}}(1 - v) - \varepsilon^{-1}(v - e^{\tilde{u}}) \right\} \quad \text{on } \Sigma,
\end{align*}

where $\Sigma$ is a compact Riemannian 2-manifold without boundary, $n \geq 0$ is an integer, $p_j \in \Sigma$ for $j = 1, \ldots, n$, $\Delta$ denotes the Laplace-Beltrami operator and $\varepsilon > 0$ a constant. We shall be interested in the asymptotic behavior of solutions when $\varepsilon \to 0$.

Physically, $e^{\tilde{u}}$ represents a density of particles; it vanishes exactly at the points $p_j$, $j = 1, \ldots, n$ (the vortex points). The function $v$ is a neutral scalar field and $\varepsilon > 0$ is the coupling constant for the Maxwell term. In particular, letting $\varepsilon \to 0$ corresponds to dropping the Maxwell term in the Lagrangian.

The limit $\varepsilon \to 0$ is meaningful in view of the following

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Theorem 1.1 ([13]). If $\varepsilon 4\pi n/|\Sigma|$ is sufficiently small, then there exist at least two solutions for (1)–(2).

The proof of Theorem 1.1 is variational. The two solutions are obtained as a local minimum and a mountain pass for a suitable functional. We refer to [13] for the detailed proof.

By a formal analysis of (1)–(2), we expect that as $\varepsilon \to 0$, $e^u$ should converge to a solution $u_\infty$ for the equation

$$-\Delta u_\infty = e^{u_\infty} (1 - e^{u_\infty}) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma.$$  

We observe that solutions for (3) correspond to vortex solutions for the Chern-Simons model introduced in [7] and [6]. In [11] we provided a rigorous proof of this formal argument, in any relevant norm. Namely, we showed

Theorem 1.2 ([11]). Suppose $(u, v)$ are (distributional) solutions for (1)–(2) with $\varepsilon \to 0$. Then there exists a solution $u_\infty$ for the equation (3) such that, up to subsequences, $(e^u, v) \to (e^{u_\infty}, e^{u_\infty})$ in $C^h(\Sigma) \times C^h(\Sigma)$, for any $h \geq 0$.

Note that $e^u$, $e^{u_\infty}$ are smooth. Theorem 1.2 completed our previous convergence result obtained with Tarantello [13], where the asymptotics for $v$ was established in the $L^2$-sense only. See also Chae and Kim [2].

At this point it is natural to seek a more general class of systems which exhibit an asymptotic behavior as in Theorem 1.2. A further motivation to this question is provided by the analysis of the $CP(1)$ Maxwell-Chern-Simons model in [3]. In [3] the authors analyze an elliptic system, whose solutions correspond to vortex solutions for the self-dual $CP(1)$ Maxwell-Chern-Simons model introduced in [4]. Their system (in a special case) is given by:

$$\Delta U = 2Q(-V + S - \frac{1 - e^U}{1 + e^U}) + 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma$$

$$\Delta V = -4Q^2(-V + S - \frac{1 - e^U}{1 + e^U}) + Q \frac{4e^U}{(1 + e^U)^2} V \quad \text{on } \Sigma$$

where $\Sigma$ and $p_1, \ldots, p_n$ are as in (1)–(2), $U, V$ are the unknown functions and $S \in \mathbb{R}$, $Q > 0$ are given constants. They prove the existence of at least one solution for (4)–(5); furthermore, they derive an asymptotic behavior as $Q \to +\infty$ analogous to that of system (1)–(2).

With this motivation, we consider (distributional) solutions $(\tilde{u}, v)$ for the system:

$$-\Delta \tilde{u} = \varepsilon^{-1}(v - f(\tilde{u})) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma$$

$$-\Delta v = \varepsilon^{-1} \left[ f'(\tilde{u})e^u(s - v) - \varepsilon^{-1}(v - f(\tilde{u})) \right] \quad \text{on } \Sigma.$$  

Here $\Sigma$ and $p_1, \ldots, p_n$ are as in (1)–(2), $f = f(t)$, $t \geq 0$ is smooth and strictly increasing, $s \in \mathbb{R}$ satisfies $f(0) < s < \sup_{t>0} f(t)$. Without loss of generality, we assume $\text{vol}\Sigma = 1$.
Clearly, when \( f(t) = t \) and \( s = 1 \), system (6)–(7) reduces to (1)–(2). On the other hand, setting \( v := V - S \), \( s := -S \), \( \varepsilon^{-1} := 2Q \), system (6)–(7) reduces to system (4)–(5) with \( f \) defined by \( f(t) = (t - 1)/(t + 1) \).

By a formal analysis of (6)–(7) we expect that, up to subsequences, \((\tilde{u}, v)\) should converge to \((\tilde{u}_\infty, f(e^{\tilde{u}_\infty}))\), where \( \tilde{u}_\infty \) is a solutions for the equation for the equation:

\[
\Delta \tilde{u}_\infty = f'(e^{\tilde{u}_\infty})e^{\tilde{u}_\infty}(s - f(e^{\tilde{u}_\infty})) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \text{ on } \Sigma.
\]

Our main result states that this is indeed the case, with respect to any relevant norm:

**Theorem 1.3 ([12])**. Let \((\tilde{u}, v)\) be (distributional) solutions to (6)–(7), with \( \varepsilon \to 0 \). There exists a (distributional) solution \( \tilde{u}_\infty \) to (8) such that a subsequence, still denoted \((\tilde{u}, v)\), satisfies:

\[
(e^{\tilde{u}}, v) \to (e^{\tilde{u}_\infty}, f(e^{\tilde{u}_\infty})) \quad \text{in } C^h(\Sigma) \times C^h(\Sigma) \text{, } \forall h \geq 0.
\]

In the rest of this note we shall outline the proof of Theorem 1.3. The detailed proof is contained in [12], although some arguments are proved here in a simpler form. Henceforth we denote by \( C > 0 \) a general constant independent of \( \varepsilon \), which may vary from line to line. Unless otherwise specified, all equations are defined on \( \Sigma \) and all integrals are taken over \( \Sigma \) with respect to the Lebesgue measure.

### 2 Proof of Theorem 1.3

In order to work in suitable Sobolev spaces, it is standard (see [14]) to define a “Green’s function” \( u_0 \), solution for the problem

\[
-\Delta u_0 = 4\pi \left(n - \sum_{j=1}^{n} \delta_{p_j}\right) \text{ on } \Sigma
\]

\[
\int_{\Sigma} u_0 = 0
\]

(see [1] for the unique existence of \( u_0 \)). Setting \( \tilde{u} = u_0 + u \), we obtain the equivalent system for \((u, v) \in H^1(\Sigma) \times H^1(\Sigma)\):

\[
-\Delta u = \varepsilon^{-1}(v - f(e^{u_0+u}))-4\pi n \quad \text{on } \Sigma
\]

\[
-\Delta v = \varepsilon^{-1}\left[f'(e^{u_0+u})e^{u_0+u}(s - v) - \varepsilon^{-1}(v - f(e^{u_0+u}))\right] \quad \text{on } \Sigma,
\]

where \( e^{u_0} \) is smooth.

The proof is obtained by a priori estimates and an inductive argument. It will be convenient to introduce the spaces \( X^k := H^k \cap L^\infty \). By the following inequality, which follows from the well-known Sobolev-Gagliardo-Nirenberg inequality (see e.g. [10]):

\[
\|D^j u\|_{L^{2k/j}} \leq C\|D^k u\|_{L^2} \|u\|_{L^\infty}^{1-j/k} \quad \forall u \in C^\infty(\Sigma),
\]

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$X^k$ is a Banach algebra for every $k \geq 0$, i.e.,
$$
\|u_1 u_2\|_{X^k} \leq C \|u_1\|_{X^k} \|u_2\|_{X^k}.
$$

It will also be convenient to set
$$
w := \varepsilon^{-1} (v - f(e^{u_0 + u}))
$$
and to consider $w$ as a third unknown function. Then the triple $(u, v, w)$ satisfies a system of the following simple form:

(12) \quad $- \Delta u = w - 4\pi u$

(13) \quad $- \varepsilon^2 \Delta v + [1 + \varepsilon c(x, u)] v = F_\varepsilon(x, u)$

(14) \quad $- \varepsilon^2 \Delta w + [1 + \varepsilon c(x, u)] w = G_\varepsilon(x, u, v, \nabla u)$

where

\begin{align*}
c(x, u) &= f'(e^{u_0 + u}) e^{u_0 + u} \\
F_\varepsilon(x, u) &= f(e^{u_0 + u}) + s \varepsilon f'(e^{u_0 + u}) e^{u_0 + u} \\
G_\varepsilon(x, u, v, \nabla u) &= f'(e^{u_0 + u}) e^{u_0 + u} (s - v) \\
&\quad + \varepsilon \left( f''(e^{u_0 + u}) e^{u_0 + u} + f'(e^{u_0 + u}) \right) e^{u_0 + u} |\nabla (u_0 + u)|^2.
\end{align*}

Furthermore, using (11), we obtain:

**Lemma 2.1.** Let $F \in C^\infty(\Sigma \times \mathbb{R})$, $G \in C^\infty(\Sigma \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2)$. Then for all $k \geq 0$ there exists constants $C_k = C_k(||u||_{L^\infty})$, $C'_k = C'_k(||u||_{L^\infty}, ||v||_{L^\infty}, ||\nabla u||_{L^\infty})$, such that:

$$
\|F(x, u)\|_{X^k} \leq C_k (1 + ||u||_{X^k})
$$

$$
\|G(x, u, v, \nabla u)\|_{X^{k-1}} \leq C'_k (1 + ||u||_{X^k}^{k-1} + ||v||_{X^{k-1}}).
$$

The basis for the induction is included in the following

**Proposition 2.1.** There exists a constant $C > 0$ independent of $\varepsilon \to 0$, such that:

(i) $\|u\|_{X^1} + \|v\|_{X^1} + \|w\|_{X^0} \leq C$

(ii) $\|u\|_{L^\infty} \leq C$

(iii) $\|\nabla u\|_{L^\infty} \leq C$

In order to prove Proposition 2.1 we need some preliminary estimates.

**Lemma 2.2.** The following estimates hold, pointwise on $\Sigma$:

(i) $f(0) \leq f(e^{\tilde{v}}) \leq s$

(ii) $f(0) \leq v \leq s$.

*Proof.* By maximum principle. The increasing monotonicity of $f$ is essential here. \qed
As a consequence of Lemma 2.2, the nonlinearity $f$ may be truncated. Therefore in what follows, without loss of generality, we assume that:

\begin{equation}
\sup_{t>0}\{|f(t)| + |f'(t)| + |f''(t)|\} \leq C.
\end{equation}

The next identity is the main step in deriving the $H^1$-estimate for $v$ and the $L^2$-estimate for $w$:

**Lemma 2.3.** The following identity holds:

\begin{equation}
\int |\nabla v|^2 + \int w^2 = \int (s - v) \left( f''(e^\bar u)e^\bar u + f'(e^\bar u) \right) e^\bar u |\nabla \bar u|^2.
\end{equation}

**Proof.** We compute:

\[ \Delta f(e^\bar u) = \left( f''(e^\bar u)e^\bar u + f'(e^\bar u) \right) e^\bar u |\nabla \bar u|^2 + f'(e^\bar u)e^\bar u \Delta \bar u. \]

Therefore $f(e^\bar u)$ satisfies the equation:

\begin{equation}
-\Delta f(e^\bar u) + \varepsilon^{-1} f'(e^\bar u) e^\bar u f(e^\bar u) = \varepsilon^{-1} f'(e^\bar u) e^\bar u v - \left( f''(e^\bar u)e^\bar u + f'(e^\bar u) \right) e^\bar u |\nabla \bar u|^2.
\end{equation}

Integrating (17), we obtain

\begin{equation}
\varepsilon^{-1} \int f'(e^\bar u) e^\bar u (v - f(e^\bar u)) = \int \left( f''(e^\bar u)e^\bar u + f'(e^\bar u) \right) e^\bar u |\nabla \bar u|^2.
\end{equation}

Now we multiply (7) by $v - f(e^\bar u)$ and integrate to obtain:

\[ \int -\Delta v(v - f(e^\bar u)) = \varepsilon^{-1} \int f'(e^\bar u) e^\bar u (s - v)(v - f(e^\bar u)) - \varepsilon^{-2} \int (v - f(e^\bar u))^2. \]

Integrating by parts and using (17) we find:

\[ \int -\Delta v(v - f(e^\bar u)) = \int |\nabla v|^2 + \int v \Delta f(e^\bar u) \]

\[ = \int |\nabla v|^2 - \varepsilon^{-1} \int v f'(e^\bar u) e^\bar u (v - f(e^\bar u)) + \int v \left( f''(e^\bar u)e^\bar u + f'(e^\bar u) \right) e^\bar u |\nabla \bar u|^2. \]

Equating left hand sides in the last two identities, we obtain

\[ \int |\nabla v|^2 + \varepsilon^{-2} \int (v - f(e^\bar u))^2 + \int v \left( f''(e^\bar u)e^\bar u + f'(e^\bar u) \right) e^\bar u |\nabla \bar u|^2 \]

\[ = \varepsilon^{-1} \int f'(e^\bar u) e^\bar u (v - f(e^\bar u)), \]

and thus identity (16) is established. \hfill \Box

We shall need some a priori estimates for solutions to

\begin{equation}
-\varepsilon^2 \Delta u + (1 + \varepsilon c) u = f.
\end{equation}

Indeed, both (13) and (14) are of the form (19).
Lemma 2.4. Let \( c, f \in X^k \) and suppose that \( u \) satisfies: (19). For every \( k \geq 0 \) there exist \( \varepsilon_k > 0, C_k > 0 \) such that

\[
\|u\|_{X^k} \leq C_k \|f\|_{X^k},
\]

for all \( \varepsilon \leq \varepsilon_k \).

Proof. The proof is an easy consequence of the following fact Let \( G_\varepsilon \) be the Green’s function for

\[
-\varepsilon \Delta x G_\varepsilon + G_\varepsilon = \delta_y \quad \text{on } \Sigma.
\]

Then \( G_\varepsilon(x, y) \to \delta_y \) weakly in the sense of measures. Note that since the operator \(-\varepsilon \Delta + 1\) is coercive, the Green’s function \( G_\varepsilon \) is uniquely defined on \( \Sigma \). By the maximum principle, \( G_\varepsilon > 0 \) on \( \Sigma \). Integrating over \( \Sigma \), we find \( \int G_\varepsilon = \int |G_\varepsilon| = 1 \). Therefore, there exists a Radon measure \( \mu \) such that \( G_\varepsilon \rightharpoonup \mu \) weakly in the sense of measures. For any \( \varphi \in C^\infty(\Sigma) \) we have:

\[
\varepsilon \int -\Delta G_\varepsilon \varphi + \int G_\varepsilon \varphi = \varphi(y).
\]

Taking limits, we find \( \int \varphi \, d\mu = \varphi(y) \) and the statement of the lemma is established.

Now we can provide the

Proof of Proposition 2.1. We begin by establishing

(20)

\[
\int e^\overline{u} |\nabla \overline{u}|^2 \leq C.
\]

Proof of the Claim. Multiplying equation (6) by \( e^\overline{u} \) and integrating by parts, we obtain

\[
\varepsilon^{-1} \int e^\overline{u} (v - f(e^\overline{u})) = \int e^\overline{u} |\nabla \overline{u}|^2 \geq 0.
\]

By the pointwise estimates in Lemma 2.2, it follows that:

(21)

\[
\varepsilon \int e^\overline{u} |\nabla \overline{u}|^2 \leq C.
\]

Multiplying (7) by \( e^\overline{u} \) and integrating, we find

(22)

\[
\varepsilon^{-1} \int e^\overline{u} (v - f(e^\overline{u})) = \int e^{2\overline{u}} f'(e^\overline{u})(s - v) + \varepsilon \int e^{\overline{u}} \Delta v.
\]

Integration by parts yields:

\[
\varepsilon \int e^{\overline{u}} \Delta v = -\int ve^{\overline{u}} (v - f(e^\overline{u})) + \varepsilon \int ve^{\overline{u}} |\nabla \overline{u}|^2.
\]

Hence, by the pointwise estimates as in Lemma 2.2, and taking into account (21), we conclude that

\[
\varepsilon \left| \int e^{\overline{u}} \Delta v \right| \leq C.
\]
Inserting into (22), recalling Lemma 2.2, we derive that
\[ \varepsilon^{-1} \int e^{\tilde{u}} (v - f(e^{\tilde{u}})) \leq C, \]
and thus it follows that
\[ \int e^{\tilde{u}} |\nabla \tilde{u}|^2 = \varepsilon^{-1} \int e^{\tilde{u}} (v - f(e^{\tilde{u}})) \leq C. \]
(20) is established.

Lemma 2.2 readily implies \( \|e^{\tilde{u}}\|_{L^\infty} \leq C \) and \( \|v\|_{L^\infty} \leq C \). In order to obtain the \( H^1 \)-estimate for \( e^{\tilde{u}} \), it suffices to observe that by Lemma 2.2–(i) and by (20) we have:
\[ \int |\nabla e^{\tilde{u}}|^2 = \int e^{2\tilde{u}} |\nabla \tilde{u}|^2 \leq C \int e^{\tilde{u}} |\nabla \tilde{u}|^2 \leq C. \]
Now we estimate \( \|\nabla v\|_{L^2} \) and \( \|\varepsilon^{-1} (v - f(e^{\tilde{u}}))\|_{L^2} \). Using identity (16), we have:
\[ \int |\nabla v|^2 + \int w^2 \leq \|s - v\|_{L^\infty} \|f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}})\|_{L^\infty} \int e^{\tilde{u}} |\nabla \tilde{u}|^2 \leq C \int e^{\tilde{u}} |\nabla \tilde{u}|^2 \leq C, \]
where we again used Lemma 2.2 and (20) in order to derive the last step. Proof of (i). Multiplying (9) by \( u - \int u \) and integrating, we have:
\[ \int |\nabla u|^2 = q \int (v - f(e^{\tilde{u}}))(u - \int u) \leq \|q(v - f(e^{\tilde{u}}))\|_{L^2} \|u - \int u\|_{L^2} \leq C \|\nabla u\|_{L^2}, \]
where the last inequality follows by Lemma 2.2 and by the Poincaré inequality. Hence \( \|\nabla u\|_{L^2} \leq C \). By Lemma 2.2–(ii), we have that \( e^{\tilde{u}} \leq C \), and thus we only have to show that \( \int u \geq -C \). To this end, we first observe that integrating (9) and (10) we obtain:
\[ \int f'(e^{u_0 + u}) e^{u_0 + u} (s - v) = q \int (v - f(e^{u_0 + u})) = 4\pi n. \]
On the other hand, we have in a straightforward manner:
\[ \int f'(e^{u_0 + u}) e^{u_0 + u} (s - v) \leq C \int e^{u_0 + u} \leq C e^f u \|e^{u_0}\|_{L^\infty} \int e^{u - f} u \leq C \int e^{u - f} u. \]
Hence, recalling the Moser-Trudinger inequality (see [1]) and the estimate for \( \|\nabla u\|_{L^2} \), we conclude that
\[ 4\pi n \leq C e^f u \int e^{u - f} u \leq C e^f u e^f |\nabla u|^2 \leq C e^f u, \]
which establishes (i). Proof of (ii). Since \( \|w\|_{L^2} \leq C \), by (i) and elliptic regularity we obtain \( \|u\|_{H^2} \leq C \). Then Sobolev embeddings yield \( \|\nabla u\|_{L^p} \leq C \), for any \( 1 \leq p < +\infty \) and \( \|u\|_{L^\infty} \leq C \), which establishes (ii). Proof of (iii). By (14), \( \|\nabla u\|_{L^p} \leq C \) and Lemma 2.4 imply that \( \|w\|_{L^p} \leq C \), for any \( 1 \leq p < +\infty \). Then (12) and Sobolev embeddings yield \( \|u\|_{W^{2,p}} \leq C \), for any \( 1 \leq p < +\infty \). For \( p > 2 \), the Sobolev embeddings yield (iii).
Proposition 2.2. For all $k \geq 0$ there exists a constant $C > 0$ (possibly depending on $k$) such that:

$$\| \tilde{u} - u_0 \|_{H^k} + \| v \|_{H^k} \leq C.$$  

Proof of Proposition 2.2. We argue by induction on $k \in \mathbb{N}_0$.

CLAIM: Suppose:

$$\| u \|_{X^k} + \| v \|_{X^k} + \| w \|_{X^{k-1}} \leq C_k.$$  

Then:

$$\| u \|_{X^{k+1}} + \| v \|_{X^{k+1}} + \| w \|_{X^k} \leq C_{k+1}.$$  

Indeed,

$$\| w \|_{X^{k-1}} \leq C \Rightarrow \| u \|_{X^{k+1}} \leq C \text{ by (12) and standard elliptic regularity}$$

$$\Rightarrow \| v \|_{X^{k+1}} \leq C \text{ by (13), Lemma 2.1 and Lemma 2.4}$$

$$\Rightarrow \| w \|_{X^k} \leq C \text{ by (14), Lemma 2.1 and Lemma 2.4}.$$  

Now Proposition 2.1, the Claim and a standard induction argument conclude the proof. \qed

Finally, we can prove our main result:

Proof of Theorem 1.3. Let $(u, v)$ be solutions to system (9)–(10), with $\varepsilon \to 0$. By the a priori estimates as stated in Proposition 2.2 and by standard compactness arguments, there exist $u_\infty, v_\infty$ such that up to subsequences $u \to u_\infty$ and $v \to v_\infty$ in $C^h$, for all $h \geq 0$. We write (9) in the form:

$$v = f(e^{u_0 + u}) + \varepsilon(-\Delta u + 4\pi n).$$

Taking limits, we find $v_\infty = f(e^{u_0 + u_\infty})$. Furthermore, taking limits in (10), we obtain

$$\varepsilon^{-1}(v - f(e^{u_0 + u})) \to f'(e^{u_0 + u_\infty})e^{u_0 + u_\infty}(s - f(e^{u_0 + u_\infty})), $$

where the convergence holds in $C^h$, for any $h \geq 0$. Consequently, taking limits in (9), we find that $u_\infty$ satisfies:

$$\Delta u_\infty = f'(e^{u_0 + u_\infty})e^{u_0 + u_\infty}(s - f(e^{u_0 + u_\infty})) - 4\pi \sum_{j=1}^{n} \delta_{p_j}.$$  

Setting $\tilde{u}_\infty = u_0 + u_\infty$, we conclude the proof of Theorem 1.3. \qed

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