A Sharp Weighted Wirtinger Inequality.

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Summary. – We obtain a sharp estimate for the best constant $C > 0$ in the Wirtinger type inequality

$$\int_0^{2\pi} \gamma^p w^2 \leq C \int_0^{2\pi} \gamma^q w'^2$$

where $\gamma$ is bounded above and below away from zero, $w$ is $2\pi$-periodic and such that $\int_0^{2\pi} \gamma^p w = 0$, and $p + q \geq 0$. Our result generalizes an inequality of Piccinini and Spagnolo.

Let $C(a, b) > 0$ denote the best constant in the following weighted Wirtinger type inequality:

$$(1) \quad \int_0^{2\pi} aw^2 \leq C(a, b) \int_0^{2\pi} bw'^2,$$

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where $w \in H^1_{\text{loc}}(\mathbb{R})$ is $2\pi$-periodic and satisfies the constraint

\begin{equation}
\int_0^{2\pi} aw = 0,
\end{equation}

and $a, b \in \mathcal{B}$ with

$$
\mathcal{B} = \{a \in L^\infty(\mathbb{R}) : a \text{ is } 2\pi\text{-periodic and } \inf a > 0\}.
$$

Here and in what follows, for every measurable function $a$ we denote by $\inf a$ and $\sup a$ the essential lower bound and the essential upper bound of $a$, respectively. For every $L > 1$, we denote

$$
\mathcal{B}(L) = \{a \in L^\infty(0, 2\pi) : a \text{ is } 2\pi\text{-periodic, } \inf a = 1 \text{ and } \sup a = L\}.
$$

Our aim in this note is to prove:

**Theorem 1.** Suppose $a = \gamma^p$ and $b = \gamma^q$ for some $\gamma \in \mathcal{B}(M)$, $M > 1$, and for some $p, q \in \mathbb{R}$ such that $p + q \geq 0$. Then

\begin{equation}
C(\gamma^p, \gamma^q) \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \gamma^{(p+q)/2} \right)^2 \frac{4}{\pi \arctan(M^{-(p+q)/4})}.
\end{equation}

If $p + q > 0$, then equality holds in (3) if and only if $\gamma(\theta) = \overline{\gamma}_{p, q}(\theta + \varphi)$ for some $\varphi \in \mathbb{R}$, where

$$
\overline{\gamma}_{p, q}(\theta) = \begin{cases} 
1, & \text{if } 0 \leq \theta < c_{p, q} \frac{\pi}{2}, \pi \leq \theta < \pi + c_{p, q} \frac{\pi}{2} \\
M, & \text{if } c_{p, q} \frac{\pi}{2} \leq \theta < \pi, \pi + c_{p, q} \frac{\pi}{2} \leq \theta < 2\pi 
\end{cases}
$$

with

$$
c_{p, q} = \frac{2}{1 + M^{-(p+q)/2}}.
$$

Furthermore, equality holds in (1)-(2) with $a(\theta) = \overline{\gamma}_{p, q}(\theta + \varphi)$ and $b(\theta) =$
\[ \overline{\gamma}_{p,q}(\theta + \varphi) \text{ if and only if } w(\theta) = \overline{w}_{p,q}(\theta + \varphi) \text{ where} \]

\[ \overline{w}_{p,q}(\theta) = \begin{cases} 
\sin \left[ \sqrt{\mu} \left( c_{p,q}^{-1} \theta - \frac{\pi}{4} \right) \right], & \text{if } 0 \leq \theta < c_{p,q} \frac{\pi}{2} \\
M^{-(p+q)/4} \cos \left[ \sqrt{\mu} \left( \frac{\pi}{2} + c_{p,q}^{-1} M^{(p-q)/2} \left( \theta - c_{p,q} \frac{\pi}{2} \right) - \frac{3\pi}{4} \right) \right], & \text{if } c_{p,q} \frac{\pi}{2} \leq \theta < \pi \\
-\sin \left[ \sqrt{\mu} \left( \pi + c_{p,q}^{-1} (\theta - \pi) - \frac{5\pi}{4} \right) \right], & \text{if } \pi \leq \theta < \pi + c_{p,q} \frac{\pi}{2} \\
-M^{-(p+q)/4} \cos \left[ \sqrt{\mu} \left( \frac{3\pi}{2} + c_{p,q}^{-1} M^{(p-q)/2} \left( \theta - \pi - c_{p,q} \frac{\pi}{2} \right) - \frac{7\pi}{4} \right) \right], & \text{if } \pi + c_{p,q} \frac{\pi}{2} \leq \theta < 2\pi 
\end{cases} \]

and \( \mu = \left( \frac{4}{\pi} \right) \arctan M^{-(p+q)} \).

If \( p + q = 0 \), then (3) is an equality for any weight function \( \gamma \). Equality is attained in (1)-(2) with \( a = \gamma^p \) and \( b = \gamma^{-p} \) if and only if

\[ w(\theta) = C \cos \left( \frac{2\pi}{2\pi} \int_0^{\theta} \gamma^p + \varphi \right), \]

for some \( C \neq 0 \) and \( \varphi \in \mathbb{R} \).

Note that when \( p = q = 0 \), Theorem 1 yields \( C(1, 1) = 1 \) according to the classical Wirtinger inequality. When \( p = q \neq 0 \), the estimate (3) reduces to the estimate obtained by Piccinini and Spagnolo in [4]. More related results may be found in [1,2,3] and in the references therein. We begin by recalling in the following lemma the Wirtinger inequality of Piccinini and Spagnolo [4].

**Lemma 1 ([4]).** Suppose \( b = a \in \mathcal{B}(L) \). Then,

\[ C(a, a) \leq \left( \frac{4}{\pi} \arctan L^{-1/2} \right)^{-2}. \]
Equality holds in (4) if and only if \( a(\theta) = \overline{a}(\theta + \varphi) \) for some \( \varphi \in \mathbb{R} \), where \( \overline{a} \) is defined by

\[
\overline{a}(\theta) = \begin{cases} 
1, & \text{if } 0 \leq \theta < \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \theta < \frac{3\pi}{2} \\
L, & \text{if } \frac{\pi}{2} \leq \theta < \pi, \quad \frac{3\pi}{2} \leq \theta < 2\pi 
\end{cases}
\]

and equality holds in (1)-(2) with \( a(\theta) = b(\theta) = \overline{a}(\theta + \varphi) \) if and only if \( \overline{w}(\theta) = \overline{w}(\theta + \varphi) \), where

\[
\overline{w}(\theta) = \begin{cases} 
\sin \left[ \sqrt{\lambda} \left( \theta - \frac{\pi}{4} \right) \right], & \text{if } 0 \leq \theta < \frac{\pi}{2} \\
L^{-1/2} \cos \left[ \sqrt{\lambda} \left( \theta - \frac{3\pi}{4} \right) \right], & \text{if } \frac{\pi}{2} \leq \theta < \pi \\
-\sin \left[ \sqrt{\lambda} \left( \theta - \frac{5\pi}{4} \right) \right], & \text{if } \pi \leq \theta < \frac{3\pi}{2} \\
-L^{-1/2} \cos \left[ \sqrt{\lambda} \left( \theta - \frac{7\pi}{4} \right) \right], & \text{if } \frac{3\pi}{2} \leq \theta < 2\pi 
\end{cases}
\]

where \( \lambda = (4\pi^{-1} \arctan L^{-1/2})^2 \).

In order to prove Theorem 1, we need the following lemma, which yields an estimate for \( C(a, b) \) for arbitrary weight functions \( a, b \).

**Lemma 2.** Let \( a, b \in \mathcal{B} \). The following estimate holds:

\[
C(a, b) \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{ab^{-1}} \, \right)^2 \left( \frac{1}{4\pi} \arctan \left( \frac{\inf ab}{\sup ab} \right)^{1/4} \right)^2
\]

If \( \sqrt{ab} \in \mathcal{B}(L), L > 1 \), then

\[
C(a, b) = \sup_{\forall a', b' \in \mathcal{B}(L)} \frac{C(a', b')}{} = \left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a'b'^{-1}} \right)^2 = \left( \frac{4\pi}{\arctan L^{-1/2}} \right)^2
\]

if and only if the following equation is satisfied:

\[
a(\theta(\tau)) b(\theta(\tau)) = \overline{a}^2(\tau + \varphi) \quad \text{a.e. } \tau \in (0, 2\pi), \text{ for some } \varphi \in \mathbb{R},
\]
where \( \theta(\tau) \) is the homeomorphism of \( \mathbb{R} \) defined by

\[
\tau(\theta) = \frac{1}{c} \int_0^\theta \sqrt{\frac{a(\tilde{\theta})}{b(\tilde{\theta})}} \, d\tilde{\theta},
\]

\( c \) is defined by

\[
c = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\frac{a(\tilde{\theta})}{b(\tilde{\theta})}} \, d\tilde{\theta},
\]

and \( \tilde{a} \) is the function defined in Lemma 1.

If \( b = a^{-1} \), then \( C(a, a^{-1}) = \left( (2\pi)^{-1} \int_0^{2\pi} a \right)^2 \) and equality is attained in (1)-(2) with \( b = a^{-1} \) if and only if \( w(\theta) = C \cos \left( 2\pi \left( \int_0^{2\pi} a \int_0^\theta a + \varphi \right) \right) \) for some \( C \neq 0 \) and \( \varphi \in \mathbb{R} \).

Proof. – Under the change of variables \( \theta = \theta(\tau) \) defined by (10)-(11), setting \( a(\tau) = a(\theta(\tau)), \, \beta(\tau) = b(\theta(\tau)), \, \xi(\tau) = w(\theta(\tau)) \), we obtain

\[
a\theta' = c\sqrt{a\beta}, \quad \beta\theta'^{-1} = c^{-1}\sqrt{a\beta},
\]

and therefore:

\[
\int_0^{2\pi} aw^2 \, d\theta = \int_0^{2\pi} a\theta' \xi^2 \, d\tau = c \int \sqrt{a\beta} \xi^2 \, d\tau
\]

\[
\int_0^{2\pi} aw \, d\theta = \int_0^{2\pi} a\theta' \xi \, d\tau = c \int \sqrt{a\beta} \xi \, d\tau = 0
\]

\[
\int_0^{2\pi} bw^2 \, d\theta = \int_0^{2\pi} \beta\theta'^{-1} \xi^2 \, d\tau = c^{-1} \int \sqrt{a\beta} \xi^2 \, d\tau.
\]

Upon substitution, (1)-(2) takes the form:

\[
\int_0^{2\pi} \sqrt{a\beta} \xi^2 \, d\tau \leq \frac{C(a, b)}{\left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{ab^{-1}} \right)^2} \int_0^{2\pi} \sqrt{a\beta} \xi^2 \, d\tau,
\]

with constraint

\[
\int_0^{2\pi} \sqrt{a\beta} \xi \, d\tau = 0.
\]
If $\sqrt{ab} \in \mathcal{B}(L)$, in view of Lemma 1 we obtain

$$
(14) \quad 
\frac{C(a, b)}{\left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{ab}^{-1} \right)^2} = C(\sqrt{\alpha \beta}, \sqrt{\alpha \beta}) \leq \left( \frac{4}{\pi} \arctan \sqrt{\frac{\inf \sqrt{\alpha \beta}}{\sup \sqrt{\alpha \beta}}} \right)^{-2} 
= \left( \frac{4}{\pi} \arctan \left( \frac{\inf ab}{\sup ab} \right)^{1/4} \right)^{-2}.
$$

This yields (7). Moreover, we have $C(\sqrt{\alpha \beta}, \sqrt{\alpha \beta}) = ((4/\pi) \arctan L^{-1/2})^{-2}$ if and only if $\sqrt{\alpha \beta} = \bar{a}(\tau + \varphi)$, for some $\varphi \in \mathbb{R}$. That is, (8) holds if and only if (9) holds.

If $b = a^{-1}$, then (12)-(13) takes the form

$$
\int_0^{2\pi} \xi^2 d\tau \leq \frac{C(a, a^{-1})}{\left( \frac{1}{2\pi} \int_0^{2\pi} a \right)^2} \int_0^{2\pi} \xi^{-2} d\tau
$$

with constraint

$$
\int_0^{2\pi} \xi d\tau = 0.
$$

Therefore, by the classical Wirtinger inequality,

$$
C(a, a^{-1}) = \left( \frac{1}{2\pi} \int_0^{2\pi} a \right)^2
$$

and equality holds in (1)-\text{(2)} with $b = a^{-1}$ if and only if $\xi(\tau) = C \cos(\tau + \varphi)$ for some $C \neq 0$ and $\varphi \in \mathbb{R}$, that is, if and only if $\omega(\theta) = C \cos \left( 2\pi \left( \int_0^{2\pi} a \right)^{-1} \int_0^\theta a + \varphi \right)$, as asserted.

**Lemma 3.** Suppose $a, b$ satisfy $\sqrt{ab} \in \mathcal{B}(L)$, $L > 1$, and (9), where $\theta(\tau)$ is defined in (10) and $c$ is defined by (11). Suppose

$$
(15) \quad a = \gamma^p, \quad b = \gamma^q
$$

for some $\gamma \in \mathcal{B}(M)$, with $M = L^{2(p+q)}$, and for some $p, q \in \mathbb{R}$ such that $p + q > 0$. Then $\gamma(\theta) = \gamma_{p, q}(\theta + \varphi)$ for some $\varphi \in \mathbb{R}$, where $\gamma_{p, q}$ is the function defined in Theorem 1.
Proof. - When \( p + q > 0 \), we have \( \gamma^{(p+q)/2} \in \mathcal{B}(L) \). In view of (9) and (15) we have
\[
\gamma(\theta(\tau)) = \overline{a}^{2(p+q)}(\tau + \psi), \quad \forall \tau \in \mathbb{R}
\]
for some \( \psi \in \mathbb{R} \). It follows that
\[
(16) \quad \theta(\tau) = c \int_0^\tau \sqrt{\frac{b(\theta(\tau))}{a(\theta(\tau))}} \, d\tau = c \int_0^\tau \overline{a}^{-\frac{(p-q)}{2}}(\tau + \psi) \, d\tau
\]
and, in view of the 2\( \pi \)-periodicity of \( a \) and \( b \),
\[
c = \left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\frac{b(\theta(\tau))}{a(\theta(\tau))}} \, d\tau \right)^{-1} = \left( \frac{1}{2\pi} \int_0^{2\pi} \overline{a}^{-\frac{(p-q)}{2}}(\tau + \psi) \, d\tau \right)^{-1}
\]
Setting
\[
h_{p, q}(\tau) = c \int_0^\tau \overline{a}^{-\frac{(p-q)}{2}}(\tau + \psi) \, d\tau,
\]
we have \( \theta(\tau - \psi) = h_{p, q}(\tau) - h_{p, q}(\psi) \) for every \( \tau \in \mathbb{R} \), and consequently \( \tau(\theta) = h_{p, q}^{-1}(\theta + h_{p, q}(\psi)) - \psi \). In view of the definition of \( \overline{a} \) with \( L = M^{(p+q)/2} \), we have:
\[
\int_0^\tau \overline{a}^{-\frac{(p-q)}{2}}(\tau + \psi) \, d\tau =
\[
\begin{cases}
\tau, & \text{if } 0 \leq \tau < \frac{\pi}{2} \\
\frac{\pi}{2} + M^{-(p-q)/2} \left( \tau - \frac{\pi}{2} \right), & \text{if } \frac{\pi}{2} \leq \tau < \pi \\
\frac{\pi}{2} \left( 1 + M^{-(p-q)/2} \right) + \tau - \pi, & \text{if } \pi \leq \tau < \frac{3\pi}{2} \\
\frac{\pi}{2} \left( 2 + M^{-(p-q)/2} \right) + M^{-(p-q)/2} \left( \tau - \frac{3\pi}{2} \right), & \text{if } \frac{3\pi}{2} \leq \tau < 2\pi
\end{cases}
\]
In particular, we derive
\[
c = \frac{2}{1 + M^{-(p-q)/2}} = c_{p, q}.
\]
It follows that \( h_{p, q}(\tau) \) is the piecewise linear homeomorphism of \( \mathbb{R} \) defined in
\[ h_{p, q}(\tau) = \begin{cases} 
 c_{p, q}\tau, & \text{if } 0 \leq \tau < \frac{\pi}{2} \\
 c_{p, q}\left[\frac{\pi}{2} + M^{-(p-q)/2}(\tau - \frac{\pi}{2})\right], & \text{if } \frac{\pi}{2} \leq \tau < \pi \\
 c_{p, q}\left[\frac{\pi}{2}(1 + M^{-(p-q)/2}) + \tau - \pi\right], & \text{if } \pi \leq \tau < \frac{3\pi}{2} \\
 c_{p, q}\left[\frac{\pi}{2}(2 + M^{-(p-q)/2}) + M^{-(p-q)/2}\left(\tau - \frac{3\pi}{2}\right)\right], & \text{if } \frac{3\pi}{2} \leq \tau < 2\pi 
\end{cases} \]

and by \( h_{p, q}(\tau + 2\pi n) = 2\pi n + h_{p, q}(\tau) \), for any \( \tau \in [0, 2\pi) \) and for any integer \( n \). Inversion yields

\[ h_{p, q}^{-1}(\theta) = \begin{cases} 
 c_{p, q}^{-1}\theta, & \text{if } 0 \leq \theta < c_{p, q}\frac{\pi}{2} \\
 \frac{\pi}{2} + c_{p, q}^{-1}M^{(p-q)/2}\left(\theta - c_{p, q}\frac{\pi}{2}\right), & \text{if } c_{p, q}\frac{\pi}{2} \leq \theta < \pi \\
 \pi + c_{p, q}^{-1}(\theta - \pi), & \text{if } \pi \leq \theta < \pi + c_{p, q}\frac{\pi}{2} \\
 \frac{3\pi}{2} + c_{p, q}^{-1}M^{(p-q)/2}\left(\theta - \pi - c_{p, q}\frac{\pi}{2}\right), & \text{if } \pi + c_{p, q}\frac{\pi}{2} \leq \theta < 2\pi 
\end{cases} \]

for \( \theta \in [0, 2\pi) \) and \( h_{p, q}^{-1}(\theta + 2\pi n) = 2\pi n + h_{p, q}^{-1}(\theta) \) for any \( \tau \in [0, 2\pi) \) and for any integer \( n \). Substitution yields \( \gamma(\theta) = \tilde{a}^{2/p+q}(h_{p, q}^{-1}(\theta + h_{p, q}(\psi))) = \tilde{a}^{2/p+q}(h_{p, q}^{-1}(\theta + \varphi)) = \overline{\gamma}_{p, q}(\theta + \varphi), \) with \( \varphi = h_{p, q}(\psi) \). □

Now we can prove Theorem 1.

**Proof of Theorem 1.** — Estimate (7) with \( a = \gamma^p \) and \( b = \gamma^q \) yields (3). Suppose \( p + q > 0 \). In view of Lemma 2 and Lemma 3 we have

\[
\frac{C(\gamma^p, \gamma^q)}{\left(\frac{1}{2\pi} \int_0^{2\pi} \gamma^{(p-q)/2}\right)^2} = \left(\frac{4}{\pi} \arctan M^{-(p+q)/4}\right)^2
\]

if and only if \( \gamma(\theta) = \overline{\gamma}_{p, q}(\theta + \varphi) \) for some \( \varphi \in \mathbb{R} \). Equality is attained in (1)-(2) with \( a(\theta) = \overline{\gamma}_{p, q}^p(\theta + \varphi) \) and \( b(\theta) = \overline{\gamma}_{p, q}^q(\theta + \varphi) \) if and only if \( \psi(\theta) = \overline{\psi}_{p, q}(\theta + \varphi) \). If \( p + q = 0 \), then the conclusion follows by Lemma 2 with \( a = \gamma^p \) and \( b = \gamma^{-p} \). □
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