Bollettino U. M. I.
(8) 8-B (2005), 259-267

A Sharp Weighted Wirtinger Inequality.

TONIA RICCIARDI (*)

Sunto. – *Si ottiene una stima ottimale per la migliore costante $C > 0$ nella disuguaglianza di tipo Wirtinger*

$$\int_0^{2\pi} \gamma^p w^2 \leqslant C \int_0^{2\pi} \gamma^q w'^2$$

dove γ è limitata superiormente e dotata di estremo inferiore positivo, w è periodica di periodo 2π e tale che $\int_0^{2\pi} \gamma^p w = 0$, e $p + q \geqslant 0$. Tale risultato generalizza una disuguaglianza di Piccinini e Spagnolo.

Summary. – *We obtain a sharp estimate for the best constant $C > 0$ in the Wirtinger type inequality*

$$\int_0^{2\pi} \gamma^p w^2 \leqslant C \int_0^{2\pi} \gamma^q w'^2$$

where γ is bounded above and below away from zero, w is 2π-periodic and such that $\int_0^{2\pi} \gamma^p w = 0$, and $p + q \geqslant 0$. Our result generalizes an inequality of Piccinini and Spagnolo.

Let $C(a, b) > 0$ denote the best constant in the following weighted Wirtinger type inequality:

$$\int_0^{2\pi} aw^2 \leqslant C(a, b) \int_0^{2\pi} bw'^2, \tag{1}$$

(*) Supported in part by Regione Campania L.R. 5/02 and by the MIUR National Project *Variational Methods and Nonlinear Differential Equations*.

where $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ is 2π-periodic and satisfies the constraint

(2) $$\int_0^{2\pi} aw = 0\,,$$

and $a, b \in \mathcal{B}$ with

$$\mathcal{B} = \{a \in L^\infty(\mathbb{R}) : a \text{ is } 2\pi\text{-periodic and } \inf a > 0\}.$$

Here and in what follows, for every measurable function a we denote by $\inf a$ and $\sup a$ the essential lower bound and the essential upper bound of a, respectively. For every $L > 1$, we denote

$$\mathcal{B}(L) = \{a \in L^\infty(0, 2\pi) : a \text{ is } 2\pi\text{-periodic}, \inf a = 1 \text{ and } \sup a = L\}.$$

Our aim in this note is to prove:

THEOREM 1. – *Suppose $a = \gamma^p$ and $b = \gamma^q$ for some $\gamma \in \mathcal{B}(M)$, $M > 1$, and for some $p, q \in \mathbb{R}$ such that $p + q \geq 0$. Then*

(3) $$C(\gamma^p, \gamma^q) \leq \left(\frac{\frac{1}{2\pi}\int_0^{2\pi} \gamma^{(p-q)/2}}{\frac{4}{\pi}\arctan\left(M^{-(p+q)/4}\right)} \right)^2.$$

If $p + q > 0$, then equality holds in (3) if and only if $\gamma(\theta) = \overline{\gamma}_{p,q}(\theta + \varphi)$ for some $\varphi \in \mathbb{R}$, where

$$\overline{\gamma}_{p,q}(\theta) = \begin{cases} 1, & \text{if } 0 \leq \theta < c_{p,q}\frac{\pi}{2}, \quad \pi \leq \theta < \pi + c_{p,q}\frac{\pi}{2} \\ M, & \text{if } c_{p,q}\frac{\pi}{2} \leq \theta < \pi, \quad \pi + c_{p,q}\frac{\pi}{2} \leq \theta < 2\pi \end{cases},$$

with

$$c_{p,q} = \frac{2}{1 + M^{-(p-q)/2}}.$$

Furthermore, equality holds in (1)-(2) with $a(\theta) = \overline{\gamma}^p_{p,q}(\theta + \varphi)$ and $b(\theta) =$

$\overline{\gamma}^q_{p,q}(\theta+\varphi)$ *if and only if* $w(\theta)=\overline{w}_{p,q}(\theta+\varphi)$ *where*

$$\overline{w}_{p,q}(\theta)=\begin{cases}\sin\left[\sqrt{\mu}\left(c_{p,q}^{-1}\theta-\dfrac{\pi}{4}\right)\right], & \text{if } 0\leqslant\theta<c_{p,q}\dfrac{\pi}{2}\\[2mm] M^{-(p+q)/4}\cos\left[\sqrt{\mu}\left(\dfrac{\pi}{2}+c_{p,q}^{-1}M^{(p-q)/2}\left(\theta-c_{p,q}\dfrac{\pi}{2}\right))-\dfrac{3\pi}{4}\right)\right], & \text{if } c_{p,q}\dfrac{\pi}{2}\leqslant\theta<\pi\\[2mm] -\sin\left[\sqrt{\mu}\left(\pi+c_{p,q}^{-1}(\theta-\pi)-\dfrac{5\pi}{4}\right)\right], & \text{if } \pi\leqslant\theta<\pi+c_{p,q}\dfrac{\pi}{2}\\[2mm] -M^{-(p+q)/4}\cos\left[\sqrt{\mu}\left(\dfrac{3\pi}{2}+c_{p,q}^{-1}M^{(p-q)/2}\left(\theta-\pi-c_{p,q}\dfrac{\pi}{2}\right)-\dfrac{7\pi}{4}\right)\right], & \text{if } \pi+c_{p,q}\dfrac{\pi}{2}\leqslant\theta<2\pi\end{cases},$$

and $\mu=\left((4/\pi)\arctan M^{-(p+q)}\right)^2$.

If $p+q=0$, *then* (3) *is an equality for any weight function* γ. *Equality is attained in* (1)-(2) *with* $a=\gamma^p$ *and* $b=\gamma^{-p}$ *if and only if*

$$w(\theta)=C\cos\left(\frac{2\pi}{\int_0^{2\pi}\gamma^p}\int_0^\theta\gamma^p+\varphi\right),$$

for some $C\neq 0$ *and* $\varphi\in\mathbb{R}$.

Note that when $p=q=0$, Theorem 1 yields $C(1,1)=1$ according to the classical Wirtinger inequality. When $p=q\neq 0$, the estimate (3) reduces to the estimate obtained by Piccinini and Spagnolo in [4]. More related results may be found in [1,2,3] and in the references therein. We begin by recalling in the following lemma the Wirtinger inequality of Piccinini and Spagnolo [4].

LEMMA 1 ([4]). – *Suppose* $b=a\in\mathcal{B}(L)$. *Then,*

$$C(a,a)\leqslant\left(\frac{4}{\pi}\arctan L^{-1/2}\right)^{-2}. \tag{4}$$

Equality holds in (4) *if and only if* $a(\theta) = \overline{a}(\theta + \varphi)$ *for some* $\varphi \in \mathbb{R}$, *where* $\overline{a}$ *is defined by*

$$\overline{a}(\theta) = \begin{cases} 1, & \text{if } 0 \leqslant \theta < \dfrac{\pi}{2}, \quad \pi \leqslant \theta < \dfrac{3\pi}{2} \\ L, & \text{if } \dfrac{\pi}{2} \leqslant \theta < \pi, \quad \dfrac{3\pi}{2} \leqslant \theta < 2\pi \end{cases} \tag{5}$$

and equality holds in (1)-(2) *with* $a(\theta) = b(\theta) = \overline{a}(\theta + \varphi)$ *if and only if* $w(\theta) = \overline{w}(\theta + \varphi)$, *where*

$$\overline{w}(\theta) = \begin{cases} \sin\left[\sqrt{\lambda}\left(\theta - \dfrac{\pi}{4}\right)\right], & \text{if } 0 \leqslant \theta < \dfrac{\pi}{2} \\ L^{-1/2}\cos\left[\sqrt{\lambda}\left(\theta - \dfrac{3\pi}{4}\right)\right], & \text{if } \dfrac{\pi}{2} \leqslant \theta < \pi \\ -\sin\left[\sqrt{\lambda}\left(\theta - \dfrac{5\pi}{4}\right)\right], & \text{if } \pi \leqslant \theta < \dfrac{3\pi}{2} \\ -L^{-1/2}\cos\left[\sqrt{\lambda}\left(\theta - \dfrac{7\pi}{4}\right)\right], & \text{if } \dfrac{3\pi}{2} \leqslant \theta < 2\pi \end{cases}, \tag{6}$$

where $\lambda = (4\pi^{-1} \arctan L^{-1/2})^2$.

In order to prove Theorem 1, we need the following lemma, which yields an estimate for $C(a, b)$ for *arbitrary* weight functions a, b.

LEMMA 2. – *Let* $a, b \in \mathcal{B}$. *The following estimate holds:*

$$C(a, b) \leqslant \left(\frac{\dfrac{1}{2\pi} \int_0^{2\pi} \sqrt{ab^{-1}}}{\dfrac{4}{\pi} \arctan \left(\dfrac{\inf ab}{\sup ab} \right)^{1/4}} \right)^2. \tag{7}$$

If $\sqrt{ab} \in \mathcal{B}(L)$, $L > 1$, *then*

$$\frac{C(a, b)}{\left(\dfrac{1}{2\pi} \int_0^{2\pi} \sqrt{ab^{-1}} \right)^2} = \sup_{\sqrt{a'b'} \in \mathcal{B}(L)} \frac{C(a', b')}{\left(\dfrac{1}{2\pi} \int_0^{2\pi} \sqrt{a'b'^{-1}} \right)^2} = \left(\frac{4}{\pi} \arctan L^{-1/2} \right)^{-2} \tag{8}$$

if and only if the following equation is satisfied:

$$a(\theta(\tau))\, b(\theta(\tau)) = \overline{a}^2(\tau + \varphi) \quad \text{a.e. } \tau \in (0, 2\pi), \quad \text{for some } \varphi \in \mathbb{R}, \tag{9}$$

where $\theta(\tau)$ *is the homeomorphism of* $\mathbb{R}$ *defined by*

$$\tau(\theta) = \frac{1}{c}\int_0^\theta \sqrt{\frac{a(\tilde\theta)}{b(\tilde\theta)}}\,d\tilde\theta\,, \tag{10}$$

c is defined by

$$c = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{\frac{a(\tilde\theta)}{b(\tilde\theta)}}\,d\tilde\theta\,, \tag{11}$$

and $\bar{a}$ *is the function defined in Lemma* **1**.

If $b = a^{-1}$, *then* $C(a, a^{-1}) = \left((2\pi)^{-1}\int_0^{2\pi} a\right)^2$ *and equality is attained in* (1)-(2) *with* $b = a^{-1}$ *if and only if* $w(\theta) = C\cos\left(2\pi\left(\int_0^{2\pi} a\right)^{-1}\int_0^\theta a + \varphi\right)$ *for some* $C \neq 0$ *and* $\varphi \in \mathbb{R}$.

PROOF. – Under the change of variables $\theta = \theta(\tau)$ defined by (10)-(11), setting $\alpha(\tau) = a(\theta(\tau))$, $\beta(\tau) = b(\theta(\tau))$, $\xi(\tau) = w(\theta(\tau))$, we obtain

$$\alpha\theta' = c\sqrt{\alpha\beta}\,, \qquad \beta\theta'^{-1} = c^{-1}\sqrt{\alpha\beta},$$

and therefore:

$$\int_0^{2\pi} aw^2\,d\theta = \int_0^{2\pi} \alpha\theta'\,\xi^2\,d\tau = c\int \sqrt{\alpha\beta}\xi^2\,d\tau$$

$$\int_0^{2\pi} aw\,d\theta = \int_0^{2\pi} \alpha\theta'\,\xi\,d\tau = c\int \sqrt{\alpha\beta}\xi\,d\tau = 0$$

$$\int_0^{2\pi} bw'^2\,d\theta = \int_0^{2\pi} \beta\theta'^{-1}\,\xi'^2\,d\tau = c^{-1}\int \sqrt{\alpha\beta}\xi'^2\,d\tau\,.$$

Upon substitution, (1)-(2) takes the form:

$$\int_0^{2\pi} \sqrt{\alpha\beta}\xi^2\,d\tau \leqslant \frac{C(a, b)}{\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{ab^{-1}}\right)^2}\int_0^{2\pi}\sqrt{\alpha\beta}\xi'^2\,d\tau\,, \tag{12}$$

with constraint

$$\int_0^{2\pi} \sqrt{\alpha\beta}\xi\,d\tau = 0\,. \tag{13}$$

If $\sqrt{ab} \in \mathcal{B}(L)$, in view of Lemma 1 we obtain

$$(14)\qquad \frac{C(a,\,b)}{\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{ab^{-1}}\right)^2} = C(\sqrt{\alpha\beta},\,\sqrt{\alpha\beta}) \leqslant \left(\frac{4}{\pi}\arctan\sqrt{\frac{\inf\sqrt{\alpha\beta}}{\sup\sqrt{\alpha\beta}}}\right)^{-2}$$

$$= \left(\frac{4}{\pi}\arctan\left(\frac{\inf ab}{\sup ab}\right)^{1/4}\right)^{-2}.$$

This yields (7). Moreover, we have $C(\sqrt{\alpha\beta},\,\sqrt{\alpha\beta}) = ((4/\pi)\arctan L^{-1/2})^{-2}$ if and only if $\sqrt{\alpha(\tau)\,\beta(\tau)} = \overline{a}(\tau+\varphi)$, for some $\varphi \in \mathbb{R}$. That is, (8) holds if and only if (9) holds.

If $b = a^{-1}$, then (12)-(13) takes the form

$$\int_0^{2\pi} \xi^2\,\mathrm{d}\tau \leqslant \frac{C(a,\,a^{-1})}{\left(\frac{1}{2\pi}\int_0^{2\pi} a\right)^2}\int_0^{2\pi}\xi'^2\,\mathrm{d}\tau$$

with constraint

$$\int_0^{2\pi}\xi\,\mathrm{d}\tau = 0\,.$$

Therefore, by the classical Wirtinger inequality,

$$C(a,\,a^{-1}) = \left(\frac{1}{2\pi}\int_0^{2\pi} a\right)^2$$

and equality holds in (1)-(2) with $b = a^{-1}$ if and only if $\xi(\tau) = C\cos(\tau+\varphi)$ for some $C \neq 0$ and $\varphi \in \mathbb{R}$, that is, if and only if $w(\theta) = C\cos\left(2\pi\left(\int_0^{2\pi} a\right)^{-1}\int_0^{\theta} a + \varphi\right)$, as asserted. ■

Lemma 3. – *Suppose a, b satisfy $\sqrt{ab} \in \mathcal{B}(L)$, $L > 1$, and (9), where $\theta(\tau)$ is defined in (10) and c is defined by (11). Suppose*

$$(15)\qquad a = \gamma^p, \qquad b = \gamma^q$$

for some $\gamma \in \mathcal{B}(M)$, with $M = L^{2/(p+q)}$, and for some p, $q \in \mathbb{R}$ such that $p + q > 0$. Then $\gamma(\theta) = \overline{\gamma}_{p,\,q}(\theta+\varphi)$ for some $\varphi \in \mathbb{R}$, where $\overline{\gamma}_{p,\,q}$ is the function defined in Theorem 1.

PROOF. – When $p+q>0$, we have $\gamma^{(p+q)/2}\in\mathcal{B}(L)$. In view of (9) and (15) we have

$$\gamma(\theta(\tau)) = \overline{a}^{2/(p+q)}(\tau+\psi), \qquad \forall\tau\in\mathbb{R}$$

for some $\psi\in\mathbb{R}$. It follows that

$$\theta(\tau) = c\int_0^\tau \sqrt{\frac{b(\theta(\overline{\tau}))}{a(\theta(\overline{\tau}))}}\,d\overline{\tau} = c\int_0^\tau \overline{a}^{-(p-q)/(p+q)}(\overline{\tau}+\psi)\,d\overline{\tau} \tag{16}$$

and, in view of the 2π-periodicity of a and b,

$$c = \left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{b(\theta(\overline{\tau}))}{a(\theta(\overline{\tau}))}}\,d\overline{\tau}\right)^{-1} = \left(\frac{1}{2\pi}\int_0^{2\pi}\overline{a}^{-(p-q)/(p+q)}(\overline{\tau})\,d\overline{\tau}\right)^{-1}.$$

Setting

$$h_{p,q}(\tau) = c\int_0^\tau \overline{a}^{-(p-q)/(p+q)}(\overline{\tau})\,d\overline{\tau},$$

we have $\theta(\tau-\psi) = h_{p,q}(\tau) - h_{p,q}(\psi)$ for every $\tau\in\mathbb{R}$, and consequently $\tau(\theta) = h_{p,q}^{-1}(\theta + h_{p,q}(\psi)) - \psi$. In view of the definition of $\overline{a}$ with $L = M^{(p+q)/2}$, we have:

$$\int_0^\tau \overline{a}^{-(p-q)/(p+q)}(\overline{\tau})\,d\overline{\tau} =$$

$$\begin{cases} \tau, & \text{if } 0\leq\tau<\dfrac{\pi}{2} \\[2mm] \dfrac{\pi}{2} + M^{-(p-q)/2}\left(\tau-\dfrac{\pi}{2}\right), & \text{if } \dfrac{\pi}{2}\leq\tau<\pi \\[2mm] \dfrac{\pi}{2}(1+M^{-(p-q)/2}) + \tau - \pi, & \text{if } \pi\leq\tau<\dfrac{3\pi}{2} \\[2mm] \dfrac{\pi}{2}(2+M^{-(p-q)/2}) + M^{-(p-q)/2}\left(\tau-\dfrac{3\pi}{2}\right), & \text{if } \dfrac{3\pi}{2}\leq\tau<2\pi \end{cases}.$$

In particular, we derive

$$c = \frac{2}{1+M^{-(p-q)/2}} = c_{p,q}.$$

It follows that $h_{p,q}(\tau)$ is the piecewise linear homeomorphism of $\mathbb{R}$ defined in

$[0, 2\pi)$ by

$$h_{p,q}(\tau) = \begin{cases} c_{p,q}\tau, & \text{if } 0 \leqslant \tau < \frac{\pi}{2} \\ c_{p,q}\left[\frac{\pi}{2} + M^{-(p-q)/2}(\tau - \frac{\pi}{2})\right], & \text{if } \frac{\pi}{2} \leqslant \tau < \pi \\ c_{p,q}\left[\frac{\pi}{2}(1 + M^{-(p-q)/2}) + \tau - \pi\right], & \text{if } \pi \leqslant \tau < \frac{3\pi}{2} \\ c_{p,q}\left[\frac{\pi}{2}(2 + M^{-(p-q)/2}) + M^{-(p-q)/2}\left(\tau - \frac{3\pi}{2}\right)\right], & \text{if } \frac{3\pi}{2} \leqslant \tau < 2\pi \end{cases}$$

and by $h_{p,q}(\tau + 2\pi n) = 2\pi n + h_{p,q}(\tau)$, for any $\tau \in [0, 2\pi)$ and for any integer n. Inversion yields

$$h_{p,q}^{-1}(\theta) = \begin{cases} c_{p,q}^{-1}\theta, & \text{if } 0 \leqslant \theta < c_{p,q}\frac{\pi}{2} \\ \frac{\pi}{2} + c_{p,q}^{-1}M^{(p-q)/2}\left(\theta - c_{p,q}\frac{\pi}{2}\right), & \text{if } c_{p,q}\frac{\pi}{2} \leqslant \theta < \pi \\ \pi + c_{p,q}^{-1}(\theta - \pi), & \text{if } \pi \leqslant \theta < \pi + c_{p,q}\frac{\pi}{2} \\ \frac{3\pi}{2} + c_{p,q}^{-1}M^{(p-q)/2}\left(\theta - \pi - c_{p,q}\frac{\pi}{2}\right), & \text{if } \pi + c_{p,q}\frac{\pi}{2} \leqslant \theta < 2\pi \end{cases},$$

for $\theta \in [0, 2\pi)$ and $h_{p,q}^{-1}(\theta + 2\pi n) = 2\pi n + h_{p,q}^{-1}(\theta)$ for any $\tau \in [0, 2\pi)$ and for any integer n. Substitution yields $\gamma(\theta) = \overline{a}^{2/(p+q)}(h_{p,q}^{-1}(\theta + h_{p,q}(\psi))) = \overline{a}^{2/(p+q)}(h_{p,q}^{-1}(\theta + \varphi)) = \overline{\gamma}_{p,q}(\theta + \varphi)$, with $\varphi = h_{p,q}(\psi)$. ■

Now we can prove Theorem 1.

PROOF OF THEOREM 1. – Estimate (7) with $a = \gamma^p$ and $b = \gamma^q$ yields (3). Suppose $p + q > 0$. In view of Lemma 2 and Lemma 3 we have

$$\frac{C(\gamma^p, \gamma^q)}{\left(\frac{1}{2\pi}\int_0^{2\pi} \gamma^{(p-q)/2}\right)^2} = \left(\frac{4}{\pi}\arctan M^{-(p+q)/4}\right)^{-2}$$

if and only if $\gamma(\theta) = \overline{\gamma}_{p,q}(\theta + \varphi)$ for some $\varphi \in \mathbb{R}$. Equality is attained in (1)-(2) with $a(\theta) = \overline{\gamma}_{p,q}^p(\theta + \varphi)$ and $b(\theta) = \overline{\gamma}_{p,q}^q(\theta + \varphi)$ if and only if $w(\theta) = \overline{w}_{p,q}(\theta + \varphi)$.

If $p + q = 0$, then the conclusion follows by Lemma 2 with $a = \gamma^p$ and $b = \gamma^{-p}$. ■

Acknowledgements. I am grateful to Professor Carlo Sbordone for many useful and stimulating discussions.

REFERENCES

[1] P. R. BEESACK, *Integral inequalities of the Wirtinger type*, Duke Math. Jour., **25** (1958), 477-498.

[2] G. CROCE - B. DACOROGNA, *On a generalized Wirtinger inequality*, Discrete Cont. Dynam. Systems, **9**, No. 5 (2003), 1329-1341.

[3] B. DACOROGNA - W. GANGBO - N. SUBÍA, *Sur une généralisation de l'inégalité de Wirtinger*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **9** (1992), 29-50.

[4] L. C. PICCININI - S. SPAGNOLO, *On the Hölder continuity of solutions of second order elliptic equations in two variables*, Ann. Scuola Norm. Sup. Pisa, **26**, No. 2 (1972), 391-402.

Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II
Via Cintia, 80126 Naples, Italy; fax: +39 081 675665
tonia.ricciardi@unina.it

Pervenuta in Redazione
il 3 gennaio 2005